A TEXT BOOK OF STATISTICS

N. M. Kapoor

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A TEXT BOOK
OF
STATISTICS

[Covering the complete syllabus of B.A./B.Sc. (Hons. & Pass) students of Indian Universities]

By

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EDUCATIONAL PUBLISHERS

888, EAST PARK ROAD, KAROL BAGH
NEW DELHI-110005 (INDIA)

BY THE SAME AUTHOR

A Text Book of Differential Equations

Price : Rs. 62-50

Edition : First, 1988
Second, 1989

Published in India by **M/s. Pitambar Publishing Company,** 888 East Park Road, Karol Bagh, New Delhi-110005, Tel. : Off. 770067 526933, Res. : 586788, 5721321, Telex : 31-65608 PTBR-IN and printed at R.B.S. Printers, F-62, Bali Nagar, New Delhi-110 015 Tel. : 501769.

PREFACE TO THE SECOND EDITION

It is a matter of great satisfaction for the author that the readers have welcomed this book and the first edition has been exhausted in less than one year's time. In this edition, one new chapter on Multiple and Partial correlation has been added. It is hoped that, in the present form, the book will prove of much greater utility to the students as well as teachers of the subject.

Suggestions for further improvement are welcome and will be most gratefully acknowledged.

N.M. Kapoor

PREFACE TO THE FIRST EDITION

Statistics today is an indispensable part of every human activity. The subject, therefore, forms part of the syllabus of degree and post-degree as well as several professional and competitive examinations.

The purpose of this book is to treat statistics as a self contained mathematical subject rigorously, avoiding non-mathematical concepts. The book assumes no previous knowledge of the subject on the part of the reader and aims at complete clarity for the beginner and such simplicity of exposition as will make the text practically self-teaching.

Although the textual material is concise and to the point, attention has been paid to the development of the underlying concepts. A serious attempt has been made to unify methods.

The subject is presented in a modulated and graded manner beginning with a fundamental core of introductory material which develops gradually from the simple and the easy to the complex and the intricate.

The examples are accompanied by problems. Some of them are simple exercises but most of them serve as additional illustrative material to the text or contain various complements. One purpose of the examples and problems is to develop the reader's intuition. Several previously treated examples show that apparently difficult problems may become almost trite once they are formulated in a natural way and put into proper context. To enhance further the utility of the book, thought provoking questions, carefully selected and systematically arranged are added at the end of each chapter.

I feel confident that the book will meet a real need.

I take this opportunity to thank the various well-known authorities of the world from whom I have drawn an inspiration. I am also grateful to all my colleagues in the Deptt. of Mathematics for the kind help they have given in the preparation of this book.

In the end I owe a debt of gratitude to the Publishers and Printers for their full co-operation.

Suggestions to further increase the utility of the book are welcome and will be most gratefully acknowledged.

N.M. Kapoor

To
My Wife

Contents

Introductory

0.1 The word statistics and the related words 'statistical' and 'statistician' have various meanings To a man in the street statistics are only figures and statistician as one who counts the number of things. To the economist, statistical stands for the quantitative and to the physicist statistical is opposite of individualistic or exact. The word statistics have either been derived from Status (Latin word) or Statistica (Italian) or Statistik (German) each of which means an "organized political state". Originally statistics was related to the collection of factual details concerning a state and this is why earlier it was known as the "Science of State Craft".

The word 'statistics' when used in plural means the numerical data collected in an orderly manner with specific end in view but in its singular meaning it means the theoretical science with techniques which deals in to collect, analyse and draw conclusions from the data.

Statistics is both a science and an art. It is a science because its methods, like other branches of science, are basically systematic and have general application and is an art in that their successful application depends to a considerable degree on the skill and experience of the statistician.

0.2. Scope of Statistics

The applications of statistics are so numerous and ever-increasing that not only it is difficult to define its scope but also unwise to do so. These days it is introducing into every branch of human knowledge. A student of science, while conducting an experiment, has to rely upon the application of statistics. In biology, testing of significance is applied to compare the effects of two drugs, the law of probability is used in radiation when the cells in the retina of the eye are exposed to light and statistical papers are used to study heart-beats through electrocardiogram.

Statistics is also used in agricultural and used in this way it is called *Agricultural Statistics.* Various Agricultural problems are solved or simplified by applying some suitable scheme for collection, analysis and the interpretation.

Statistics is an important member of the mathematical family. It is regarded as a branch of applied mathematics which specialises in data statistical techniques are of great assistance in the defence strategies to plan maximum destruction with minimum effect. Also

for the day-to-day functioning Government depends upon statistics. In insurance also statistics is used. Calculation of premiums and annuity etc., is wholly a statistical work based on the theory of probability and expectation.

0.3. Distrust of Statistics

Inspite of the very valuable service that statistics renders, there is some amount of misgiving in the minds of a few people with regard to its reliability and usefulness. It is said :

(1) "With statistics anything can be proved"

(2) There are three kinds of lies : namely (*i*) lies, (*ii*) damned lies and (*iii*) statistics—wicked in the order of their meaning.

0.4. Limitations of Statistics

Statistics has its own limitations. It cannot be applied to all kinds of phenomena and cannot be made to answer all queries. Few of its main limitations are listed below :

(1) It deals only with those subjects of inquiry which can be measured quantitatively and can be expressed numerically.

(2) It deals only with aggregates of facts and no importance is attached to individual items.

(3) Statistical data is only approximately and not mathematically correct.

(4) Statistics can be used to establish wrong conclusions and therefore can be used only by experts.

The methods by which statistical data are analyzed are called *statistical methods*. These methods range from the most elementary descriptive devices which may be understood by the common man to those complicated mathematical procedures which can be apprehended only by the expert theoreticians. The mathematical theory which is the basis of these methods is called the *theory of statistics* or *mathematical statistics*. The purpose of this text is to discuss the fundamental principles and theory of statistics in simple and easily comprehensible manner.

1

Frequency Distribution and Measures of Central Tendency

1.1. Introduction

By a variable is meant a quantity which assumes different values. A variable may be continuous or discrete. If a variable can take any numerical value within a certain range it is called a continuous variable. A variable, for which there is a gap between its two successive values, is called discrete. Heights, weights are examples of continuous variable and marks, test scores etc., are discrete variables.

In this and next few chapters only discrete variables will be considered although the results obtained will also be true for continuous variables.

1.2. Frequency Distribution

A frequency distribution is one where the values of the variable and the number of times each value is taken are put together. The number of times the value is taken is called the *frequency* of that value. The sum of all the frequencies is called **total frequency**.

When the values of the variable are presented in the form of groups, the representation is called **grouped frequency distribution**. The groups are called **classes** and the boundary figures are called **class limits**. The figures on the left are called **lower limits** and those on the right are called **upper limits**. The difference between the two limits is called class-interval or width of the class. The frequency of the class is called **class frequency**. The values midway between lower and upper limits are called **mid-values** or **central values**.

For computational purposes, it is assumed that variate takes mid values only. Thus, a frequency distribution can be taken in the form

$$\begin{array}{c} x \rightarrow \\ f \rightarrow \end{array} \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ f_1 & f_2 & \cdots & f_n \end{pmatrix}$$

where $x_1, x_2, \ldots, x_n$ are the values of the variable x with frequencies $f_1, f_2, \ldots, f_n$.

Let $f_1+f_2+\ldots+f_n=N$

Then, N is total frequency.

1.3. Measures of location or central tendency

These are statistical constants which give an idea about the concentration of the values in the central part of the distribution. It can be thought of as the value of the variable which is representative of the entire distribution. The following are the various measures of central tendency.

(*i*) **Arithmetic Mean.** *It is defined by*

$$\bar{x}=\frac{x_1 f_1+x_2 f_2+\ldots\ldots+x_n f_n}{f_1+f_2+\ldots\ldots f_n}$$

To obtain it for a given data, calculations are simplified by taking the variate defined by

$$u=\frac{x-a}{h}$$

where 'a' and 'h' are to be chosen suitably. Then A.M. is given by

$$\bar{x}=a+h\,\bar{u}$$

where $\bar{u}$ is A.M. of u.

Ex. 1-1. *Show that the algebraic sum of the deviations of a set of values from their arithmetic mean is zero*

Sol. Let $x_1, x_2, \ldots, x_n$ be the set of values with frequencies $f_1, f_2 \ldots f_n$.

Let $N=f_1+f_2+\ldots+f_n$

Let $\bar{x}$ be A.M. Then $\bar{x}=\frac{1}{N}\sum_{i=1}^{n} f_i x_i$

Now $\sum_{i=1}^{n} f_i(x_i-\bar{x})=\sum_{i=1}^{n} f_i x_i-\bar{x}\sum_{i=1}^{n} f_i$

$$=N\bar{x}-\bar{x}N$$

$$=0.$$

Ex. 1-2. *Show that the arithmetic mean of the first n natural numbers is $\frac{n+1}{2}$*

Sol.
$$\text{A.M.}=\frac{1+2+\ldots+n}{n}$$
$$=\frac{n(n+1)}{2n}=\frac{n+1}{2}$$

Ex. 1·3. *Calculate the mean of each of the following sequences of binomial co-efficients :*

(*i*) 1, 5, 10, 5, 1

(*ii*) 1, 6, 15, 20, 15, 6, 1

(*iii*) 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1

Sol.

(*i*) $\text{A.M.}=\frac{\Sigma x}{n}=\frac{1+5+10+5+1}{5}=\frac{22}{5}=4{\cdot}4$

(*ii*) $\text{A.M.}=\frac{\Sigma x}{n}=\frac{1+6+15+20+15+6+1}{7}=\frac{64}{7}=9{\cdot}143$

(*iii*) $\text{A.M.}=\frac{\Sigma x}{n}$

$$=\frac{1+10+45+120+210+252+210+120+45+10+1}{11}$$

$$=\frac{1024}{11}=93{\cdot}09.$$

Ex. 1·4. *From the data given below, calculate mean.,*

S.N.	*Marks*	*S.N.*	*Marks*
1	17	10	18
2	32	11	20
3	35	12	22
4	33	13	11
5	15	14	15
6	21	15	35
7	41	16	23
8	32	17	38
9	11	18	12

Sol. (*i*) ***Direct Method.***

Arithmetic mean (A.M.)$=\frac{\Sigma x}{n}$

$$=\frac{17+32+35+33+15+21+41+32+11+18+20+22+11+15+35+23+38+12}{18}$$

$$=\frac{431}{18}=23{\cdot}94.$$

(*ii*) *Short cut method.*

S.N.	x *Marks*	$x-23=u$	*S.N.*	x *Marks*	$x-23=u$
1	17	-6	10	18	-5
2	32	9	11	20	-3
3	35	12	12	22	-1
4	33	10	13	11	-12
5	15	-8	14	15	-8
6	21	-2	15	35	12
7	41	18	16	23	0
8	32	9	17	38	15
9	11	-12	18	12	-11
					17

$\therefore\quad$ A.M.$=23+\frac{\Sigma u}{n}=23+\frac{17}{18}$

$$=\frac{414+17}{18}=\frac{431}{18}=23{\cdot}94.$$

Ex. 1-5. *Calculate the mean from the following data :*

Size	*Frequency*	*Size*	*Frequency*
4—8	6	24—28	12
8—12	10	28—32	10
12—16	18	32—36	6
16—20	30	36—40	2
20—24	15		

Sol.

Size	*Frequency*	*Mid points* x	$u=\dfrac{x-22}{4}$	uf
4—8	6	6	—4	—24
8—12	10	10	—3	—30
12—16	18	14	—2	—36
16—20	30	18	—1	—30
20—24	15	22	0	0
24—28	12	26	1	12
28—32	10	30	2	20
32—36	6	34	3	18
36—40	2	38	4	8
	109			—62

$$\therefore \quad \text{A.M.}=22+\left(\frac{\Sigma fu}{N}\right)\times 4=22-\frac{62}{109}\times 4$$

$$=19{\cdot}725.$$

Ex. 1·6 *Calculate the mean from the following data :*

Income between (in Rs.)	*No. of persons.*	*Income between (in Rs.)*	*No. of persons.*
100—200	15	100—500	83
100—300	33	100—600	100
100—400	63		

The data is given in the form of cumulative frequency distribution. For calculating mean, it is to be converted into an ordinary frequency distribution.

Now the number of persons having income between 100–200=15 and the number of persons having income between 100–300=33.

$\therefore$ The number of persons having income between 200–300=33–15=18.

In a similar manner the frequencies of groups 300—400, 400—500 etc., can be calculated.

Income	Given Freq. (c.f.)	Frequency (f)	(x) Mid pt.	$U=\left(\frac{x-350}{100}\right)$	Uf
100—200	15	15	150	—2	—30
200—300	33	33—15=18	250	—1	—18
300—400	63	63—33=30	350	0	0
400—500	83	83—63=20	450	1	20
500—600	100	100—83=17	550	2	34
		100			6

$$\therefore \qquad \text{A.M.}=350+\frac{6}{100}\cdot 100$$

$$=356$$

Ex. 1-7. *Calculate the mean for the data given below :*

Marks	*No. of students*	*Marks*	*No. of students*
More than 0	100	More than 40	25
" " 10	90	" " 50	15
" " 20	75	" " 60	5
" " 30	50	" " 70	0

Sol. The data is given in cummulative distribution type. For calculating mean it is to be first converted into ordinary frequency distribution.

Now the number of students getting marks more than

'0'=100

and the number of students getting marks more than

'10'=90

∴ The number of students getting marks between

0—10=100—90=10.

Proceeding likewise the frequencies of other classes 10—20. 20—30 etc., can be obtained.

In the end since there is no student getting marks more than 70, the last class in the table will be 60—70.

Thus we have the table :

Note. Classes below are closed from right.

Sol. Calculation of A.M.

Marks	*Given Freq.*	*Frequencies* (f)	*Mid pt.* (x)	$U=\frac{x-35}{10}$	Uf
0—10	100	100—90=10	5	—3	—30
10—20	90	90—75=15	15	—2	—30
20—30	75	75—50=25	25	—1	—25
30—40	50	50—25=25	35	0	0
40—50	25	25—15=10	45	1	10
50—60	15	15— 5=10	55	2	20
60—70	5	5	65	3	15
		100			—40

$$\therefore \qquad \text{A.M.}=35+\frac{(-40)}{100}\times 10$$

$$=31.$$

Ex. 1-8. *The following table gives the frequency distribution of monthly salaries of 70 employees of company X.*

Salary (in Rs.)	*No. of Employees*
100—119	8
120—139	10
140—159	16
160—179	15
180—199	10
200—239	8
240—259	3
	70

Compute the arithmetic mean.

Sol. The class intervals are given in inclusive forms. A value less than 119·5 will be counted in first class and a value greater than or equal to it in the second. So for calculation the class 100—119 is as good as 99·5—119·5 and so on. Thus we have the following table :

Salary	*Frequency* (f)	*Mid value* (x)	$d=\left(\frac{x-169\cdot5}{10}\right)$	df
99·5—119·5	8	109·5	—6	—48
119·5—139·5	10	129·5	—4	—40
139·5—159·5	16	149·5	—2	—32
159·5—179·5	15	169·5	0	0
179·5—199·5	10	189·5	2	20
199·5—239·5	8	219·5	5	40
239·5—259·5	3	249·5	8	24
	70			—36

$$\therefore \quad \text{A.M.}=169\cdot5-\frac{36}{70}\times10$$

$$=169\cdot5-\frac{36}{7}$$

$$=169\cdot5-5\cdot143=164\cdot357$$

Ex. 1-9. *Illustrate by an example the effect of*

(*i*) *adding 'a' to every item.*

(*ii*) *subtracting 'a' from every item.*

(*iii*) *Multiplying every item by 'a'.*

(*iv*) *Dividing every item by 'a'.*

on the arithmetic mean of a series,

Sol. Let the series be

$$2,\ 3,\ 4,\ 5,\ 6$$

$$\therefore \quad \text{A.M.}=\frac{2+3+4+5+6}{5}=\frac{20}{5}=4$$

(*i*) By adding '*a*' to every item, the new series is

$$2+a,\ 3+a,\ 4+a,\ 5+a,\ 6+a$$

$$\therefore \quad \text{A.M.}=\frac{(2+a)+(3+a)+(4+a)+(5+a)+(6+a)}{5}$$

$$=4+a.$$

so that arithmetic mean also increases by '*a*'.

(*ii*) By subtracting '*a*' from each item, tl.. new series is

$$2-a,\ 3-a,\ 4-a,\ 5-a,\ 6-a$$

$$\therefore \quad \text{A.M.}=\frac{(2-a)+(3-a)+(4-a)+(5-a)+(6-a)}{5}$$

$$=4-a.$$

so that arithmetic mean also diminishes by 'a'.

(*iii*) By multiplying each item by 'a', the new series is

$$2a,\ 3a,\ 4a,\ 5a,\ 6a$$

$$\therefore \quad \text{A.M.}=\frac{2a+3a+4a+5a+6a}{5}$$

$$=4a.$$

so that arithmetic mean is also multiplied by 'a'.

(*iv*) By dividing each item by 'a', the new series is

$$\frac{2}{a},\ \frac{3}{a},\ \frac{4}{a},\ \frac{5}{a},\ \frac{6}{a}.$$

$$\therefore \quad \text{A.M.}=\frac{1}{5}\left(\frac{2}{a}+\frac{3}{a}+\frac{4}{a}+\frac{5}{a}+\frac{6}{a}\right)$$

$$=\frac{4}{a}$$

so that arithmetic mean is also divided by 'a'.

Ex. 1·10. *If m_1, m_2 be the arithmetic means for two series of sizes n_1 and n_2 respectively. Find the A.M. of the series obtained on combining them.*

Sol. Let $x_1, x_2, \ldots x_{n_1}$ and $y_1, y_2, \ldots y_{n_2}$ be the items of two series respectively. Then

$$m_1=\frac{1}{n_1}\left(x_1+x_2+\ldots+x_{n_1}\right)$$

$$m_2=\frac{1}{n_2}\left(y_1+y_2+\ldots+y_{n_2}\right)$$

Let m be the A.M. of the combined series.

Then $$m=\frac{1}{n_1+n_2}\left\{\left(x_1+x_2+\ldots+x_{n_1}\right)+\left(y_1+y_2+\ldots+y_{n_2}\right)\right\}$$

$$=\frac{1}{n_1+n_2}\{m_1n_1+m_2n_2\}$$

Ex. 1·11. *The mean of the marks obtained in an examination by a group of 100 students was found to be 49·46. The mean of the marks obtained in the same examination by another group of 200 students was 52·32. Find the mean of the marks obtained by both the groups of students taken together.*

Sol. Here $m_1=49{\cdot}46,\quad n_1=100$

$m_2=52{\cdot}32,\quad n_2=200$

$$\therefore\quad m=\frac{4946+10464}{100+200}$$

$$=\frac{15410}{300}$$

$$=51{\cdot}37.$$

Ex. 1-12. *Two groups of students reported mean weights of 162 and 148 pounds respectively. When would the mean weights of both groups together be 155 pounds ?*

Sol. Here $m=155$, $m_1=162$, $m_2=148$. Let n_1 and n_2 be the sizes of two groups.

Then $$m=\frac{n_1m_1+m_2n_2}{n_1+n_2}$$

$$\therefore\quad 155=\left(\frac{n_1}{n_1+n_2}\right)162+\left(\frac{n_2}{n_1+n_2}\right)(148)$$

$$=\frac{n_1}{(n_1+n_2)}162+\left(1-\frac{n_1}{n_1+n_2}\right)(148)$$

$$=\left(\frac{n_1}{n_1+n_2}\right)(162-148)+148$$

$$\therefore\quad 14\left(\frac{n_1}{n_1+n_2}\right)=155-148=7$$

$$\therefore\quad \frac{n_1}{n_1+n_2}=\frac{7}{14}=\frac{1}{2}$$

$$\therefore\quad 2n_1=n_1+n_2$$

or $$n_1=n_2$$

$\therefore$ Two groups must be of same size.

Ex. 1-13. *The mean annual salary paid to all employees of a company was Rs. 5000. The mean annual salaries paid to male and female employees were Rs. 5200 and Rs. 4200 respectively. Determine the percentages of males and females employed by the company.*

Sol. Here $m=5000$, $m_1=5200$, $m_2=4200$. Let n_1 and n_2 be the number of male and female employees

Now $$m=\frac{n_1m_1+n_2m_2}{n_1+n_2}$$

$$\therefore\quad 5000=\left(\frac{n_1}{n_1+n_2}\right)(5200)+\left(\frac{n_2}{n_1+n_2}\right)(4200)$$

or
$$50=\left(\frac{n_1}{n_1+n_2}\right)(52)+\left(1-\frac{n_1}{n_1+n_2}\right)(42)$$
$$=\left(\frac{n_1}{n_1+n_2}\right)(52-42)+42$$

or $\left(\frac{n_1}{n_1+n_2}\right)(10)=8$

or $$\frac{n_1}{n_1+n_2}=\frac{8}{10}$$

$$\therefore \quad \frac{n_2}{n_1+n_2}=1-\frac{n_1}{n_1+n_2}=1-\frac{8}{10}=\frac{2}{10}$$

$\therefore$ no. of male employees$=80\%$

and no. of female employees$=20\%$.

Ex. 1·14. *The population of five towns A, B, C, D, E was 20,000 ; 26,000 ; 23,000 ; 25,000 and 24,000 respectively. The average income of the resident for the respective towns was 280, 270, 240, 230 and 300.*

Find out the average income per head for all the towns combined.

Sol. Let n_1, n_2, n_3, n_4, n_5 be the population of five towns and m_1, m_2, m_3, m_4, m_5 be the average income of the resident for the respective towns.

Then

$n_1=20{,}000,\ n_2=26{,}000,\ n_3=23{,}000,\ n_4=25{,}000,\ n_5=24{,}000$

and $m_1=280,\ m_2=270,\ m_3=240,\ m_4=230,\ m_5=300.$

Let m be the average income per head for all the towns combined.

Then
$$m=\frac{m_1n_1+m_2n_2+m_3n_3+m_4n_4+m_5n_5}{n_1+n_2+n_3+n_4+n_5}$$
$$=\frac{5600+7020+5520+5750+7200}{20+26+23+25+24}$$
$$=\frac{31090}{118}=263{\cdot}47.$$

Ex. 1·15. *What is the average of daily wages for the workers of the two factories combined :*

	Factory A	*Factory B*
No. of wage earners	*250*	*200*
Average daily wage	*Rs. 2·00*	*Rs. 2·50*

Sol. Let m be the average of daily wages for the workers of the two factories combined.

Then $$m=\frac{(250)(2)+(200)(2\cdot5)}{250+200}$$
$$=\frac{50+50}{45}=\frac{100}{45}$$
$$=\text{Rs. } 2\cdot22.$$

Ex. 1-16. *The mean marks of 100 students were found to be 40. Later on it was discovered that a score of 53 was misread as 83. Find the corrected mean corresponding to the corrected score.*

Sol. Let x be the variable for the marks.

Then $$\bar{x}=40$$
and $$n(=\text{size})=100$$
$$\frac{1}{n}\ \Sigma x=40$$
$$\therefore \quad \Sigma x=40\times100=4000$$
Corrected value of $\Sigma x=4000-83+53$
$$=3970$$
$$\therefore \quad \text{Corrected mean}=\frac{3970}{100}=39\cdot7$$

(ii) Geometric and Harmonic Means

Geometric Mean : *It is defined by*
$$G=\left\{x_1^{f_1}.x_2^{f_2}\ldots\ldots x_n^{f_n}\right\}^{1/N}$$
where $N=f_1+f_2+\ldots+f_n$.

For a given data it is obtained by finding A.M. of $\log x$ *and then taking 'antilog'.*

Harmonic Mean : *It is defined by*
$$\frac{1}{H}=\frac{f_1.\frac{1}{x_1}+f_2.\frac{1}{x_2}+\ldots+f_n\frac{1}{x_n}}{f_1+f_1+\ldots\ldots+f_n}$$
where H is the harmonic mean.

For a given data it is obtained by finding A.M. of $\frac{1}{x}$ *and then taking its reciprocal.*

Ex. 1-17. *Find the geometric mean of the series 1, 2, 4, 8,...*2^n.

Sol. Let x be the variable. Then values of x are
1, 2, 4, 8,......2^n

$\therefore$ Values of $y=\log_{10} x$ are

$0, \log_{10}2, 2\log_{10}2, 3\log_{10}2, \ldots n\log_{10}2.$

$$\therefore \log_{10}(\text{G.M.})=\frac{0+\log_{10}2+2\log_{10}2+3\log_{10}2+\ldots+n\log_{10}2}{(n+1)}$$

$$=(\log_{10}2)\frac{1}{n+1}\{1+2+\ldots+n\}$$

$$=(\log_{10}2)\frac{1}{n+1}\cdot\frac{n(n+1)}{2}$$

$$=\frac{n}{2}\log_{10}2$$

$$=\log_{10}2^{n/2}$$

$\therefore$ G.M.$=2^{n/2}$.

Ex. 1-18. *Calculate geometric and Harmonic mean from the following data :*

6·5, 169·0, 11·0, 112·5, 14·2, 75·5, 35·5, 215·0.

Sol

x	$\log_{10} x$	$\frac{1}{x}$
6·5	0·8129	0·1539
169·0	2·2279	0·0059
11·0	1·0414	0·0909
112·5	2·0512	0·0089
14·2	1·1523	0·0704
75·5	1·8779	0·0133
35·5	1·5502	0·0282
215·0	2·3324	0·0047
	13·0462	0·3762

$$\log_{10}(\text{G.M.})=\frac{1}{n}(\Sigma\log_{10} x)=\frac{13\cdot0462}{8}=1\cdot630775$$

$$\simeq 1\cdot6308$$

$\therefore$ G.M.$=42\cdot74$

$$\frac{1}{\text{H.M.}}=\frac{\left(\sum\frac{1}{x}\right)}{n}=\frac{0\cdot3762}{8}$$

H.M.$=21\cdot27$.

Ex. 1-19. *Find the geometric mean and harmonic mean for the data of Ex. 1-5.*

Class Intervals	*Freq.* (f)	*Mid points* x	$\log_{10} x$	$\frac{1}{x}$	$f.(\log_{10} x)$	$f.\left(\frac{1}{x}\right)$
4— 8	6	6	0·778151	0·166667	4·668906	1·000002
8—12	10	10	1·000000	0·100000	10·000000	1·000000
12—16	18	14	1·146128	0·071429	20·630304	1·285722
16—20	30	18	1·255273	0·055556	37·65819	1·666680
20—24	15	22	1·342423	0·045455	20·136345	0·681825
24—28	12	26	1·414973	0·038462	16·979676	0·461544
28—32	10	30	1·477121	0·033333	14·77121	0·333330
32—36	6	34	1·531479	0·029412	9·188874	0·176472
36—40	2	38	1·579784	0·026316	3·159568	0·052632
	109				137·193073	6·658207

$$\therefore \quad \log_{10}(\text{G.M.}) = \frac{\Sigma f(\log_{10} x)}{N}$$

$$= \frac{137 \cdot 193073}{109} = 1 \cdot 25865 \simeq 1 \cdot 2587$$

$$\therefore \quad \text{G.M.} = 18 \cdot 14$$

and $$\frac{1}{\text{H.M.}} = \frac{\left(\sum \frac{f}{x}\right)}{N} = \frac{6 \cdot 658207}{109}$$

$$\therefore \quad \text{H.M.} = \frac{109}{6 \cdot 658207} = 16 \cdot 37.$$

Ex. 1-20. *In previous Ex. find quadratic mean.*

Class-Intervals	*Frequency* (f)	*Mid points* x	x^2	fx^2
4— 8	6	6	36	216
8—12	10	10	100	1000
12—16	18	14	196	3528
16—20	30	18	324	9720
20—24	15	22	484	7260
24—28	12	26	676	8112
28—32	10	30	900	9000
32—36	6	34	1156	6936
36—40	2	38	1444	2888
	109			48660

$$\therefore \quad \text{Quadratic mean} = \frac{(\Sigma fx^2)}{N}$$

$$= \frac{48660}{109} = 446{\cdot}42.$$

Ex. 1·21. *If g_1 and g_2 be the geometric means of two series of n_1 and n_2 items. Find the geometric mean of the series obtained on combining them.*

Sol. Let $x_1, x_2 \ldots\ldots x_{n_1}$ and $y_1, y_2 \ldots\ldots y_{n_2}$ be the items of two series respectively.

Then

$$g_1 = \left\{ x_1 \,.\, x_2 \ldots\ldots x_{n_1} \right\}^{1/n_1}$$

and

$$g_2 = \left\{ y_1 \,.\, y_2 \ldots\ldots y_{n_2} \right\}^{1/n_2}.$$

Let g be the G.M. of the combined series.

Then $$g = \left(x_1.x_2 \ldots\ldots x_{n_1} . y_1.y_2 \ldots\ldots y_{n_2} \right)^{\frac{1}{n_1+n_2}}$$

$$= \left\{ \left(x_1.x_2 \ldots\ldots x_{n_1} \right) \left(y_1.y_2 \ldots\ldots y_{n_2} \right) \right\}^{\frac{1}{n_1+n_2}}.$$

$$= \left\{ g_1^{\,n_1} \,.\, g_2^{\,n_2} \right\}^{\frac{1}{n_1+n_2}}$$

$$= (g_1)^{\frac{n_1}{n_1+n_2}} \quad (g_2)^{\frac{n_2}{n_1+n_2}}$$

Ex. 1·22. *If $x_1, x_2 \ldots\ldots x_n$ be non-zero positive numbers with A.M. 'A', G.M. 'G' and H.M. 'H', show that*

$$A \geqslant G \geqslant H.$$

Sol. $$A = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n}, \; G = (x_1.x_2 \ldots\ldots x_n)^{1/n}$$

and

$$\frac{1}{H} = \frac{\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots\ldots + \frac{1}{x_n} \right)}{n}$$

From inequalities

$$\frac{x_1+x_2+\ldots\ldots+x_n}{n} \geqslant (x_1.x_2\ldots\ldots x_n)^{1/n}$$

$\therefore \quad A \geqslant G.$

Also $$\frac{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}}{n} \geqslant \left(\frac{1}{x_1}.\frac{1}{x_2}\ldots\ldots\frac{1}{x_n}\right)^{1/n}$$

$\therefore \quad \frac{1}{H} \geqslant \frac{1}{G}$ or $G \geqslant H$

$\therefore \quad A \geqslant G \geqslant H.$

Ex. 1-23 *A variate takes values a, ar, $ar^2\ldots\ldots ar^{n-1}$ each with frequency unity. Show that A.M. 'A' is $\frac{a(1-r^n)}{n(1-r)}$ and G.M. 'G' is $ar^{\frac{n-1}{2}}$ and the H.M. 'H' is $\frac{an(1-r)r^{n-1}}{1-r^n}$. Prove that $AH=G^2$. Prove also that $A \geqslant G \geqslant H$.*

Sol.

$$A=\frac{a(1+r+r^2+\ldots\ldots+r^{n-1})}{n}=\frac{a(1-r^n)}{n(1-r)}$$

$$G=(a.ar.ar^2\ldots ar^{n-1})^{1/n}=ar^{\frac{1+2+\ldots\ldots+(n-1)}{n}}$$

$$=ar^{\frac{n-1}{2}}$$

$$\frac{1}{H}=\frac{1}{na}\left\{1+\frac{1}{r}+\frac{1}{r^2}+\ldots\ldots\frac{1}{r^{n-1}}\right\}=\frac{1}{na}.\frac{1-\frac{1}{r^n}}{1-\frac{1}{r}}$$

$\therefore \quad H=\frac{na(1-r)r^{n-1}}{1-r^n}$

$\therefore \quad A.H.=a^2r^{n-1}=G^2.$

(*iii*) **Partition Values** : *These are the values of the variate which divide the total frequency into a number of equal parts. Some important partition values are quartiles, deciles, percentiles, quintiles etc. For a grouped dist these are given by*

$$Q_i=L+\left\{\frac{i\frac{N}{4}-c}{f}\right\}h \quad i=1, 2, 3$$

Q_2 is known as median.

$$D_j = L + \left\{ \frac{j\,\frac{N}{100} - c}{f} \right\} . h \qquad j = 1, 2 \ldots\ldots 9$$

$$P_k = L + \left\{ \frac{k\frac{N}{100} - c}{f} \right\} . h \qquad k = 1, 2 \ldots\ldots 99$$

and $$Q_l = L + \left\{ \frac{l\frac{N}{5} - c}{f} \right\} . h \qquad l = 1, 2, 3, 4$$

where L=Lower limit of the class in which partition value lies

h=Width of the class.

f=Frequency of the class.

c=Cummulative frequency upto and including the class preceding the class in which partition value lies.

N=Total Frequency.

Ex. 1-24. *Find out the median of the following items.*
25, 15, 23, 40, 27, 25, 23, 25 and 20.

Sol. Items arranged in ascending order of magnitude :

S N.	*Size of the items*
1	15
2	20
3	23
4	23
5	25
6	25
7	25
8	27
9	40

If n is the number of items,

Median=size of $\left(\frac{n+1}{2}\right)$th item

=size of $\left(\frac{9+1}{2}\right)$=5th item

=25.

Ex. 1-25. *From the data of ex. 1-4 find out the median and Quartiles.*

Sol. Given figures arranged in ascending order are :

S.N.	*Marks*	*S.N*	*Marks*
1	11	10	22
2	11	11	23
3	12	12	32
4	15	13	32
5	15	14	33
6	17	15	35
7	18	16	35
8	20	17	38
9	21	18	41

$$\text{Median} = \text{value of } \left(\frac{18+1}{2}\right)\text{th item}$$

$$= \text{value of } 9{\cdot}5\text{th item}$$

$$= \frac{\text{value of 9th item} + \text{value of 10th item}}{2}$$

$$= \frac{21+22}{2} = \frac{43}{2} = 21{\cdot}5$$

$$Q_1 = \text{value of } \left(\frac{18+1}{4}\right)\text{th item}$$

$$= \text{value of } (4{\cdot}75)\text{th item}$$

$$= \text{value of 4th item} + \frac{3}{4}\,(\text{value of 5th item} - \text{value of 4th item})$$

$$= 15 + \frac{3}{4}\,(15-15)$$

$$= 15$$

$$Q_3 = \text{value of } \frac{3(18+1)}{4}\text{th item}$$

$$= \text{value of } (14{\cdot}25)\text{th i em}$$

$$= \text{value of 14th item} + \frac{1}{4}\,(\text{value of 15th item} - \text{value of 14th item})$$

$$= 33 + \frac{1}{4}(35-33)$$

$$= 33{\cdot}5.$$

Ex. 1-26. *From the data given below find out the median and the two Quartiles :*

Wages in Rs. :	20	21	22	23	24	25	26	27	28
No. of workers :	8	10	11	16	20	25	15	9	6

Sol.

Calculation of Median and Quartiles

Wages in Rs.	*Frequency No. of workers*	*Cummulative Frequency*
20	8	8
21	10	18
22	11	29
23	16	45
24	20	65
25	25	90
26	15	105
27	9	114
28	6	120

$$\text{Median} = \text{value of } \left(\frac{120+1}{2}\right)\text{th item}$$

$$= \text{value of } (60\cdot5)\text{th item.}$$

$$= \frac{\text{value of 60th item} + \text{value of 61st item}}{2}.$$

From the above table, there are 45 items up to 23 and 65 items up to 24.

$\therefore$ Value of item from 46th to 65th is 24.

$\therefore$ Value of 60th and 61st items each is 24.

$$\therefore \text{ Median} = \frac{24+24}{2} = 24.$$

$$Q_1 = \text{value of } \left(\frac{120+1}{4}\right)\text{th item}$$

$$= \text{value of } (30\cdot25)\text{th item}$$

$$= \text{value of 30th item} + \frac{1}{4}\,(\text{value of 31st item} - \text{value of 30th item})$$

$$= 23 + \frac{1}{4}\,(23-23)$$

$$= 23$$

$$Q_3 = \text{value of } \frac{3}{4}(120+1)\text{th item}$$
$$= \text{value of } (90{\cdot}75)\text{th item}$$
$$= \text{value of 90th item} + \frac{3}{4}(\text{value of 91st item} - \text{value of 90th item})$$
$$= 25 + \frac{3}{4}(26-25)$$
$$= 25{\cdot}75.$$

Ex. 1·27. *Find out the median, quartiles, 3rd quintile, 5th octile, 7th decile from the following data :*

Monthly Rent in Rs.	*No. of families paying the Rent.*
20— 40	6
40— 60	9
60— 80	11
80—100	14
100—120	20
120—140	15
140—160	10
160—180	8
180—200	7

Sol.

Class-Intervals	*Frequency*	*Cumulative Frequency*
20— 40	6	6
40— 60	9	15
60— 80	11	26
80—100	14	40
100—120	20	60
120—140	15	75
140—160	10	85
160—180	8	93
180—200	7	100

$$\text{The Median} = \text{value of } \left(\frac{100}{2}\right)\text{th item}$$
$$= \text{value of 50th item}$$

which lies in 100—120.

Applying interpolation formula,

$$\text{Median} = 100 + \frac{(120-100)}{20}(50-40)$$
$$= 100 + 10 = 110$$

$$Q_1 = \text{value of } \left(\frac{100}{4}\right)\text{th item}$$

$$= \text{value of 25th item}$$

which lies in (60—80)

$$\therefore \quad Q_1 = 60 + \frac{(80-60)}{11}\{25-15\}$$

$$= 60 + \frac{20}{11}(10)$$

$$= 60 + \frac{200}{11} = 60 + 18{\cdot}2 = 78{\cdot}2$$

$$Q_3 = \text{value of } \frac{3}{4}(100)\text{th item}$$

$$= \text{value of (75)th item}$$

which lies in 120—140

$$\therefore \quad Q_3 = 120 + \frac{140-120}{15}(75-60) = 140$$

$$\text{3rd quintile} = \text{value of } \frac{3}{5}(100)\text{th item}$$

$$= \text{value of 60th item}$$

which lies in 100—120

$$\therefore \quad \text{4th quintile} = 100 + \frac{120-100}{20}(60-40) = 120.$$

$$\text{5th octile} = \text{value of } \frac{5}{8}(100)\text{th item}$$

$$= \text{value of } 62{\cdot}5$$

which lies in (120—140)

$$\therefore \quad \text{5th octile} = 120 + \frac{140-120}{15}(62{\cdot}5-60)$$

$$= 120 + \frac{4}{3}(2{\cdot}5) = 123{\cdot}3$$

$$\text{7th Decile} = \text{value of } \frac{7}{10}(100)\text{th item}$$

$$= \text{value of 70th item}$$

which lies in 120—140

$$\therefore \quad \text{7th Decile} = 120 + \frac{140-120}{15}(70-60)$$

$$= 120 + \frac{4}{3}(10) = 133{\cdot}3.$$

Ex. 1-28. *Find the median for the data :*

Monthly Wages (in Rs.)	50—55	55—60	60—65	65—70	70—75	75—80	80—100
No. of Workers :	6	10	22	30	16	12	15

Sol.

Class-Intervals	*Frequency*	*Cummulative Frequency*
50— 55	6	6
55— 60	10	16
60— 65	22	38
65— 70	30	68
70— 75	16	84
75— 80	12	96
80—100	15	111

Median=value of $\left(\frac{111}{2}\right)$th item

=value of 55·5th item

which lies in 65—70

$$\therefore \text{ Median} = 65 + \frac{70-65}{30}(55{\cdot}5 - 38)$$

$$= 65 + \frac{1}{6}(17{\cdot}5) = 67{\cdot}92.$$

Ex. 1-29. *Find the median and Quartiles from the data of Ex. 1-6.*

Sol.

Class Intervals	*Frequency*	*Cummulative Frequency*
100—200	15	15
200—300	18	33
300—400	30	63
400—500	20	83
500—600	17	100

$$\text{Median} = \text{Value of } \left(\frac{100}{2}\right)\text{th item}$$

$$= \text{Value of 50th item}$$
which lies in 300—400

$$\therefore \quad \text{Median} = 300 + \frac{400-300}{30}(50-33)$$

$$= 300 + \frac{100}{30}(17) = 356{\cdot}67$$

$$Q_1 = \text{Value of } \left(\frac{100}{4}\right)\text{th item}$$

$$= \text{Value of 25th item}$$
which lies in 200—300.

$$\therefore \quad Q_1 = 200 + \frac{300-200}{18}(25-15)$$

$$= 200 + \frac{100}{18}(10) = 255{\cdot}55$$

$$Q_3 = \text{Value of } \left[\frac{3}{4}(100)\right]\text{th item}$$

$$= \text{Value of (75)th item}$$
which lies in 400—500

$$\therefore \quad Q_3 = 400 + \frac{500-400}{20}(75-63)$$

$$= 400 + 5(12) = 460$$

Ex. 1·30. *Compute the Median of data in Ex. 1·8.*

Sol.

Class Intervals	*Frequency*	*Cummulative Frequency*
100—119	8	8
120—139	10	18
140—159	16	34
160—179	15	49
180—199	10	59
200—239	8	67
240—259	3	70

$$\text{Median} = \text{Value of } \left(\frac{70}{2}\right)\text{th item}$$

$$= \text{Value of 35th item}$$

which lies in 160—179.

As class intervals are of inclusive type, the real limits of the group are 159·5 to 179·5.

$$\therefore \qquad \text{Median} = 159{\cdot}5 + \frac{179{\cdot}5 - 159{\cdot}5}{15}(35-34)$$

$$= 159{\cdot}5 + \frac{20}{15} = 160{\cdot}83$$

[redacted]

(iv) **Mode** : *It is that value of the variate for which frequency is maximum. For grouped dist it is given by*

$$Mode = L + \frac{f_m - f_1}{2f_m - f_1 - f_2}.h \qquad \text{(I Method)}$$

$$Mode = L + \frac{f_2}{f_1 + f_2}\, h \qquad \text{(II Method)}$$

Second Method is used where first fails.

where $\quad$ *L = Lower limit of the class in which mode lies i.e., Modal class.*

f_m = *Frequency of modal class.*

f_1 = *Frequency of the class preceding the modal class.*

f_2 = *Frequency of the class following the modal class.*

For a moderately skew distribution mode is given by

Mode = 3 Median − 2 Mean.

Ex. 1-31. *From the data of Ex. 1-4 find the mode :*

Sol. Converting the data in to ordinary Frequency dist :

Marks	*Frequency*	*Marks*	*Frequency*
11	2	22	1
12	1	23	1
15	2	32	2
17	1	33	1
18	1	35	2
20	1	38	1
21	1	41	1

Location of Mode by grouping

Marks	Frequency (1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
11	√ 2	√ 3		√ 5			√ 6				√ 7				
12	1		√ 3		√ 4			5				√ 6			
15	√ 2	√ 3				√ 4			√ 5				6		
17	1		2	3						4				5	
18	1	2			3		4								5
20	1		2			3		4			6				
21	1	2		3					√ 5			√ 6			
22	1		2		√ 4					√ 5			√ 7		
23	1	√ 3				√ 4	√ 6							√ 7	
32	√ 2		√ 3	√ 5				√ 6							√ 7
33	1	√ 3			√ 4				√ 5						
35	√ 2		√ 3			√ 4									
38	1	2													
41	1														

The frequencies in column (1) are first added in two's in columns (2) and (3), then in three's in columns (4), (5), (6), in four's in columns (7), (8), (9), (10) and in fives in columns (11), (12), (13), (14), (15). The maximum frequency in each column is indicated by putting a sign '√' above the figure.

To find out the point of maximum concentration the data can be arranged in the shape of table below :

Analysis Table

Col.	11	12	15	17	18	20	21	22	23	32	33	35	38	41
(1)	√		√							√		√		
(2)	√	√	√	√					√	√	√	√		
(3)		√	√							√	√	√	√	
(4)	√	√	√							√	√	√		
(5)		√	√	√				√	√	√	√	√	√	
(6)			√	√	√				√	√	√	√	√	√
(7)	√	√	√	√					√	√	√	√		
(8)										√	√	√	√	
(9)			√	√	√	√	√	√	√	√	√	√	√	√
(10)								√	√	√	√			
(11)	√	√	√	√	√									
(12)		√	√	√	√	√	√	√	√	√	√			
(13)								√	√	√	√	√		
(14)									√	√	√	√	√	
(15)										√	√	√	√	√
	5	7	10	7	4	2	2	5	9	14	13	12	7	3

In this table marks are taken along the horizontal and columns (1), (2) etc., along the vertical. Since according to column (1), mode should be either 11 or 15 or 32 or 35, in the row (1) the signs '√' are put under 11, 15, 32 and 35. Similarly since according to column (2), mode should be either 11 or 12 or 15 or 17 or 23 or 32 or 33 or 35, in row (2) the sign '√' are put under 11, 12 etc.

In this way the whole table is completed.

Since value 32 occurs the largest number of times, mode is 32.

Ex. 1-32. *Find out mode for the data of Ex. 1-6.*

Income	Frequency (1)	(2)	(3)	(4)	(5)	(6)	Columns (1)	(2)	(3)	(4)	(5)	(6)	
100—200	15	33		√ 63						√			1
200—300	18		√ 48		√ 68				√	√	√		3
300—400	√ 30	√ 50				√ 67	√	√	√	√	√	√	6
400—500	20		37					√			√	√	3
500—600	17											√	1

$\therefore$ Model class is 300—400

$$\therefore \quad \text{Mode} = 300 + \frac{30-18}{60-18-20}(100) = 354{\cdot}55$$

Ex. 1-33. *Find the mode from the data of Ex. 1-8.*

Sol.

Salary	*Frequency*						*Columns*						
	(1)	(2)	(3)	(4)	(5)	(6)	(1)	(2)	(3)	(4)	(5)	(6)	
100—119	8	18								√			1
120—139	10		√ 26	√ 34					√	√	√		3
140—159	√ 16	√ 31			√ 41		√	√	√	√	√	√	6
160—179	15		25			√ 41		√			√	√	3
180—199	10	18		33								√	1
200—239	8		11		21								
240—259	3												

$\therefore$ Model class is 140—159

i.e., 139·5—159·5

$$\therefore \quad \text{Mode}=139\cdot5+\frac{16-10}{32-10-15}\,(20)$$

$$=139\cdot5+\frac{(6)(20)}{7}=156\cdot64$$

Ex. 1·34. *Following is the distribution of the size of certain farms selected at random from a district. Calculate the mode of the distribution :*

Central size of the farms (in acre)	*No. of farms*
10	8
20	12
30	17
40	29
50	31
60	5
70	3

Sol. Since central size increases by 10 throughout the data, each class interval must be of size 10. Hence various class-intervals are 5—15, 15—25 etc.

(*See Table on page 1.30*)

Model class is 35—45.

$$\therefore \quad \text{Model}=35+\frac{29-17}{58-17-31}\,(10)$$

$$=35+\frac{120}{10}=47$$

which lies outside the class-interval 35—45.

$\therefore$ The formula fails.

In such cases we use the second formula, *i.e.*

$$\text{Mode}=L+\frac{f_2}{f_1+f_2}\,h$$

By this formula,

$$\text{Mode}=35+\frac{31}{17+31}\,(10)$$

$$=35+\frac{310}{48}$$

$$=35+6\cdot46=41\cdot46.$$

Ex. 1·35. *If the mode and mean of a moderately asymmetrical series are respectively 16 inches and 20·2 inches. Compute the most probable median.*

Class intervals	Frequency (1)	(2)	(3)	(4)	(5)	(6)	Columns (1)	(2)	(3)	(4)	(5)	(6)	
5—15	8	20		37									
15—25	12		29		√ 58						√		1
25—35	17	√ 46				√ 77		√			√	√	3
35—45	29		√ 60	√ 65				√	√	√	√	√	5
45—55	√ 31	36			39		√		√	√		√	4
55—65	5		8							√			1
65—75	3												

Sol. For a moderately asymmetrical series, the relation connecting mean, median and mode is

$$\text{Mode}=\text{Mean}-3(\text{Mean}-\text{Median})$$

$$=3\text{ Median}-2\text{ Mean}$$

$$\therefore \quad \text{Median}=\frac{1}{3}(\text{Mode}+2\text{ Mean})$$

$$=\frac{1}{3}(16+40{\cdot}4)$$

$$=\frac{56{\cdot}4}{3}=18{\cdot}8\text{ inches.}$$

Ex. 1-36. *If the mode and mean of a moderately asymmetrical series are, respectively 16″ and 15·6″, what would be its most probable median.*

Sol.
$$\text{Mean}=15{\cdot}6''$$
$$\text{Mode}=16''$$

$$\text{Median}=\frac{1}{3}(\text{mode}+2\text{ mean})$$

$$=\frac{1}{3}(16+31{\cdot}2)$$

$$=\frac{47{\cdot}2}{3}=15{\cdot}73$$

Ex. 1-37. *The dist $x_1, x_2 \ldots x_n$ with frequencies $f_1, f_2 \ldots f_n$ is transformed into the dist $X_1, X_2 \ldots X_n$ with the same corresponding frequencies by the relation $X=ax+b$ where 'a' and 'b' are constants. Show that the mean, median and mode are given in terms of those of the first dist by the same transformation.*

Sol. By given, $X=ax+b$

$$\therefore \quad X_i=ax_i+b$$

$$\therefore \quad \bar{X}=\frac{1}{N}\sum_{i=1}^{n} f_iX_i=\frac{1}{N}\sum_{i=1}^{n} f_i(ax_i+b)$$

$$=a.\frac{1}{N}\sum_{i=1}^{n} f_ix_i+b.\frac{1}{N}\sum_{i=1}^{n} f_i=a\bar{x}+b$$

where $\bar{X}$ and $\bar{x}$ are *A.Ms.*

Since for median the value of the variable corresponding to the middle item is to be obtained and this middle item remains middle item in the transformed dist, median of the transformed dist is given by same transformation.

Similar argument holds for mode.

Ex. 1-38. *From the following data find the missing frequency :*

No. of Tablets :	4—8	—12	—16	—20	—24	—28	—32	—36	—40
No. of persons cured :	11	13	16	14	—	9	17	6	4

The average number of tablets to cure fever was 19·9.

Sol. Let the missing frequency be 'a'.

Class-Interval	*Frequency* (f)	*Mid point* (x)	xf
4—8	11	6	66
8—12	13	10	130
12—16	16	14	224
16—20	14	18	252
20—24	a	22	$22a$
24—28	9	26	234
28—32	17	30	510
32—36	6	34	204
36—40	4	38	152
	$90+a$		$22a+1772$

Now A.M. of tablets for all the persons $=19{\cdot}9$.

$\therefore$ Total number of tablets for all the persons

$$=19{\cdot}9\,(90+a)$$
$$=19{\cdot}9\,a+1791$$

$\therefore \quad 19{\cdot}9a+1791=22a+1772$

or $\quad 2{\cdot}1a=19$

or $\quad a=9{\cdot}0.$

Ex. 1·39. *The following table gives the marks obtained by 30 students of a class in certain paper :*

Digits (Division) of class intervals

Marks	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*Total*
30—39	2	1	2	2	—	1	—	1	1	—	10
40—49	—	1	—	2	—	—	4	—	—	—	7
50—59	—	—	—	—	3	1	—	3	—	1	8
60—69	1	1	—	—	1	—	1	—	—	1	5

Calculate the mean and median of the series : (a) by using only the total of class-intervals (b) by using the entire data.

Sol. The data indicates that 4 students get 46 marks, 3 students get 54 marks, 3 students get 57 marks and so on.

Caculation of Mean

Marks	Total marks for each student 0	1	2	3	4	5	6	7	8	9	Total marks	Total Freq. (f)	Mid point (x)	xf	C Freq.
30—39	60	31	64	66	—	35	—	37	38	—	331	10	34·5	345·0	10
40—49	—	41	—	86	—	—	184	—	—	—	311	7	44·5	311·5	17
50—59	—	—	—	—	162	55	—	171	—	59	447	8	54·5	436·0	25
60—69	60	61	—	—	64	—	66	—	—	69	320	5	64·5	322·5	30
Total	120	133	64	152	226	90	250	208	38	128	1409	30		1415	

Mean : (*a*) Using totals of classes

$$= \frac{\Sigma fx}{\Sigma f} = \frac{1415}{30} = 47{\cdot}2$$

$$\simeq 47 \text{ (approx)}$$

(*b*) Using entire data.

$$=\frac{\Sigma x}{n}=\frac{1409}{30}=46{\cdot}97$$

$$=47 \text{ (approx).}$$

Median

(*a*) Using only totals of class intervals.

Median has $\frac{30}{2}=15$ items below it *i.e.*, it lies in 40—49 *i.e.*, 39·5—49·5 (taking real limits).

$$\therefore \quad \text{Median}=39{\cdot}5+\frac{10}{7}(15-10)$$

$$=39{\cdot}5+\frac{50}{7}=46{\cdot}64.$$

(*b*) Using entire data.

$$\text{Median}=\text{value of}\left(\frac{30+1}{2}\right)\text{th item}$$

$$=\text{value of } 15{\cdot}5\text{th item}$$

$$=\frac{\text{value of 15th item}+\text{value of 16th item}}{2}.$$

From Cummulative frequencies it is clear that 15th and 16th items lie in 40—49. There are 10 items up to 39.

So counting in the group 40—49 the various items we see that

$$15\text{th item}=46=16\text{th item}$$

$$\therefore \quad \text{Median}=46.$$

Ex. 1·40. *Show that in finding the arithmetic mean of a set of readings on a thermometer, it does not matter whether we measure the temperature in centigrade or Fahrenheit degrees, but that in finding the G.M. it does matter.*

Sol. Let a set of N thermometric readings in Centigrade degrees be $C_1, C_2 \ldots\ldots C_N$ and the corresponding readings in Fahrenheit degrees be $F_1, F_2 \ldots\ldots F_N$.

The relation between Centigrade and Fahrenheit readings is

$$F=32+\frac{9}{5}C.$$

where C corresponds to Centigrade readings and F to Fahrenheit readings.

$$\therefore \quad F_r=32+\frac{9}{5}C_r \qquad r=1, 2 \ldots\ldots N.$$

Now the A.M. of the N readings in Centigrade degrees

$$=\frac{C_1+C_2+\ldots+C_N}{N}=\bar{C}\text{ (say)}$$

and the same in Fahrenheit degrees

$$=\frac{F_1+F_2+\ldots+F_N}{N}=\bar{F}\text{ (say)}$$

$$\therefore\quad \bar{F}=\frac{1}{N}\left(F_1+F_2+\ldots+F_N\right)$$

$$=\frac{1}{N}\left\{\left(32+\frac{9}{5}C_1\right)+\left(32+\frac{9}{5}C_2\right)+\ldots\ldots+\left(32+\frac{9}{5}C_N\right)\right\}$$

$$=32+\frac{9}{5}\left(\frac{C_1+C_2+\ldots\ldots+C_N}{N}\right)=32+\frac{9}{5}\bar{C}.$$

=Fahrenheit equivalent of $\bar{C}$. (A.M. of Centigrade readings).

G.M. of the readings in Fahrenheit

$=(F_1, F_2\ldots\ldots F_N)^{1/N}$

$$=\left\{\left(32+\frac{9}{5}C_1\right)\left(32+\frac{9}{5}C_2\right)\ldots\ldots\left(32+\frac{9}{5}C_N\right)\right\}^{1/N}\quad\ldots(1)$$

G.M. of the readings in Centigrade

$=(C_1.C_2\ldots\ldots C_N)^{1/N}$

$\therefore$ Fahrenheit equivalent of the geometric mean of the readings in Centigrade

$$=32+\frac{9}{5}\left(C_1C_2\ldots\ldots C_N\right)^{1/N}\quad\ldots(2)$$

But (1) and (2) are not same.

$\therefore$ Fahrenheit equivalent of the G.M. of the Centigrade readings is not the same as the G.M. of the Farenheit readings.

$\therefore$ The given statement follows.

Weighted Average. *If w_1. $w_2\ldots\ldots w_n$ be the weights of values $x_1, x_2\ldots\ldots x_n$, then*

Weighted arithmetic mean $=\dfrac{w_1x_1+w_2x_2+\ldots+w_nx_n}{w_1+w_2+\ldots+w_n}$

Weighted geometric mean = *Antilog* $\left\{\dfrac{w_1\log x_1+\ldots\ldots w_n\log x_n}{w_1+\ldots\ldots+w_n}\right\}$

and Weighted harmonic mean = *Reciprocal* $\left\{\dfrac{w_1\dfrac{1}{x_1}+\ldots+w_n.\dfrac{1}{x_n}}{w_1+\ldots+w_n}\right\}.$

Ex. 1·41. *Show that the weighted arithmetic mean of first n natural numbers when weights are equal to the corresponding numbers is $\frac{2n+1}{3}$.*

Sol.

$$\text{Weighted A.M.} = \frac{1\cdot 1+2\cdot 2+\ldots+n.n}{1+2+\ldots+n}$$

$$= \frac{1^2+2^2+\ldots+n^2}{\frac{n(n+1)}{2}}$$

$$= \frac{n(n+1)(2n+1)}{6} \cdot \frac{2}{n(n+1)}$$

$$= \frac{2n+1}{3}$$

Ex. 1·42. *From the following results of two colleges A and B find out which of the two is better :*

	A		*B*	
Exam.	*Appeared*	*Passed*	*Appeared*	*Passed*
M.A.	100	90	240	200
M.Sc.	60	45	200	160
B.A.	120	75	160	100
B.Sc.	200	150	200	140
Total	480	360	800	600

Sol. Since the number of students appearing for M.A., M.Sc., B.A. and B.Sc. widely differ, simple arithmetic average of pass percentages of the college will not give the correct idea of pass percentage of a college for all the examinations taken together. So we take the weighted average of pass percentages, weights being the number of students appeared for each examination.

For college A,

Pass Percentage for M.A. $=90\%$

" " " M.Sc. $= \frac{45}{60} \times 100 = 75\%$

" " " B.A. $= \frac{75}{120} \times 100 = 62\cdot5\%$

" " " B.Sc. $= \frac{150}{200} \times 100 = 75\%$.

For college B,

Pass percentage for M.A. $= \frac{200}{240} \times 100 = \frac{250}{3}$ %

„ „ „ M.Sc. $= \frac{160}{200} \times 100 = 80\%$

„ „ „ B.A. $= \frac{100}{160} \times 100 = 62\cdot5\%$

„ „ „ B.Sc. $= \frac{140}{200} \times 100 = 70\%$

A			*B*		
(x) *Pass %*	*No. of students appeared (weight) w*	*xw*	(x) *Pass %*	*No. of students appeared (weight) w*	*xw*
90	100	9000	$\frac{250}{3}$	240	20000
75	60	4500	80	200	16000
62·5	120	7500	62·5	160	10000
75	200	15000	70	200	14000
	480	36000		800	60000

∴ Pass percentage of college $A = \frac{36000}{480} = 75\%$

Pass percentage of college $B = \frac{60000}{800} = 75\%$

Since pass percentages for two colleges *A* and *B* are same, none is better than the other.

Ex. 1·43. *Find the weighted geometric mean from the following data :*

Group	*Index No.*	*Weight*
Food	125	7
Clothing	133	5
Fuel and light	141	4
House Rent	173	1
Miscellaneous	182	3

Sol.

Group	*Index No.* x	*Weight* (w)	$\log_{10} x$	$(w \log_{10} x)$
Food	125	7	2·0969	14·6783
Clothing	133	5	2·1239	10·6195
Fuel and light	141	4	2·1492	8·5968
House Rent	173	1	2·2380	2·2380
Miscellaneous	182	3	2·2601	6·7803
		20		42·9129

$\therefore$ Weighted geometric mean is given by

$$\log_{10} G = \frac{42{\cdot}9129}{20} = 2{\cdot}1457$$

$$\therefore \qquad G = 139{\cdot}8 \simeq 140.$$

Ex. 1-44. *A train starts from rest and travels successive quarters of a kilometre at average speeds of 12, 16, 24, 48 k.m. per hour. The average speed over the whole k.m. is 19·2 k.m. per hour and not 25 k.m. per hour. Explain.*

Sol. Here average speeds for each quarter of a k.m. are given. To find out the average speed over the total distance, first the total time taken by the train is to be calculated by dividing distances by average speeds and then the total distance is to be divided by the total time. This procedure is equivalent to finding the weighted harmonic mean of average speeds weights being the respective distances.

Here distances travelled in four cases are same each being equal to $\frac{1}{4}$ k.m. So here equal weighted or simple harmonic mean is the appropriate method of averaging.

$\therefore$ Average speed over the whole mile

$$= \frac{1}{\frac{1}{4}\cdot\frac{1}{12}+\frac{1}{4}\cdot\frac{1}{16}+\frac{1}{4}\cdot\frac{1}{24}+\frac{1}{4}\cdot\frac{1}{48}}$$

$$= \frac{(4)(48)}{4+3+2+1} = 19{\cdot}2$$

$\therefore$ Average speed over the whole k.m. is 19·2 *k.m.p.h.* and not 25 *k m.p.h.* which is simply the A.M. or weighted (weights being $\frac{1}{4}$ each) A.M. of four average speeds 12, 16, 24 and 48.

Ex. 1·45. *A cyclist covers his first three k.m. at an average speed of 8 k.m p.h., another two k.m. at 3 k.m.p.h. and the last two k.m. at 3 k.m.p.h. Find his average speed for the entire journey.*

Sol. The average speed for the entire journey is the weighted harmonic mean of the speeds with distances as weights.

$\therefore$ The average speed for the entire journey

$$=\frac{3+2+2}{3.\frac{1}{8}+2.\frac{1}{3}+2.\frac{1}{3}}=\frac{7}{\frac{3}{8}+\frac{4}{3}}$$

$$=\frac{(7)(24)}{9+32}=\frac{168}{41}=4{\cdot}1 \text{ k m.p.h.}$$

Ex. 1·46. *Mr X travels from A to B at an average speed of 30 k.m.p.h. and returns from B to A at an average speed of 60 k.m. per hour. Find the average speed of Mr X for the entire trip.*

Sol. Let x be the distance between A and B

Then average speed for the entire trip

$$=\frac{2x}{x.\frac{1}{30}+x.\frac{1}{60}}=\frac{(2)(60)}{2+1}$$

$$=40.$$

$\therefore$ Average speed for the entire trip

$=40$ *m.p.h.*

Ex. 1·47. *An aeroplane flies round a square the sides of which measure 100 k.m. each. The aeroplane covers at a speed of 100 k.m. per hour the first side, at 200 k.m.p h. the second side, at 300 k m.p.h. the third side and 400 k.m.p.h. the fourth side. What is the average speed of the aeroplane around the square ?*

Sol. Average speed$=\dfrac{400}{100\left(\frac{1}{100}+\frac{1}{200}+\frac{1}{300}+\frac{1}{400}\right)}$

$$=\frac{4(1200)}{12+6+4+3}$$

$$=\frac{4800}{25}=192 \text{ k.m.p.h.}$$

Ex. 1·48. *You take a trip which entails travelling 900 k.m. by train at an average speed of 60 k.m p.h., 3000 k.m. by boat at an average speed of 25 k.m.p.h., 400 k.m. by plane at 350 k.m.p.h. and finally 15 k m. by taxi at 25 k.m.p.h. What is your average speed for the entire distance (4315 k.m.) ?*

Sol. The average speed

$$=\frac{900+3000+400+15}{900.\frac{1}{60}+3000.\frac{1}{25}+400.\frac{1}{350}+15.\frac{1}{25}}$$

$$=\frac{4315}{15+120+\frac{8}{7}+\frac{3}{5}}$$

$$=\frac{(4315)(35)}{525+4200+40+21}$$

$$=\frac{(4315)(35)}{4786}=\frac{151025}{4786}=31{\cdot}56\ k.m.p.h$$

Ex. 1-49. *A man travels 50 k.m. at a speed of 20 k.m.p.h. and then returns at speed of 30 k.m.p.h. What is his average speed for the whole journey ?*

Sol. Average speed for the whole journey

$$=\frac{50+50}{\left(50.\frac{1}{20}\right)+\left(\frac{50}{30}\right)}$$

$$=\frac{(2)(60)}{3+2}=24\ k.m.p.h.$$

Ex. 1-50. *The price of a commodity increased by 5% from 1948 to 1949, 8% from 1949 to 1950 and 77% from 1950 to 1951. The average increase from 1948 to 1951 is quoted as 26% and not 30%. Explain this statement and verify the arithmetic.*

Sol. Let the price of the commodity in the beginning of 1948 be x.

Then price in the beginning of 1949

$$=\left(\frac{105}{100}x\right)$$

Since the price from 1949 to 1950 increases by 8%, the price in the beginning of 1950$=\left(\frac{108}{100}\right)\left(\frac{105}{100}x\right)$

Similarly the price in the beginning of 1951

$$=\left(\frac{108}{100}\right)\left(\frac{105}{100}\right)x.\left(\frac{177}{100}\right).$$

$\therefore$ The price at the end of 1950 or in the beginning of 1951

$$=\frac{(105)(108)(177)}{(100)^3}x$$

Let r be the average rate of increase. Then

$$\frac{(105)(108)(177)}{(100)^3}x=x(1+r)^3$$

or $$1+r=\frac{1}{100}\sqrt[3]{(105)(108)(177)}$$

$\therefore$ $$\log_{10}(1+r)=\frac{1}{3}(-6+2\cdot0212+2\cdot0334+2\cdot2480)$$

$$=\frac{1}{3}(0\cdot3026)=0\cdot1009$$

$\therefore$ $$1+r=1\cdot262$$

$\therefore$ $$r=0\cdot262 \text{ or } 26\%.$$

$\therefore$ Average price rise was 26%.

The A.M. of the rise in price is

$$\frac{5+8+77}{3}=30\%.$$

If this be the rise in price in each year, the price at the end of 1950 would be $\left(\frac{130}{100}\right)^3 x$ which is much higher than the value obtained from the given data *i.e.*, $x\left(\frac{105}{100}\right)\left(\frac{108}{100}\right)\left(\frac{177}{100}\right)$. But if the rate of rise in price is taken to be 26%, the price at the end of 1950 would be $x\left(\frac{126}{100}\right)^3$ which is nearly equal to the value obtained from the given data.

$\therefore$ Average price rise was 26% and not 30%.

Ex. 1·51. *Find the average rate of increase in population which in the first decade had increased 20%, in the next 30% and in the third 45%.*

Sol. Let x be the population in the beginning. Then the population at the end of first decade$=\left(\frac{120}{100}\right)x$.

Since the population in the next decade increases by 30%, the population at the end of second decade$=\left(\frac{130}{100}\right)\left(\frac{120}{100}\right)x$.

Similarly, the population at the end of third decade

$$=\left(\frac{145}{100}\right)\left(\frac{130}{100}\right)\left(\frac{120}{100}\right)x.$$

Let r be the average rate of increase per year.

Then

$$\frac{(145)(130)(120)}{(100)^3}\,x=x(1+r)^{30}$$

$$\therefore \log_{10}(1+r)=\frac{1}{30}\{-6+2{\cdot}1614+2{\cdot}1139+2{\cdot}0792\}$$

$$=\frac{1}{30}\{0{\cdot}3545\}\simeq 0{\cdot}0118$$

$$\therefore \qquad 1+r=1{\cdot}028$$

$$\therefore \qquad r=0{\cdot}028 \quad \text{or} \quad 2{\cdot}8\%\simeq 3\%.$$

Ex. 1-52. *A machine is assumed to depreciate 40% in value in the first year, 25% in the second year and 10% per annum for the next three years, each percentage being calculated on the diminishing value. What is the average percentage depreciation for the five years ?*

Sol. Let x be the value of the machine in the beginning,

Then price at the end of first year

$$=\left(\frac{60}{100}\right)x.$$

The price at the end of 2nd year

$$=\left(\frac{75}{100}\right)\left(\frac{60}{100}\right)x.$$

The price at the end of fifth year

$$=\left(\frac{75}{100}\right)\left(\frac{60}{100}\right)\left(\frac{90}{100}\right)^3 x$$

Let r be the average rate of depreciation per year.

Then $$\frac{(75)(60)(90)^3}{(100)^5}\,x=x(1-r)^5$$

$$\therefore \log_{10}(1-r)=\frac{1}{5}\{-10+1{\cdot}8751+1{\cdot}7782+3(1{\cdot}9542)\}$$

$$=\frac{1}{5}\{-10+1{\cdot}8751+1{\cdot}7782+5{\cdot}8626\}$$

$$=\bar{1}{\cdot}9032$$

$$\therefore \qquad (1-r)=0{\cdot}8002$$

$$\therefore \qquad r=19{\cdot}98\%$$

$$\simeq 20\%.$$

Ex. 1-53. *The age distribution of the members of a certain children's club is as follows :*

Age on last birthday (in yrs.)	*Frequency*
4	5
5	9
6	18
7	35
8	42
9	32
10	15
11	7
12	3

There is a member A s.t. there are twice as many members older than A as there are younger than A. Estimate his age (in years up to two places of decimals).

Sol. Since Ages on last birthday are given,

No. of persons who are in 4—5 group = 5

No. of persons who are in 5—6 group = 9

and so on.

Age	*Frequency (f)*	*Cumulative Freq.*
4— 5	5	5
5— 6	9	14
6— 7	18	32
7— 8	35	67
8— 9	42	109
9—10	32	141
10—11	15	56
11—12	7	163
12—13	3	166

Age of A = size of $\left(\frac{166}{3}\right)$th item

$= \text{size of } \left(55\frac{1}{3}\right)$th item

which lies in 7—8

$\therefore$ Age of $A = 7 + \frac{(8-7)}{35}\left(55\frac{1}{3} - 32\right)$

$$=7+\frac{1}{35}\left(23\frac{1}{3}\right)$$

$$=7+\frac{70}{105}$$

$$=7{\cdot}67.$$

Ex. 1-54. *An incomplete freq. dist. is given below :*

Variate :	10—20	20—30	30—40	40—50	50—60	60—70	70—80	Total
Freq. :	12	30	?	65	?	25	18	229

Given that median value is 46, find the missing frequencies using the median formula.

Sol. Median=46 lies in 40—50 class

$$\therefore \quad 46=40+10\left\{\frac{\frac{229}{2}-(x+42)}{65}\right\}$$

where x is the freq. of the class 30 – 40.

$\therefore \quad x=33{\cdot}5=34.$

$\therefore$ Freq. of class 50 – 60 = 229 – (Sum of remaining frequencies)

$=229-184=45.$

EXERCISE 1·1

Find arithmetic means of following datas :

1. Gold output (in millions of pounds) for different years.

94	95	96	93	87	79	73	69	68	67
78	82	83	89	95	103	108	117	130	97

(Ans. 90·15)

2.

x :	0	1	2	3	4	5	6	7	8
f :	1	9	26	59	72	52	29	7	1

where x denotes the number of heads and f their frequencies when eight coins are tossed 256 times. **(Ans : 3·97)**

3.

Age Group	*No. of persons*	*Age Group*	*No. of persons*
25—30	1	50—55	53
30—35	2	55—60	126
35—40	4	60—65	163
40—45	10	65—70	35
45—50	21	70—75	6
		75—80	1

(Ans. 58·66)

4. Find A.M. of data in ex. 1-27. (Ans. 110)

5.

Wts (in lbs.)	*Freq.*	*Wts. (in lbs.)*	*Freq.*
90—100	10	130—140	51
100—110	37	140—150	35
110—120	65	150—160	18
120—130	80	160—170	4

(Ans. 125·73)

6.

Wage (in Rs.)	*No. of Employees*	*Wage (in Rs.)*	*No. of Employees*
50—55	250	30—35	800
45—50	300	25—30	1100
40—45	400	20—25	1700
35—40	450		

(Ans. 31·15)

7.

No. of days absent	*No. of students*	*No. of days absent*	*No. of students*
Less than 5	29	Less than 30	487
" " 10	124	" " 35	493
" " 15	349	" " 40	497
" " 20	442	" " 45	500
" " 25	478		

(Ans. 13·51)

8. Find out the median from the following data :

Age group (in years)	15—20	20—25	25—30	30—35	35—40
No. of men	5	9	82	58	49
Age group (in years)	40—45	45—50	50—55		
No. of men	28	6	3		

(Ans. 32·07)

9. From the following data find out the median and the quartiles :

Marks	*No. of students*	*Marks*	*No. of students*
0—5	4	20—25	25
5—10	6	25—30	22
10—15	10	30—35	18
15—20	10	35—40	5

(Ans. 24 ; 17·5 ; 29·54)

10. From the table given below find out the median and the quartiles :

Size	11—15	16—20	21—25	26—30	31—35	36—40
Freq.	7	10	13	26	35	40
Size	41—45	46—50				
Freq.	11	5				

(Ans. 33, 26·8, 37·9)

11. Find the Quartiles, 20th percentiles and the 8th decile of hts. from the following table :

ht. (in inches)	*No. of students*	*ht. (in inches)*	*No. of students*
58	15	63	22
59	20	64	20
60	32	65	10
61	35	66	8
62	33		

(**Ans.** 60 ; 63 ; 60 and 63)

12. Find out the median and quartiles of data in Ex. 7.
(**Ans.** 12·8 ; 10·02 ; 16·4)

13. Find out the median, quartiles, 6th decile, 70th percentile and 3rd quartile for the data in Ex. 1-7.
(**Ans.** 124·75 ; 114·3 ; 136·47 ; 128·5 ; 133·53 and 21·85)

14. The following table gives the dist of forms according to their sizes in a given region. Calculate the median and the quartiles (size of the form is rounded to the nearest acre) :

Farm size (acres)	*No. of forms*	*Farm size (acres)*	*No. of farms*
0— 40	394	161—200	169
41— 80	461	201—240	113
81—120	391	241 and over	148
121—160	334		

(**Ans.** 95·85 ; 49·91 ; 151·82)

15. Find mode from the following data :

Wage (in Rs.)	20	21	22	23	24	25	26	27	28
No. of workers	8	10	11	16	20	25	15	9	6

(**Ans.** 25)

16. Find mode of date in solved Ex. 1-5. (**Ans.** 17·78)
17. Find mode for the data in Ex. 7. (**Ans.** 12·48)
18. Find mode of data in Ex. 1-6. (**Ans.** 110·91)
19. Find mode of data in Ex. 1-7. (**Ans.** 123·41)
20. Find mode of data given below :

5 students get less than 3 marks
12 „ „ „ „ 6 „
25 „ „ „ „ 9 „
30 „ „ „ „ 12 „ (**Ans.** 7·29)

21. The consumption of petrol by a motor was 'a gallan' for 20 k.m. while going up from planes to a hill station and 'a gallon'

for 24 miles while coming down. What particular average would you consider appropriate for finding the average consumption in miles per gallon for up and down journey and why ?

$\left(\textbf{Ans. } 21\frac{9}{11} \text{ m.p.h. per gallon} \right)$

22. Under what conditions weighted average is

(*i*) equal to simple average

(*ii*) greater than simple average.

(*iii*) less than simple average.

Illustrate your answer with the help of examples.

23. The following is the dist of 136 individuals by 10 year age groups. Calculate that measure of central tendency which will appropriately describe the dist.

Age-group	*No. of persons*	*Age-group*	*No. of persons*
0— 9	48	40—49	13
10—19	26	50—59	4
20—29	27	60—69	3
30—39	11	70 and over	4

(**Median**+17·2)

24. The table below shows the age dist of heads of families in country *A* during the year 1977.

Age	*No. (in millions)*	*Age*	*No. (in millions)*
under 25	2·22	55—64	6·63
25—29	4·05	66—74	4·16
30—34	5·08	75 and over	1·66
35—44	10·45		
45—54	9·47		

Do you think that in this case median is a better measure of central tendency than the mean ? Give reasons.

25. The daily expenditure of 100 families is given as under :

Expenditure	: 0—10	10—20	20—30	30—40	40—50
No. of families	: 14	?	27	?	15

The median and mode for the distribution are Rs. 25 and Rs. 29 respectively. Calculate the missing frequencies. (**Ans.** 33, 11)

2

Measures of Dispersion and Skewness

2.1. Introduction

In the preceding chapter several measures used to describe the central tendency of a frequency distribution were discussed. These measures have their limitations and may conceal much pertinent factual information. It is also possible that these measures of central tendency may give results which are quite misleading Thus, a measure of central tendency alone is not enough to give a correct picture of a distribution and for this some additional information is required. The following information is needed :

(1) The extent of scatterpeness of items around central tendency. This is called *dispersion.*

(2) The direction of scatteredness. This is called *skewness.*

(3) The extent to which the distribution is more peaked or more flat-topped than the normal distribution. This is called *kurtosis.*

2.2. Measures of Dispersion

The object of measuring dispersion is to obtain a single summary figure which adequately exhibits the extent of the scatter of the variable values. Various measures of dispersion are :

(1) **Range, Interquartile range and Quartile deviation.**

Range. It is difference between the greatest and least values of the variate.

Interquartile range. It is the difference between the upper and lower quartiles.

i.e., $Q_3 - Q_1$.

Quartile Deviation. *It is defined to be* $\frac{Q_3 - Q_1}{2}$, *where* Q_3 *and* Q_1 *are quartiles.*

Quartile Co-efficient of Dispersion. *It is defined to be*

$$\frac{Q_3-Q_1}{Q_3+Q_1}$$

Ex. 2·1. *Compute Quartile Deviation and the coefficient of dispersion from the following data :*

Size	*Frequency*	*Size*	*Frequency*
4—8	6	24—28	12
8—12	10	28—32	10
12—16	18	32—36	6
16—20	30	36—40	2
20—24	15		

Sol.

Size Class Interval	*Frequency*	*C. Frequency*
4—8	6	6
8—12	10	16
12—16	18	34
16—20	30	64
20—24	15	79
24—28	12	91
28—32	10	101
32—36	6	107
36—40	2	109

Q_1 has $\frac{109}{4}=27{\cdot}25$ items below it

$\therefore$ It lies in 12—16

$$\therefore \quad Q_1=12+\frac{4}{18}(27{\cdot}25-16)$$

$$=12+\frac{4}{18}(11{\cdot}25)$$

$$=12+\frac{45}{18}$$

$$=12+2{\cdot}5$$

$$=14{\cdot}5.$$

Q_3 has $\frac{3}{4}$ (109)=81·75 items below it.

$\therefore$ It lies in 24—28

$$\therefore \quad Q_3=24+\frac{4}{12}(81{\cdot}75-79)$$

$$=24+\frac{1}{3}(2{\cdot}75)$$

$$=24+0{\cdot}92=24{\cdot}92$$

$$\therefore \quad \text{Q.D.}=\frac{Q_3-Q_1}{2}=\frac{24{\cdot}92-14{\cdot}5}{2}$$

$$=\frac{10{\cdot}42}{2}=5{\cdot}21$$

and Co-efficient of dispersion

$$=\frac{Q_3-Q_1}{Q_3+Q_1}=\frac{24{\cdot}92-14{\cdot}5}{24{\cdot}92+14{\cdot}5}$$

$$=\frac{10{\cdot}42}{39{\cdot}42}=0{\cdot}26$$

(ii) **Mean Deviation.** *It is defined by* $M.D.=\frac{1}{N}\Sigma f\,|\,x-a\,|$ *where 'a' is a point from which the deviations are to be taken.*

Co-efficient of Mean-Deviation. *It is defined to be*

$$\frac{\text{Mean-deviation about 'a'}}{a}.$$

If nothing is mentioned usually mean deviation about median should be calculated.

Short Cut Method. *M D. is calculated more easily by the formulae*

$$M.D. \text{ about 'a'}=\frac{\Sigma f\,|\,x-b\,|+(a-b)\left(\sum\limits_{x<a} f-\sum\limits_{x>a} f\right)}{N}$$

$$\text{and M.D. about median } (M)=\frac{1}{N}\left\{\sum_{x>M} fx-\sum_{x<M} fx\right\}$$

Ex. 2-2. *Find the mean deviation for the following data :*

Height (in inches)	No. of students	Height (in inches)	No. of students
58	15	59	20
60	32	61	35
62	33	63	22
64	20	65	10
66	8		

Sol. It is not given about which the mean deviation is to be calculated. So mean deviation about median is to be calculated.

Calculation of Mean Deviation

Height x	*No. of students Freq.* (f)	*c.f.*	$d=\|x-61\|$	fd
58	15	15	3	45
59	20	35	2	40
60	32	67	1	32
61	35	102	0	0
62	33	135	1	33
63	22	157	2	44
64	20	177	3	60
65	10	187	4	40
66	8	195	5	40
				334

$$\text{Median}=\text{Value of }\left(\frac{195+1}{2}\right)\text{th item}$$

$$=\text{Value of 98th item}$$

$$=61$$

$$\text{Mean Deviation}=\frac{334}{195}$$

$$=1{\cdot}71.$$

Theorem 2.2-1 *Show that the mean deviation from the median is less than that measured from any other value.*

Sol. Let x be the variable.

Let $x_1, x_2, \ldots\ldots x_n$ be the values arranged in ascending order. Let M be the median. Then by def.,

$$\text{Mean deviation about } M=\frac{1}{n}\sum_{i=1}^{n}|x_i-M|$$

$$=\frac{1}{n}\sum_{x\leq M}|x-M|+\frac{1}{n}\sum_{x>M}|x-M|$$

$$=\frac{1}{n}\sum_{x<M}(M-x)+\frac{1}{n}\sum_{x>M}(x-M)$$

$$=\frac{1}{n}\sum_{x<M}(M-a+a-x)+\frac{1}{n}\sum_{x>M}(x-a+a-M)$$

where 'a' is any other point.

$$\therefore \quad \text{M.D. about } M=\frac{1}{n}\sum_{x<M}(M-a)+\frac{1}{n}\sum_{x<M}(a-x)$$
$$+\frac{1}{n}\sum_{x>M}(x-a)+\frac{1}{n}\sum_{x>M}(a-M)$$

$$=\frac{1}{n}\sum_{x<M}(M-a)+\frac{1}{n}\sum_{x<M}(a-x)$$
$$+\frac{1}{n}\sum_{x>M}(x-a)-\frac{1}{n}\sum_{x>M}(M-a)$$

As M is the median, the number of items for which $x<M$ is equal to the number of items for which $x>M$.

$$\therefore \quad \sum_{x<M}(M-a)=\sum_{x>M}(M-a)$$

$$\therefore \quad \text{M.D. about } M=\frac{1}{n}\sum_{x<M}(a-x)+\frac{1}{n}\sum_{x>M}(x-a)$$

(*i*) Let $a<M$

$$=\frac{1}{n}\sum_{x<a}(a-x)+\frac{1}{n}\sum_{a<x<M}(a-x)$$
$$+\frac{1}{n}\sum_{x>a}(x-a)-\frac{1}{n}\sum_{M>x>a}(x-a)$$

$$=\left\{\frac{1}{n}\sum_{x<a}(a-x)+\frac{1}{n}\sum_{x>a}(x-a)\right\}-\frac{2}{n}\sum_{a<x<M}(x-a)$$

$$=\frac{1}{n}\sum_{i=1}^{n}|x_i-a|-\frac{2}{n}\sum_{a<x<M}(x-a)$$

As in second term, $x>a$,

$$\frac{2}{n}\sum_{a<x<M}(x-a)$$

is non negative.

$\therefore$ Mean deviation about median is less than that measured from any other value.

(*ii*) Let $a>M$.

Here M.D. about $M=\frac{1}{n}\sum_{x<M}(a-x)+\frac{1}{n}\sum_{x>M}(x-a)$

$$=\frac{1}{n}\sum_{x<a}(a-x)-\frac{1}{n}\sum_{M<x<a}(a-x)$$

$$+\frac{1}{n}\sum_{x>a}(x-a)+\frac{1}{n}\sum_{a>x>M}(x-a)$$

$$=\frac{1}{n}\sum_{i=1}^{n}|x_i-a|-\frac{2}{n}\sum_{M<x<a}(a-x)$$

Since second term is non-negative, mean deviation about median is less than that measured from any other value.

Ex. 2·3. *Find the mean deviation about median from the following data :*

S.N.	Marks	S.N.	Marks	S.N.	Marks
1	17	7	41	13	11
2	32	8	32	14	15
3	35	9	11	15	35
4	33	10	18	16	23
5	15	11	20	17	38
6	21	12	22	18	12

(*i*) *by direct method.*

(*ii*) *by short cut method.*

Sol. Arranging Marks in ascending order :

S.N.	*Marks (x)*	$\vert x-21{\cdot}5 \vert$	*S.N.*	*Marks (x)*	$\vert x-21{\cdot}5 \vert$
1	11	10·5	10	22	0·5
2	11	10·5	11	23	1·5
3	12	9·5	12	32	10·5
4	15	6·5	13	32	10·5
5	15	6·5	14	33	11·5
6	17	4·5	15	35	13·5
7	18	3·5	16	35	13·5
8	20	1·5	17	38	16·5
9	21	0·5	18	41	19·5
	140			291	151·0

$$\text{Median}=\text{Value of } \frac{18+1}{2}=9{\cdot}5\text{th item}$$

$$=\frac{\text{Value of 9th item}+\text{Value of 10th item}}{2}$$

$$=\frac{21+22}{2}=21{\cdot}5.$$

(*i*) By Direct method,

Mean deviation about median

$$=\frac{151}{18}=8{\cdot}4$$

(*ii*) By short cut method,

Mean deviation about median

$$=\frac{1}{N}\left\{\sum_{x>M} fx-\sum_{x<M} fx\right\}$$

Now $\sum\limits_{x>M} fx=291$ and $\sum\limits_{x<M} fx=140$

$\therefore$ M.D. about median$=\dfrac{151}{18}=8{\cdot}4$

Ex. 2-4. *Show that the mean deviation about the mean $\bar{x}$ of the variate x can be written in the form*

$$\frac{2}{N}\left[\bar{x}\sum_{x_i<\bar{x}} f_i-\sum_{x_i<\bar{x}} f_i x_i\right]$$

where f_i is the frequency of the value x_i.

Sol. Mean deviation about mean is given by

$$S=\frac{1}{N}\sum_{i} f_i \mid x_i-\bar{x} \mid$$

$$=\frac{1}{N}\sum_{x_i<\bar{x}} f_i(\bar{x}-x_i)+\frac{1}{N}\sum_{x_i>\bar{x}} f_i(x_i-\bar{x})$$

Now $\quad \sum_{i} f_i(x_i-\bar{x})=\sum_{i} f_ix_i-\bar{x}\sum_{i} f_i=N\bar{x}-N\bar{x}=0$

$\therefore \quad \sum_{x_i<\bar{x}} f_i(x_i-\bar{x})+\sum_{x_i>\bar{x}} f_i(x_i-\bar{x})=0$

or $\quad \sum_{x_i>\bar{x}} f_i(x_i-\bar{x})=\sum_{x_i<\bar{x}} f_i(\bar{x}-x_i)$

$$\therefore \quad S=\frac{2}{N}\sum_{x_i<\bar{x}} f_i(\bar{x}-x_i)=\frac{2}{N}\left[\bar{x}\sum_{x_i<\bar{x}} f_i-\sum_{x_i<\bar{x}} f_ix_i\right]$$

(*iii*) **Variance.** *It is defined by*

$$\mu_2=\frac{1}{N}\Sigma f(x-\bar{x})^2$$

Standard Deviation. *It is the positive square root of the variance.*

Mean Square Deviation. *Mean square deviation about the pt 'a' is defined by*

$$\mu_2'(a)=\frac{1}{N}\Sigma f(x-a)^2$$

Root Mean Square Deviation. *It is the positive square root of mean square deviation.*

Co-efficient of Variation. *It is defined to be*

$$100\times\frac{(s.d.)}{mean}$$

Co-efficient of Dispersion. *It is defined by* $\frac{s.d.}{mean}$

For a given data s. d. is obtained by the formula

$$s.d.=h\sqrt{\frac{1}{N}\Sigma fX^2-\left(\frac{1}{N}\Sigma fX\right)^2}$$

where $\quad X=\frac{x-a}{h}.$

Ex. 2·5. *Calculate the mean and s.d. of the following values of the world's annual gold output (in millions of pounds) for 20 different years :*

94	95	96	93	87	79	73	69	68	67
78	82	83	89	95	103	108	117	130	97

Also calculate the percentage of cases lying outside the mean at distances $\pm\sigma$, $\pm 2\sigma$, $\pm 3\sigma$ *where* σ *denotes the s.d.*

Sol. Arranging the data in ascending order :

Out put (x) Arranged in order	$X=x-90$	X^2	*Out put (x) Arranged in order*	$X=x-90$	X^2
67	−23	529	93	3	9
68	−22	484	94	4	16
69	−21	441	95	5	25
73	−17	289	95	5	25
78	−12	144	96	6	36
79	−11	121	97	7	49
82	− 8	64	103	13	169
83	− 7	49	108	18	324
87	− 3	9	117	27	729
89	− 1	1	130	40	1600
	−125	2131		128	2982

$$\therefore \quad \Sigma X = 128-125=3$$

$$\Sigma X^2 = 2982+2131=5113$$

$$\therefore \quad \text{A.M.}=90+\left(\frac{3}{20}\right)=90+0{\cdot}15$$

$$=90{\cdot}15 \text{ million pounds}$$

$$\text{S.D.}=\sqrt{\frac{1}{n}\,\Sigma X^2-\left(\frac{1}{n}\,\Sigma X\right)^2}$$

$$=\sqrt{\frac{1}{20}(5113)-\left(\frac{1}{20}.3\right)^2}$$

$$=\frac{1}{20}\sqrt{102260-9}$$

$$=\frac{1}{20}\sqrt{102251}$$

$$=\frac{319{\cdot}767}{20}=15{\cdot}99 \text{ million pounds.}$$

Now $\quad \text{mean}\pm\sigma=90{\cdot}15\pm15{\cdot}99$

$$=106{\cdot}14,\ 74{\cdot}16$$

$\therefore$ No. of cases outside the range 74·16 to 106·14 = 7

$\therefore$ Percentage of cases outside the mean at distances $\pm\sigma$

$$=\frac{7}{20}\times 100=35\%.$$

No. of cases outside the range.

mean $\pm 2\sigma$ *i.e.*, $90{\cdot}15 \pm 31{\cdot}98$ or $58{\cdot}17$ to $122{\cdot}13=1$.

$\therefore$ Percentage of cases outside the mean at distances $\pm 2\sigma$

$$=\frac{1}{20}\times 100=5\%$$

No. of cases outside the range mean $\pm 3\sigma$

i.e., $90{\cdot}15 \pm 47{\cdot}97$ or $42{\cdot}18$ to $138{\cdot}12=0$

$\therefore$ Percentage of cases outside mean $\pm 3\sigma=0\%$.

Ex. 2-6. *The distribution of maximum loads in tons supported by cables produced in a factory is shown below. Compute the standard deviation and the co-efficient of variation of the distribution :*

Max. load (in tons)	*No. of cables*
9·3— 9·7	2
9·8—10·2	5
10·3—10·7	12
10·8—11·2	17
11·3—11·7	14
11·8—12·2	6
12·3—12·7	3
12·8—13·2	1
	60

Sol.

Class intervals	*Freq.* (f)	*Mid points* (x)	$X=x-11{\cdot}0$	$u=\frac{X}{0{\cdot}5}$	uf	u^2f
9·3— 9·7	2	9·5	−1·5	−3	−6	18
9·8—10·2	5	10·0	−1·0	−2	−10	20
10·3—10·7	12	10·5	−0·5	−1	−12	12
10·8—11·2	17	11·0	0	0	0	0
11·3—11·7	14	11·5	0·5	1	14	14
11·8—12·2	6	12·0	1·0	2	12	24
12·3—12·7	3	12·5	1·5	3	9	27
12·8—13·2	1	13·0	2·0	4	4	16
	60				11	131

$$\text{S.D.}=(0{\cdot}5)\sqrt{\frac{131}{60}-\left(\frac{11}{60}\right)^2}$$

$$=\frac{(0{\cdot}5)}{60}\sqrt{7860-121}$$

$$=\frac{1}{120}\sqrt{7739}$$

$$\log_{10}(\text{S.D.})=\frac{1}{2}\log_{10}(7739)-\log_{10}(120)$$

$$=\frac{1}{2}(3{\cdot}8887)-2{\cdot}0792$$

$$=1{\cdot}94435-2{\cdot}0792$$

$$=1{\cdot}86515\simeq\overline{1}{\cdot}8652$$

$\therefore$ $$\text{S.D.}=0{\cdot}7331\simeq0{\cdot}733 \text{ tons}$$

$$\text{A.M.}=11{\cdot}0+(0{\cdot}5)\left(\frac{11}{60}\right)$$

$$=11+\frac{11}{120}=\frac{1331}{120}$$

$$=11{\cdot}092 \text{ tons.}$$

$\therefore$ Co-efficient of variation

$$=\left(\frac{\text{S.D.}}{\text{A.M.}}\right)(100)$$

$$=\left(\frac{0{\cdot}733}{11{\cdot}092}\right)(100)$$

$$=\frac{73300}{11092}=6{\cdot}61\%$$

Ex. 2-7. (*a*) *Find out the coefficient of variation if*

Var=148·6

Mean=40.

(*b*) *If in a series which is not highly skewed the mean deviation is 7·8, what would be the approximate value of its s.d.*

Sol. (*a*) $\text{S.D.}=\sqrt{148{\cdot}6}=12{\cdot}19$

Co-efficient of variation

$$=\left(\frac{\text{S.D.}}{\text{A.M.}}\right)(100)$$

$$=\left(\frac{12\cdot19}{40}\right)(100)=\frac{1219}{40}$$

$$=30\cdot5\%$$

(*b*) M.D.$=7\cdot8$

But M.D.$=\frac{4}{5}$ (S.D.)

$\therefore$ S.D.$=\frac{5}{4}$ (M.D.) $=\frac{5}{4}(7\cdot8)$

$$=\frac{39}{4}=9\cdot75.$$

Ex. 2-8. *Calculate s.d. of data given below by the method of summation (using cumulative frequency of less than type) :*

Marks	*Students*	*Marks*	*Students*
80—84	1	50—54	6
75—79	1	45—49	6
70—74	1	40—44	6
65—69	4	35—39	3
60—64	4	30—34	0
55—59	7	25—29	1
			40

Sol.

Mid points (*x*)	*Freq.* (*f*)	*First cumulation* (*c. Freq*)	*Second cumulation* (*c. Freq. of c. Freq.*)
27	1	1	1
32	0	1	2
37	3	4	6
42	6	10	16
47	6	16	32
52	6	22	54
57	7	29	83
62	4	33	116
67	4	37	153
72	1	38	191
77	1	3	230
82	1	40	270
	40	270	1154

Standard deviation by the method of summation is given by

$$SD=h\sqrt{2F_2-F_1-F_1^2}$$

where h=common class-interval

F_1=The Sum of Cumulative Frequencies (less than type) divided by the number of items.

F_2=The sum of Cumulative Frequencies (less than type) of the Cumulative Frequencies (less than type) divided by the number of items.

Here $h=5$

$$F_1=\frac{270}{40}$$

$$F_2=\frac{1154}{40}$$

$$\therefore \quad S.D.=5\sqrt{\frac{2308}{40}-\frac{270}{40}-\left(\frac{270}{40}\right)^2}$$

$$=\frac{1}{8}\sqrt{92320-10800-72900}$$

$$=\frac{1}{8}\sqrt{8620}=\frac{1}{8}\ (92{\cdot}844)$$

$$=11{\cdot}6055\simeq 11{\cdot}61.$$

Ex. 2·9. *Calculate the s.d. of data given below by the method of summation (using more than type cumulative frequency).*

Sol.

Mid points (x)	*Freq.* (f)	*First Cumulation (c. Freq.)*	*Second Cumulation (c. Freq. of c. Freq.)*
75	12	230	1045
85	18	218	815
95	35	200	597
105	42	165	397
115	50	123	232
125	45	73	109
135	20	28	36
145	8	8	8
	230	1045	3239

S.D. by the method of summation is given by

$$S.D.=h\sqrt{2F_2-F_1^2-F_1}$$

where h=common class-interval

F_1=The sum of cumulative Frequencies (more than type) divided by the number of items.

F_2=The sum of cumulative Frequencies (more than type) of the cumulative frequencies (more than type) divided by the number of items.

Here $h=10$

$$F_1=\frac{1045}{230}$$

$$F_2=\frac{3239}{230}$$

$\therefore$
$$S.D.=10\sqrt{\frac{2(3239)}{230}-\frac{1045}{230}-\left(\frac{1045}{230}\right)^2}$$

$$=\frac{1}{23}\sqrt{(6478)(230)-(1045)(230)-(1045)^2}$$

$$=\frac{1}{23}\sqrt{1489940-240350-1092025}$$

$$=\frac{1}{23}\sqrt{157565}=17{\cdot}258$$

Ex. 2-10. *The following table gives the fluctuations in the prices of shares of two companies A and B. Find out which of them shows greater variability?*

Share A :	318	322	325	312	324	315	308	319
Share B :	2542	2542	2534	2532	2545	2530	2566	2550

Sol. Arranging the values in ascending order :

Share A			*Share B*		
x	$d_1=x-318$	d_1^2	y	$d_2=y-2542$	d_1^2
308	−10	100	2530	−12	144
312	−6	36	2532	−10	100
315	−3	9	2534	−8	64
318	0	0	2542	0	0
319	1	1	2542	0	0
322	4	16	2545	3	9
324	6	36	2550	8	64
325	7	49	2566	24	576
	−1	247		5	957

For Share *A*,

$$A.M. = 318 - \frac{1}{8} = 318 - 0{\cdot}125$$

$$= 317{\cdot}875$$

$$S.D. = \sqrt{\frac{247}{8} - \left(-\frac{1}{8}\right)^2}$$

$$= \frac{1}{8}\ \sqrt{1976 - 1} = \frac{1}{8}\sqrt{1975}$$

$$= \frac{1}{8}\ (44{\cdot}441) = 5{\cdot}555$$

$$\text{Coefficient of variation} = \left(\frac{S.D.}{A.M.}\right)(100)$$

$$= \left(\frac{5{\cdot}555}{317{\cdot}875}\right)(100)$$

$$= 17{\cdot}5\%.$$

For Share *B*,

$$A.M. = 2542 + \frac{5}{8} = 2542{\cdot}625$$

$$S.D. = \sqrt{\frac{957}{8} - \left(\frac{5}{8}\right)^2}$$

$$= \frac{1}{8}\sqrt{7656 - 25} = \frac{1}{8}\ \sqrt{7631}$$

$$= \frac{1}{8}\ (87{\cdot}36) = 10{\cdot}92$$

$$\text{Co-efficient of variation} = \left(\frac{10{\cdot}92}{2542{\cdot}625}\right)(100)$$

$$= \frac{1092000}{2542625} = 0{\cdot}43\%.$$

Since co-efficient of variation for share *A* is greater than that for share *B*, share *A* shows greater variability.

Ex. 2-11. *On a final examination in statistics, the mean marks of a group of 150 students were 78 and the s.d. was 8·0. In Economics, however the mean marks of the group were 73 and the s.d. was 7·6. In what subject was there greater variability?*

Sol.

Coefficient of variation for statistics paper

$$=\left(\frac{8\cdot0}{78}\right)(100)=\frac{800}{78}$$
$$=10\cdot3\%.$$

Coefficient of variation for Economics paper

$$=\left(\frac{7\cdot6}{73}\right)(100)=\frac{760}{73}$$
$$=10\cdot4\%$$

$\therefore$ In Economics there was greater variability.

Ex. 2.12. *Show that if the variable takes the values 0, 1, 2,.. n with frequencies proportional to the binomial co-efficients 1, $^nc_1$, $^nc_2$,... ^nc_n respectively, then the mean of the dist is $\frac{n}{2}$, the mean square deviation about $x=0$ is $\frac{n(n+1)}{4}$ and the variance is $\frac{n}{4}$.*

Sol.
$$A.M.=\frac{0.1+1.{}^nc_1+2.{}^nc_2+3.{}^nc_3+\ldots+n.{}^nc_n}{1+{}^nc_1+{}^nc_2+\ldots\ldots+{}^nc_n}$$

$$=\frac{n\left\{1+(n-1)+\frac{(n-1)(n-2)}{2!}+\ldots+1\right\}}{(1+1)^n}$$

$$=n\left\{\frac{1+{}^{n-1}c_1+{}^{n-1}c_2+\ldots\ldots+{}^{n-1}c_{n-1}}{2^n}\right\}$$

$$=n.\frac{2^{n-1}}{2^n}=\frac{n}{2}$$

$$\mu_2'(0)=\frac{0^2.1+1^2.{}^nc_1+\ldots\ldots+n^2.{}^nc_n}{2^n}$$

$$=\frac{1}{2^n}\sum_{x=0}^{n}x^2.{}^nc_x=\frac{1}{2^n}\sum_{x=0}^{n}\{x(x-1)+x\}.{}^nc_x$$

$$=\frac{1}{2^n}\sum_{x=0}^{n}x(x-1).{}^nc_x+\frac{1}{2^n}\sum_{x=0}^{n}x.{}^nc_x$$

$$=\frac{1}{2^n}\{2.1.{}^nc_2+3.2.{}^nc_3+\ldots+n(n-1){}^nc_n\}+\frac{n}{2}$$

$$=\frac{1}{2^n}\,n(n-1)(1+1)^{n-2}+\frac{n}{2}=\frac{n(n-1)}{4}+\frac{n}{2}$$

$$=\frac{n(n+1)}{4}$$

$$\therefore\quad \mu_2=\frac{n(n+1)}{4}-\frac{n^2}{4}=\frac{n}{4}.$$

Ex. 2-13. *Find the mean deviation from the mean and the s.d. of the A.P. a, a+d,......a+2nd. and prove that the latter is greater than the former.*

Sol. A.M. is given by

$$\bar{x}=\frac{a+(a+d)+\ldots+(a+2nd)}{2n+1}=a+d\left\{\frac{1+2+\ldots\ldots+2n}{2n+1}\right\}$$

$$=a+d\,\frac{2n(2n+1)}{2(2n+1)}=a+nd$$

$\therefore$ Mean deviation from the mean

$$=\frac{1}{2n+1}\,[\{\,|-nd|+|-(n-1)d|+\ldots\ldots+|d|\,\}$$
$$+\{\,|d|+\ldots+|nd|\,\}]$$

$$=\frac{2d}{2n+1}\,\{1+2+\ldots\ldots+n\}=\frac{n(n+1)d}{(2n+1)}$$

$$\text{S.D.}=\sqrt{\frac{1}{2n+1}[\{(-nd)^2+(-(n-1)d)^2+\ldots(-d)^2\}+\{d^2+\ldots+n^2d^2\}]}$$

$$=\sqrt{\frac{2d^2}{2n+1}\,\{1^2+2^2+\ldots+n^2\}}=d\sqrt{\frac{2n(n+1)(2n+1)}{6}\cdot\frac{1}{(2n+1)}}$$

$$=d\sqrt{\frac{n(n+1)}{3}}.$$

Consider

$$(\text{Mean deviation})^2-\text{Variance}=a^2\left\{\frac{n^2(n+1)^2}{(2n+1)^2}-\frac{n(n+1)}{3}\right\}$$

$$=\frac{n(n+1)d^2}{3(2n+1)^2}\{3n(n+1)-(4n^2+4n+1)\}$$

$$=-\frac{n(n+1)d^2}{3(2n+1)^2}(n^2+n+1)<0$$

$\therefore$ Mean deviation $<$ *S.D.*

Ex. 2-41. *If r be the range and* $S=\left\{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}^{\frac{1}{2}}$ *be*

the s.d. of a set of observations $x_1, x_2, \ldots\ldots x_n$ then show that

$$S < r\left(\frac{n}{n-1}\right)^{\frac{1}{2}}$$

Sol. Let $x_r = \max.\ (x_1, x_2, \ldots\ldots x_n)$

and $\qquad x_k = \min.\ (x_1, x_2, \ldots\ldots x_n)$

Then $\qquad r = x_r - x_k$

Now $\qquad \bar{x} = \dfrac{x_1 + x_2 + \ldots\ldots + x_n}{n} > \dfrac{x_k + x_k + \ldots\ldots + x_k}{n} = x_k$

$\therefore \qquad (x_i - \bar{x})^2 \leqslant (x_i - x_k)^2$

$$\therefore\ S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 < \frac{1}{n-1}\sum_{i=1}^{n}(x_i - x_k)^2$$

$$\leqslant \frac{1}{n-1}\left\{\sum_{i=1}^{n}(x_r - x_k)^2\right\}$$

$$= \frac{nr^2}{n-1} \qquad (\because\ x_r - x_k = r)$$

$$\therefore \qquad S < r\left\{\frac{n}{n-1}\right\}^{\frac{1}{2}}$$

Theorem 2.2-2. *Show that the root mean square deviation is least when deviations are measured from the mean.*

Sol. Consider the freq. dist.

$$\begin{matrix} x \rightarrow \\ f \rightarrow \end{matrix}\begin{pmatrix} x_1\ x_2 \ldots\ldots x_n \\ f_1\ f_2 \ldots\ldots f_n \end{pmatrix}$$

where $\qquad N = \displaystyle\sum_{i=1}^{n} fi$

Let 'a' be an arbitrary point. Then

$$\mu_2'(a) = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - a)^2 = \frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x} + \bar{x} - a)^2$$

where $\bar{x}$ is the A.M.

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\{(x_i - \bar{x})^2 + (\bar{x} - a)^2 + 2(x_i - \bar{x})(\bar{x} - a)\}$$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})^2+(\bar{x}-a)^2.\ \frac{1}{N}\sum_{i=1}^{n} f_i$$

$$+2(\bar{x}-a)\frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})$$

$$=\mu_2+(\bar{x}-a)^2$$

$\therefore \qquad \mu_2'(a)-\mu_2=(\bar{x}-a)^2 > 0$

or $\qquad \mu_2'(a) > \mu_2$

$\therefore \qquad \sqrt{\mu_2'(a)} > \sqrt{\mu_2}$

$\therefore$ The root mean square deviation is least when deviations are measured from the mean.

Ex. 2-15. *In a series of measurements we obtain m_1 values of magnitude x_1, m_2 values of magnitude x_2 and so on. If $\bar{x}$ is the mean value of all the measurements, prove that*

$$S.D.=\sqrt{\frac{\Sigma m_r\,(k-x_r)^2}{\Sigma m_r}-\delta^2}$$

where $\bar{x}=k+\delta$ and k is any constant.

Sol. $\qquad \mu_2=\mu_2'(k)-(\bar{x}-k)^2=\mu_2'(k)-\delta^2$

$\therefore \qquad S.D.=\sqrt{\frac{\Sigma m_r\,(k-x_r)^2}{\Sigma m_r}-\delta^2}$

Theorem 2.2-3. *Show that S.D. is independent of origin but not of scale.*

Sol. Consider the freq. dist.

$$\begin{matrix} x\to \\ f\to \end{matrix}\begin{pmatrix} x_1\ x_2 \ldots\ldots x_n \\ f_1\ f_2 \ldots\ldots f_n \end{pmatrix}$$

where $\qquad N=\sum_{i=1}^{n} f_i$

The transformation corresponding to change of origin and scale is

$$U=\frac{x-a}{h}$$

where 'a' corresponds to change of origin and h to change in scale.

$\therefore \qquad x=a+Uh$

Let $\bar{x}$ be the A.M. of x and $\bar{U}$ that of U.

Then $$\bar{x}=\frac{1}{N}\sum_{i=1}^{n} f_i(x_i)=\frac{1}{N}\sum_{i=1}^{n} f_i\,(a+U_i\,h)$$

$$=a+h\frac{1}{N}\sum_{i=1}^{n} f_iU_i=a+h\bar{U}$$

$\therefore$ $$\mu_2=\frac{1}{N}\sum_{i=1}^{n} f_i\{x_i-\bar{x}\}^2=h^2.\frac{1}{N}\sum_{i=1}^{n} f_i(U_i-\bar{U})^2$$

$$=h^2.\mu_2 \text{ for } U$$

$\therefore$ μ_2 for $x=h^2.\mu_2$ for U

$\therefore$ Variance and hence *s.d* is independent of origin but not of scale

Ex. 2-16. *From a sample of n observations, the A.M. and variance are calculated. It is then found that one of the values x_1 is in error and should be replaced by x_1'. Show that the adjustment to the variance to correct this error is*

$$\frac{1}{n}(x_1'-x_1)\left(x_1'+x_1-\frac{x_1'-x_1+2T}{n}\right)$$

where T is the total of original observations.

Sol. Let $\bar{x}$ and σ^2 be the calculated values of A.M. and variance.

Then $\Sigma x_i=n\bar{x}$

and $\Sigma(x_i-\bar{x})^2=n\sigma^2$

i.e., $\Sigma x_i^2-n\bar{x}^2=n\sigma^2$

$\therefore$ $\Sigma x_i^2=n\sigma^2+n\bar{x}^2$

Now corrected value of $\Sigma x_i=(\Sigma x_i-x_1+x_1')$

Corrected value of $\Sigma x_i^2=(\Sigma x_i^2-x_1^2+x_1'^2)$

$\therefore$ Corrected value of A.M. $=\frac{1}{n}\{\Sigma x_i-x_1+x_1')$

$$=\left(\bar{x}+\frac{x_1'-x_1}{n}\right)$$

and corrected value of $\frac{1}{n}\Sigma x_i^2=\frac{1}{n}\{\Sigma x_i^2-x_1^2+x_1'^2\}$

$$=\sigma^2+\bar{x}^2+\frac{x_1'^2-x_1^2}{n}$$

$\therefore$ Corrected value of variance

$$=\sigma^2+\bar{x}^2+\frac{x_1'^2-x_1^2}{n}-\left(\bar{x}+\frac{x_1'-x_1}{n}\right)^2$$

$\therefore$ Adjustment to the variance to correct the error

$$=\text{corrected value of variance }-\sigma^2$$

$$=\bar{x}^2+\frac{x_1'^2-x_1^2}{n}-\left(\bar{x}+\frac{x_1'-x_1}{n}\right)^2$$

$$=\frac{1}{n}(x_1'-x_1)\left\{x_1'+x_1-2\bar{x}-\frac{x_1'-x_1}{n}\right\}$$

Now $\qquad \bar{x}=\frac{1}{n}\Sigma x_i=\frac{T}{n} \qquad (\because T=\Sigma x_i)$

$\therefore$ Adjustment to the variance

$$=\frac{1}{n}(x_1'-x_1)\left\{x_1'+x_1-\frac{x_1'-x_1+2T}{n}\right\}$$

Ex. 2-17. *For a frequency distribution of marks in History of 200 candidates (grouped in intervals 0—5, 5—10,......etc.) the mean and s.d. were found to be 40 and 15. Later it was discovered that the score 43 was misread as 53 in obtaining the frequency dist. Find the corrected mean and s.d. corresponding to the corrected frequency dist.*

Sol. Since the score 43 was misread as 53 and the scores 43 and 53 lie in intervals 40—45 and 50—55, in the calculation of mean and s.d. variate value was taken to be 52·5 instead of the actual value 42·5.

Now if x be the variate,

$$\Sigma x=(40)(200)=8000$$

$\therefore$ Corrected value of

$$\Sigma x=8000-52{\cdot}5+42{\cdot}5$$
$$=7990$$

$\therefore$ Corrected value of mean

$$=\frac{7990}{200}=39{\cdot}95$$

Also $\qquad s.d.=15$

$\therefore \qquad Var(x)=225$

$\therefore \qquad \Sigma(x-\bar{x})^2=(225)(200)=45000$

or $\qquad \Sigma x^2-N\bar{x}^2=45000$

$\therefore \qquad \Sigma x^2=45000+(200)(1600)=365000$

$\therefore$ Corrected value of Σx^2

$$=365000-(52{\cdot}5)^2+(42{\cdot}5)^2=364050$$

$\therefore$ Corrected value of $\Sigma(x-\bar{x})^2$=(corrected value of Σx^2)
$-N$(corrected mean)^2

$$=364050-200(39\cdot95)^2=44849\cdot5$$

$\therefore$ Corrected variance

$$=\frac{44849\cdot5}{200}=224\cdot2475$$

$\therefore$ Corrected $s.d.=\sqrt{224\cdot2475}=14\cdot97$

Ex. 2-18. *The mean of 5 observations is 4·4 and the variance is 8·24. If three of the five observations are 1, 2 and 6, find the other two.*

Sol. Let x_1 and x_2 be other observations. Then

$$(4\cdot4)(5)=(1+2+6)+(x_1+x_2)$$

or
$$x_1+x_2=22-9=13 \qquad ...(1)$$

and
$$(8\cdot24)(5)=(1-4\cdot4)^2+(2-4\cdot4)^2+(6-4\cdot4)^2$$
$$(x_1-4\cdot4)^2+(x_2-4\cdot4)^2$$

$\therefore$ $x_1^2+x_2^2=97\cdot0$

Now $2(x_1^2+x_2^2)=(x_1+x_2)^2+(x_1-x_2)^2$

$\therefore$ $x_1-x_2=5$ (taking positive sign) $\qquad$ .. (2)

From (1) and (2)

$$x_1=9,\ x_2=4$$

Ex. 2-19. *If the mean and s.d. of a variate x are m and σ respectively, obtain the mean and s.d. of $\frac{ax+b}{c}$ where a, b and c are constants.*

Sol. Let $\qquad U=\dfrac{ax+b}{c}$

Let $\bar{U}$ and σ_u be the mean and *s.d.* of *U*.

Then $\bar{U}=\dfrac{1}{N}\sum f\left(\dfrac{ax+b}{c}\right)=\dfrac{1}{c}\left\{a\,\dfrac{1}{N}\sum fx+b\dfrac{1}{N}\sum f\right\}$

$$=\frac{a\bar{x}+b}{c}$$

and $\qquad \sigma_u^2=\dfrac{1}{N}\sum f(U-\bar{U})^2=\dfrac{a^2}{c^2}\,\dfrac{1}{N}\sum f(x-\bar{x})^2=\dfrac{a^2}{c^2}\,\sigma^2$

$\therefore$ $\qquad \sigma_u=\left|\dfrac{a}{c}\right|\sigma$

Theorem 2.2-4. *Show that the s.d. is not less than the mean deviation from the mean.*

Sol. Consider the freq. dist.

$$\begin{matrix} x \to \\ f \to \end{matrix} \begin{pmatrix} x_1 & x_2 \ldots\ldots x_n \\ f_1 & f_2 \ldots\ldots f_n \end{pmatrix}$$

where
$$N = \sum_{i=1}^{n} f_i$$

Let $\bar{x}$ be the A.M. Then it is to be proved that *s.d.* $\nless$ mean deviation from the mean

i.e., *S.D.* $\geqslant$ Mean deviation from the mean

$$\text{i.e.,} \sqrt{\frac{1}{N} \sum_{i=1}^{n} f_i(x_i - \bar{x})^2} \geqslant \frac{1}{N} \sum_{i=1}^{n} f_i \mid x_i - \bar{x} \mid$$

$$\text{i.e.,} \qquad N \sum_{i=1}^{n} f_i y_i^2 \geqslant \left(\sum_{i=1}^{n} f_i y_i \right)^2$$

where $\qquad y_i = \mid x_i - \bar{x} \mid$

$$\text{i.e.,} (f_1 + f_2 + \ldots + f_n)(f_1 y_1^2 + \ldots + f_n y_n^2) \geqslant (f_1 y_1 + f_2 y_2 + \ldots + f_n y_n)^2$$

$$\text{i.e.,} \qquad f_1 f_2 (y_1 - y_2)^2 + \ldots\ldots \geqslant 0$$

which is true.

Ex. 2-20. *Show that if the deviations are small compared with the mean so that* $\left(\frac{x}{M}\right)^3$ *and higher powers of* $\left(\frac{x}{M}\right)$ *may be neglected,*

(*i*) $G = M\left(1 - \frac{\sigma^2}{2M^2}\right)$ *where 'G' is the G.M. and 'M' the A.M. and 'σ' the s.d.*

(*ii*) $H = M\left(1 - \frac{\sigma^2}{M^2}\right)$ *where H is the H.M.*

(*iii*) $H + M = 2G$.

(*iv*) $M^2 - G^2 = \sigma^2$.

(*v*) $MH = G^2$.

(*vi*) *mean* $(\sqrt{x}) = \sqrt{M}\left(1 - \frac{\sigma^2}{8M^2}\right)$.

Sol. (*i*) By def.

$$\log G = \frac{1}{N} \sum_{i=1}^{n} f_i \log x_i$$

Let $X_i = x_i - M$ so that $x_i = X_i + M$

$$\therefore \log G = \frac{1}{N}\sum_{i=1}^{n} f_i \log (X_i + M)$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i \left\{ \log M + \log\left(1 + \frac{X_i}{M}\right)\right\}$$

$$= \log M + \frac{1}{N}\sum_{i=1}^{n} f_i \log\left(1 + \frac{X_i}{M}\right)$$

Applying expansion of $\log\left(1+\frac{X_i}{M}\right)$ and neglecting $\left(\frac{X_i}{M}\right)^3$ and higher powers

$$\log G = \log M + \frac{1}{N}\sum_{i=1}^{n} f_i \left\{\frac{X_i}{M} - \frac{1}{2}\,\frac{X_i^2}{M^2}\right\}$$

$$= \log M + \frac{1}{M}\cdot\frac{1}{N}\sum_{i=1}^{n} f_i X_i - \frac{1}{2M^2}\cdot\frac{1}{N}\sum_{i=1}^{n} f_i X_i^2$$

$$\therefore \quad \log\frac{G}{M} = -\frac{1}{2M^2}\sigma^2 \qquad \left(\because \quad \sum_{i=1}^{n} f_i X_i = 0\right)$$

$$\therefore \quad G = Me^{-\frac{1}{2M^2}\sigma^2} = M\left(1 - \frac{\sigma^2}{2M^2}\right)$$

(Applying the expansion of $e^{-\frac{\sigma^2}{2M^2}}$ and neglecting higher powers)

(*ii*) By def. $\dfrac{1}{H} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{n} f_i x_i$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i \frac{1}{X_i + M} = \frac{1}{M}\cdot\frac{1}{N}\sum_{i=1}^{n} f_i \left(1 + \frac{X_i}{M}\right)^{-1}$$

$$=\frac{1}{M}\frac{1}{N}\sum_{i=1}^{n} f_i\left\{1-\frac{X_i}{M}+\frac{X_i^2}{M^2}\right\}=\frac{1}{M}\left(1+\frac{\sigma^2}{M^2}\right)$$

$$\therefore \qquad H=M\left\{1+\frac{\sigma^2}{M^2}\right\}^{-1}=M\left(1-\frac{\sigma^2}{M^2}\right)$$

(*iii*) From (*ii*) $H+M=M\left(2-\frac{\sigma^2}{M^2}\right)=2M\left(1-\frac{\sigma^2}{2M^2}\right)$

$$=2G$$

(*iv*) From (*i*) $G^2=M^2\left(1-\frac{\sigma^2}{2M^2}\right)=M^2\left(1-\frac{\sigma^2}{M^2}\right)$

(neglecting higher power)

$$=M^2-\sigma^2$$

$$\therefore \qquad M^2-G^2=\sigma^2$$

(v) From (*ii*) $MH=M^2-\sigma^2=G^2$ [from (*iv*)]

(*vi*) mean $(\sqrt{x})=\frac{1}{N}\sum_{i=1}^{n} f_i\sqrt{x_i}$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i\{X_i+M\}^{\frac{1}{2}}=\sqrt{M}.\frac{1}{N}\sum_{i=1}^{n} f_i\left(1+\frac{X_i}{M}\right)^{\frac{1}{2}}$$

$$=\sqrt{M}.\frac{1}{N}\sum_{i=1}^{n} f_i\left\{1+\frac{1}{2}\frac{X_i}{M}-\frac{1}{8}\frac{X_i^2}{M^2}\right\}$$

$$=\sqrt{M}.\left\{1-\frac{1}{8M^2}\sigma^2\right\}$$

Ex. 2-21. *Show that, if the deviations are small compared with the mean M so that* $\left(\frac{x}{M}\right)^3$ *and higher powers may be neglected.*

$$V=\sqrt{\frac{2(M-G)}{M}}$$

where V is the co-efficient of variation.

Sol. From last Ex.,

$$G=M\left(1-\frac{\sigma^2}{2M^2}\right)$$

$$\therefore \qquad 2(M-G)=\frac{\sigma^2}{M}$$

$$\therefore\quad V = \text{co-efficient of variation} = \frac{s.d.}{\text{mean}} = \frac{\sigma}{M}$$

$$= \sqrt{\frac{2(M-G)}{M}}$$

2·3. Combining no. of Distributions. *If k-distributions with respective means $m_1, m_2, \ldots\ldots m_k$, sizes $n_1, n_2 \ldots\ldots n_k$ and s.d.s. $\sigma_1, \sigma_2, \ldots\ldots \sigma_k$ be combined together, the mean and s.d. of the new distribution are given by*

$$m = \frac{n_1 m_1 + n_2 m_2 + \ldots\ldots + n_k m_k}{n_1 + n_2 + \ldots\ldots + n_k}$$

and
$$\sigma^2 = \frac{n_1 \sigma_1^2 + n\ \sigma_2^2 + \ldots\ldots + n_k \sigma_k^2}{n_1 + n_2 + \ldots\ldots n_k}$$

$$+ \frac{n_1(m-m_1)^2 + n_2(m-m_2)^2 + \ldots\ldots n_k(m-m_k)^2}{n_1 + n_2 + \ldots\ldots + n_k}$$

Ex. 2-22. *The standard deviations of two sets containing n_1 and n_2 members are σ_1 and σ_2 respectively being measured from their respective means m_1 and m_2. If the two sets are grouped together as one set of (n_1+n_2) members, show that the s.d. σ of this set measured from its mean is given by*

$$\sigma^2 = \frac{n_1 \sigma_1^2 + n_2 \sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1+n_2)^2}(m_1 - m_2)^2$$

Sol. Let $x_1, x_2, \ldots\ldots x_{n_1}$ and $y_1, y_2 \ldots y_{n_2}$ be the members of two sets. Then by def.

$$m_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i \quad \text{and} \quad m_2 = \frac{1}{n_2} \sum_{j=1}^{n_2} y_j$$

Let m be the mean of the grouped set. Then

$$m = \frac{1}{n_1+n_2} \left\{ \sum_{i=1}^{n_1} x_i + \sum_{j=1}^{n_2} y_j \right\} = \frac{n_1 m_1 + n_2 m_2}{n_1 + n_2}$$

Now
$$\sigma^2 = \frac{1}{n_1+n_2} \left\{ \sum_{i=1}^{n_1} (x_i - m)^2 + \sum_{j=1}^{n_2} (y_j - m)^2 \right\}$$

$$= \frac{1}{n_1+n_2} \left\{ \sum_{i=1}^{n} (x_i - m_1 + m_1 - m)^2 + \sum_{i=1}^{n} (y_j - m_2 + m_2 - m)^2 \right\}$$

$$=\frac{1}{n_1+n_2}\Big[\Big\{\sum_{i=1}^{n_1}(x_i-m_1)^2+2(m_1-m)\sum_{j=1}^{n_1}(x_i-m_1)$$

$$+\sum_{i=1}^{n_1}(m_1-m)^2\Big\}+\Big\{\sum_{j=1}^{n_2}(y_j-m_2)^2+2(m_2-m)\sum_{i=1}^{n_2}(y_j-m_2)$$

$$+\sum_{j=1}^{n_2}(m_2-m)^2\Big\}\Big]$$

$$=\frac{1}{n_1+n_2}\Big\{n_1\sigma_1^2+n_1(m_1-m)^2+n_2\sigma_2^2+(m_2-m)^2\,n_2\Big\}$$

$$=\frac{n_1\sigma_1^2+n_2\sigma_2^2}{n_1+n_2}+\frac{n_1(m_1-m)^2+n_2(m_2-m)^2}{n_1+n_2}$$

$$\left(\because\quad \sigma_1^2=\frac{1}{n_1}\sum_{i=1}^{n_1}(x_i-m_1)^2 \text{ etc.}\right)$$

Now $\quad (m_1-m)^2=\dfrac{n_2^2(m_1-m_2)^2}{(n_1+n_2)^2}$

and $\quad (m_2-m)^2=\dfrac{n_1^2(m_1-m_2)^2}{(n_1+n_2)^2}$

$\therefore \quad \sigma^2=\dfrac{n_1\sigma_1^2+n_2\sigma_2^2}{n_1+n_2}+\dfrac{n_1n_2(m_1-m_2)^2}{(n_1+n_2)^2}$

Ex. 2-23. *An analysis of the monthly wages paid to workers in two firms A and B, belonging to the same industry, gives the following results :*

	Firm A	*Firm B*
No. of wage earners	*586*	*648*
Average monthly wage	*Rs. 52·5*	*Rs. 47·5*
Variance of the dist. of wages	*100*	*121*

(a) Which firm A or B pays out the larger amount as monthly wages ?

(b) In which firm A or B is there greater variability in individual wages ?

(c) What are the measures of (i) average monthly wage and the variability in individual wages of all the workers in the firm A and B taken together ?

Sol. (*a*) Firm *A* pays$=(52\cdot5)\,(586)=30765$ Rs. monthly

Firm *B* pays$=(47\cdot5)\,(648)=$Rs 30780 monthly

$\therefore$ Firm *B* pays more as monthly wages.

(*b*) Co-efficient of variation for Firm *A*

$$=\left(\frac{\sqrt{100}}{52\cdot5}\right)(100)=\frac{1000}{52\cdot5}=19\cdot05\%$$

Co-efficient of variation for firm *B*

$$=\left(\frac{\sqrt{121}}{47\cdot5}\right)(100)=\frac{1100}{47\cdot5}=23\cdot16\%$$

$\therefore$ Firm *B* has greater variability in the individual wages.

(*c*) Average monthly wage and the variability in individual wages of all the workers in the firm *A* and *B* taken together, are the A.M. and co-efficient of variation of the wages paid by Firms *A* and *B* together.

Let m and σ be the A.M. and *s.d.* of the wages paid by *A* and *B* together.

Then $$m=\frac{(586)\,(52\cdot6)+(648)(47\cdot5)}{586+648}=49\cdot87$$

and $$\sigma^2=\frac{(586)(100)+(648)(121)}{1234}+\frac{(586)(49\cdot87-52\cdot5)^2+648(49\cdot87-47\cdot5)^2}{1234}$$

$$=117\cdot26$$

$\therefore$ Co-efficient of variation for firms *A* and *B* taken together

$$=\frac{\sqrt{117\cdot26}}{49\cdot87}\times100=\frac{1082\ 87}{49\cdot87}$$

$$=21\cdot7\%.$$

Ex. 2·24. *The first of two samples has 100 items with mean 15 and s.d. 3. If the whole group has 250 items with mean 15·6 and s.d. $\sqrt{13\cdot44}$, find the s.d. of the second group.*

Sol. Let m_2 be the mean of second group

Then $$15\cdot6=\frac{(100)(15)+(150)m_2}{250}$$

$\therefore$ $$m_2=\frac{(250)(15\cdot6)-100(15)}{150}=16$$

Let σ be the standard deviation of second group. Then

$$\therefore \qquad 13{\cdot}44=\frac{[(100)(9)+(150)\sigma^2]+100(0{\cdot}6)^2+150(0{\cdot}4)^2}{250}$$

$$\therefore \qquad \sigma=4.$$

Ex. 2-25. *The mean and s.d. of 63 children on an arithmetic test are respectively 27·6 and 7·1. To them are added a new group of 26 who have had less training and whose mean is 19 2 and s.d. 6·2. How will the values of the combined group differ from those of the original 63 children as to the following (i) the mean (ii) the s.d.*

Sol. Mean m and *s.d.* σ of the combined group are given by

$$(i) \qquad m=\frac{(63)(27{\cdot}6)+(26)(19{\cdot}2)}{63+26}=25{\cdot}1$$

$\therefore$ The A.M. is decreased by $27{\cdot}6-25{\cdot}1=2{\cdot}5$

$$(ii)\ \sigma^2=\frac{(63)(7{\cdot}1)^2+(26)(6{\cdot}2)^2}{63+26}+\frac{63(25{\cdot}1-27{\cdot}6)^2+26(25{\cdot}1-19{\cdot}2)^2}{63+26}$$

$\therefore \quad \sigma=7{\cdot}8$ (approx)

$\therefore$ The *s.d.* is increased by

$7{\cdot}8-7{\cdot}1=0{\cdot}7$ **(approx).**

Ex. 2-26 *A distribution consists of three components with frequencies of 200, 250 and 300 having means 25, 10 and 15 and s.d. of 3, 4 and 5 respectively. Show that the mean of the combined distribution is 16 and s.d. 7·2 approximately.*

Sol. Let m and σ be the mean and *s.d.* of the combined distribution.

$$\text{Then} \qquad m=\frac{(25)(200)+(10)(250)+(15)(300)}{200+250+300}$$

$$=\frac{5000+2500+4500}{750}=\frac{12000}{750}=16$$

$$\text{and} \qquad \sigma^2=\frac{(200)(3^2)+(250)(4^2)+(300)(5^2)}{200+250+300}$$

$$+\frac{200\{16-25\}^2+250\{16-10\}^2+300\{16-15\}^2}{200+250+300}$$

$$=\frac{(1800+4000+7500)+(16200+9000+300)}{750}$$

$$=\frac{38800}{750}=51{\cdot}73$$

$$\therefore \qquad \sigma=\sqrt{51{\cdot}73}=7{\cdot}2 \text{ (approx)}$$

2·4. Moments. *The rth moment about the point 'a' is defined by*

$$\mu_r'(a)=\frac{1}{N}\,\Sigma f(x-a)^r$$

If 'a' is A.M., rth moment about 'a' is denoted by μ_r.

Factorial Moments. *Factorial moment of order 'r' about the origin is defined by*

$$\mu'_{(r)}=\frac{1}{N}\,\Sigma f x^{(r)}$$

where $$x^{(r)}=x(x-1)\ldots\ldots\ldots(x-r+1)$$

Absolute Moments. *Absolute moment of order r about an arbitrary point 'a' is defined to be*

$$\frac{1}{N}\,\Sigma f\,|\,x-a\,|^r$$

Pearson's β and γ Co-efficients. These co-efficients are defined by

$$\beta_1=\frac{\mu_3^2}{\mu_2^3},\qquad \beta_2=\frac{\mu_4}{\mu_2^2}$$

$$\gamma_1=\sqrt{\beta_1}\quad\text{and}\quad \gamma_2=\beta_2-3$$

Moments about mean in terms of moments about any other point are given by

$$\mu_r=\mu_r'-{}^rc_1\,\mu'_{r-1}\,\mu_1'+{}^rc_2\,\mu_{r-2}\,\{\mu_1'\}^2+\ldots\ldots+(-1)^r\,{}^rc_r\,\{\mu_1'\}^r$$

Shappard's Corrections to Moments of Grouped Frequency Distribution.

No correction applied to odd order moments i.e., μ_1' *(corrected)*$=\mu_1'$ *and* μ_3 *(corrected)*$=\mu_3$ *and* μ_2 *(corrected)*$=\mu_2-\dfrac{h^2}{12}$

$$\mu_4\ (\text{corrected})=\mu_4-\frac{1}{2}h^2\mu_2+\frac{7}{240}h^4$$

Theorem. 2.4-1 *Show that moments about mean are independent of origin but not of scale.*

Sol. The transformation corresponding to change in origin and scale is $u=\dfrac{x-a}{h}$

$\therefore \qquad x=a+uh$

$\therefore \qquad \bar{x}=a+\bar{u}\,h$

where $\bar{x}$ and $\bar{u}$ are A.Ms..

$$\therefore \quad \mu_r \text{ of } x=\frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})^r=h^r.\ \frac{1}{N}\sum_{i=1}^{n} f_i(u_i-\bar{u})^r$$

Theorem 2.4-2. *Express rth moment about mean in terms of various moments about an arbitrary pt. 'a'.*

Sol. Consider a freq. dist.

$$\begin{matrix} x\rightarrow \\ f\rightarrow \end{matrix}\begin{pmatrix} x_1\ x_2 \ldots\ldots x_n \\ f_1\ f_2 \ldots\ldots f_n \end{pmatrix}$$

where $N=\sum_{i=1}^{n} f_i$

Let $\bar{x}$ be the A.M. Then

$$\mu_r=\frac{1}{N}\sum_{i=1}^{n} f_i\,(x_i-\bar{x})^r=\frac{1}{N}\sum_{i=1}^{n} f_i[(x_i-a)+(a-\bar{x})]^r$$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i[(x_i-a)^r+{}^rc_1(x_i-a)^{r-1}(a-\bar{x})$$

$$+{}^rc_2(x_i-a)^{r-2}(a-\bar{x})^2+\ldots+{}^rc_r(a-\bar{x})^r]$$

$$=\mu_r'(a)+{}^rc_1\mu'_{r-1}(a)(a-\bar{x})+{}^rc_2\mu'_{r-2}(a)(a-\bar{x})^2+\ldots\ldots$$
$$+{}^rc_r(a-\bar{x})^r$$

Now $a-\bar{x}=-\dfrac{1}{N}\displaystyle\sum_{i=1}^{n} f_i(x_i-a)=-\mu_1'(a)$

$$\therefore \qquad \mu_r=\mu_r'(a)-{}^rc_1\mu_r'{}_{-1}(a)\ \mu'(a)+{}^rc_2\mu_{r-2}(a)\{\mu_1'(a)\}^2+\ldots\ldots$$
$$+(-1)^r\ {}^rc_r\{\mu_1'(a)\}^r$$

Theorem 2.4-3. *Express rth moment about a pt. 'a' in terms of various moments about any other pt. 'b'*

Sol. $\mu_r'(a)=\dfrac{1}{N}\displaystyle\sum_{i=1}^{n} f_i(x_i-a)^r=\dfrac{1}{N}\displaystyle\sum_{i=1}^{n} f_i\{x_i-b)+(b-a)\}^r$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i[(x_i-b)^r+{}^rc_1(x_i-b)^{r-1}\,(b-a)$$

$$+{}^rc_2(x_i-b)^{r-2}(b-a)^2+\ldots+{}^rc_r(b-a)^r]$$

$$=\mu_r'(b)+{}^rc_1(b-a)\mu'_{r-1}(b)+{}^rc_2(b-a)^2\mu'_{r-2}(b)$$
$$+\ldots\ldots+{}^rc_r(b-a)^r.$$

Ex. 2-27. *Calculate the first four moments about the mean of the following dist, also calculate β_1 and β_2.*

x values in cm. are the mid-points of intervals :

x :	2·0	2·5	3·0	3·5	4·0	4·5	5·0
f :	5	38	65	92	70	40	0

Sol.

Variable x	f	$d=\frac{x-3\cdot5}{0\cdot5}$	$d.f$	$d^2.f$	$d^3.f$	$d^4.f$
2·0	5	−3	−15	45	−135	405
2·5	38	−2	−76	152	−304	608
3·0	65	−1	−65	65	−65	65
3·5	92	0	0	0	0	0
4·0	70	1	70	70	70	70
4·5	40	2	80	160	320	640
5·0	0	3	0	0	0	0
	310		−6	492	−114	1788

$$\therefore \quad \mu_1'(3\cdot5)=\frac{\Sigma d.f}{N}\times(0\cdot5)=-\frac{6}{310}\times(0\cdot5)$$

$$=-\frac{3}{310}=-0\cdot01$$

$$\mu_2'(3\cdot5)=\frac{\Sigma d^2 f}{N}(0\cdot5)^2=\frac{492}{310}(0\cdot25)$$

$$=\frac{123}{310}=0\cdot397.$$

$$\mu_3'(3\cdot5)=\frac{\Sigma d^3 f}{N}\times(0\cdot5)^3$$

$$=-\frac{114}{310}(0\cdot125)=-0\cdot046$$

$$\mu_4'(3\cdot5)=\frac{\Sigma d^4 f}{N}\times(0\cdot5)^4=\frac{1788}{310}(0\cdot0625)$$

$$=\frac{1788}{310}\times\frac{1}{16}=\frac{447}{1240}=0\cdot360.$$

Moments about the A.M. are :

$\mu_1=0$

$\mu_2=\mu_2'(3\cdot5)-[\mu_1'(3\cdot5)]^2$

$=0\cdot397-0\cdot0001=0\cdot3969\simeq0\cdot40$

$\mu_3=\mu_3'(3\cdot5)-3\mu_2'(3\cdot5)\mu_1'(3\cdot5)+2[\mu_1'(3\cdot5)]^3$

$=-0\cdot046-3(0\cdot397)(-0\cdot01)+2(-0\cdot01)^3$

$=-0\cdot034=-0\cdot03$ (approx)

$\mu_4=\mu_4'(3\cdot5)-4\mu_3'(3\cdot5)\mu_1'(3\cdot5)+6\mu_2'(3\cdot5)[\mu_1'(3\cdot5)]^2-3[\mu_1'(3\cdot5)]^4$

$=0\cdot360-4(-0\cdot046)(-0\cdot01)+6(0\cdot397)(-0\cdot01)^2-3(-0\cdot01)^4$

$=0\cdot358=0\cdot36$ (approx).

$$\beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{(-0\cdot034)^2}{(0\cdot397)^3}=0\cdot02$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{0\cdot358}{(0\cdot397)^2}=2\cdot27$$

Ex. 2·28. *The first four moments of a distribution about the value 4 are —1·5, 17, —30, 108. Calculate the moments about the mean.*

Sol. Moments about the mean are :

$\mu_1=0$

$\mu_2=\mu_2'-\mu_1'^2=17-(-1\cdot5)^2$

$=17-2\cdot25=14\cdot75$

$\mu_3=\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3$

$=-30-3(17)(-1\cdot5)+2(-1\cdot5)^3$

$=-30+76\cdot5-6\cdot75$

$=39\cdot75$

$\mu_4=\mu_4-4\mu_3'\mu_1'+6\mu_2'\mu_1'^2-3\mu_1'^4$

$=108-4(-30)(-1\cdot5)+6(17)(-1\cdot5)^2-3(-1\cdot5)^4$

$=108-180+229\cdot5-15\ 1875$

$=142\cdot3125\simeq142\cdot3.$

Ex. 2·29. *The first four moments of a distribution about the value 5 of the variable are 2, 20, 40 and 50. Obtain as far as possible the various characteristics of this distribution on the basis of the information given.*

Sol. The first four moments about the mean are :

$\mu_1=0$

$\mu_2=\mu_2'-\mu_1'^2=20-4=16$

$\mu_3=\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3$

$=40-3(20)(2)+2(2)^3$

$$=40-120+16$$
$$=-64$$
$$\mu_4==\mu_4-4\mu_3'\mu_1'+6\mu'_2\mu_1'^2-3\mu_1'^4$$
$$=50-4(40)(2)+6(20)(2)^2-3(2)^4$$
$$=50-320+480-48$$
$$=530-368=162$$

$$\text{A.M.}=\mu_1'(0)=\frac{1}{N}\Sigma fx$$
$$=\frac{1}{N}\Sigma f[(x-5)+5]$$
$$=\frac{1}{N}\Sigma f(x-5)+5$$
$$=\mu_1'(5)+5=2+5=7.$$
$$\beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{(-64)^2}{(16)^3}=1$$
$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{162}{(16)^2}=\frac{81}{128}=0{\cdot}63.$$
$$\gamma_1=\sqrt{\beta_1}=1$$
$$\gamma_2=\beta_2-3=0{\cdot}63-3=-2{\cdot}37.$$

Ex. 2·30. *The first three moments of a distribution about the value 2 of the variable are 1, 16, −40. Find as far as you can, the various characteristics of this dist on the basis of the information given.*

Sol.

$$\text{A.M.}=\frac{1}{N}\Sigma fx=\frac{1}{N}\Sigma f(x-2)+2]$$
$$=\frac{1}{N}\Sigma f(x-2)+2$$
$$=\mu_1'(2)+2$$
$$=1+2=3.$$

The first three moments about the A.M. are given by

$$\mu_1=0$$
$$\mu_2=\mu_2'-\mu_1'^2=16-1=15$$
$$\mu_3=\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3$$
$$=-40-3(16)(1)+2(1)^3$$
$$=-40-48+2=-86$$

$$\beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{(-86)^2}{(15)^3}=2{\cdot}19$$

$$\therefore \qquad \gamma_1=\sqrt{\beta_1}=\sqrt{2{\cdot}19}=1{\cdot}48.$$

Ex. 2·31. *The first four moments of a distribution are 1, 4, 10 and 46 respectively. Compute the first four central moments and beta constants. Comment upon the nature of the dist.*

Sol. The first four central moments are given by :

$$\mu_1=0$$
$$\mu_2=\mu_2'-\mu_1'^2=4-1=3$$
$$\begin{aligned}\mu_3&=\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3\\&=10-3(4)(1)+2(1)^3\\&=10-12+2=0\end{aligned}$$
$$\begin{aligned}\mu_4&=\mu_4-4\mu_3'\mu_1'+6\mu_2'\mu_1'^2-3\mu_1'^4\\&=46-4(10)(1)+6(4)(1)^2-3(1)^4\\&=46-40+24-3\\&=27\end{aligned}$$

$$\therefore \qquad \beta_1=0,\ \beta_2=\frac{\mu_4}{\mu_2^2}=\frac{27}{9}=3.$$

Since $\beta_1=0$, $\beta_2=3$, the distribution must be normal.

Ex. 2·32. *For a distribution of 250 heights, calculations showed that the mean, standard deviation, β_1 and β_2 were 54 inches, 3 inches, 0 and 3 inches respectively. It was however discovered on checking that the two items 64 and 50 in the original data were wrongly written in place of correct values 62 and 52 inches respectively. Calculate the correct frequency constants.*

Sol. Let x be the variable and N be the total frequency. Then $N=250$.

Then $\Sigma x=(250)(54)$

$\therefore$ Corrected value of $\Sigma x=(250)(54)-64-50+62+52$
$=(250)(54)$

$\therefore$ Corrected A.M.$=54$.

Variance$=9$

or $$\frac{1}{N}\Sigma(x-\bar{x})^2=9$$

or $$\Sigma(x-\bar{x})^2=(250)(9)=2250$$

$$\begin{aligned}\therefore \text{ Corrected } \Sigma(x-\bar{x})^2&=2250-(64-54)^2-(50-54)^2\\&\qquad+(62-54)^2+(52-54)^2\\&=2250-100-16+64+4\\&=2202\end{aligned}$$

$\therefore$ Corrected variance $=\dfrac{2202}{250}=8{\cdot}808$

$$\beta_1=0=\frac{\mu_3^2}{\mu_2^3}$$

$\therefore \qquad \mu_3=0$

$\therefore \qquad \Sigma(x-\bar{x})^3=0$

$\therefore$ Corrected $\Sigma(x-\bar{x})^3$

$$=0-(64-54)^3-(50-54)^3+(62-54)^3+(52-54)^3$$
$$=0-1000+64+512-8$$
$$=-432$$

$\therefore$ Corrected $\mu_3=\dfrac{-432}{250}=-1{\cdot}728$

$\therefore$ Corrected $\beta_1=\dfrac{(\text{corrected } \mu_3)^2}{(\text{corrected } \mu_2)^3}$

$$=\frac{(-1{\cdot}728)^2}{(8{\cdot}808)^3}$$
$$=0{\cdot}004$$

$$\beta_2=3=\frac{\mu_4}{\mu_2^2}=\frac{\mu_4}{(9)^2}$$

$\therefore \qquad \mu_4=243$

$\therefore \quad \Sigma(x-\bar{x})^4=(243)(250)=60750$

$\therefore$ Corrected $\Sigma(x-\bar{x})^4=60750-(64-54)^4-(50-54)^4+(62-54)^4+(52-54)^4$

$$=60750-10000-256+4096+16$$
$$=54606$$

$\therefore$ Corrected $\mu_4=\dfrac{54606}{250}=218{\cdot}424$

$\therefore$ Corrected $\beta_2=\dfrac{\text{corrected } \mu_4}{(\text{corrected } \mu_2)^2}=\dfrac{218{\cdot}424}{(8{\cdot}808)^2}$

$$=\frac{218{\cdot}4}{(8{\cdot}808)^2} \text{ (approx)}$$
$$=2{\cdot}815.$$

Ex. 2-33. *Second, third and fourth central moments of a variable characteristics are 19·67, 29·26 and 866·0 respectively. Calculate the beta constants correct to three decimal places.*

Sol.
$$\beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{(29{\cdot}26)^2}{(19{\cdot}67)^3}=0{\cdot}113 \text{ (approx.)}$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{866{\cdot}0}{(19{\cdot}67)^2}=2{\cdot}238.$$

Theorem 2.4-4. *Show that for a symmetrical distribution all moments about the mean of odd order are zero.*

Sol. For a symmetrical distribution the frequencies are symmetrically distributed about the mean *i.e.*, the values equidistant from the mean have equal frequencies. Let x be the variable and $\bar{x}$ its A.M.

Let $$y = x - \bar{x}.$$

Let x_1, x_2 be the values of x equidistant from $\bar{x}$.

Then the quantities $(x_1 - \bar{x})$ and $(x_2 - \bar{x})$ are equal in magnitude but opposite in signs. Let these quantities be y_1 and $-y_1$. Then since the distribution is symmetrical, the values $-y_1$ and y_1 of y have same frequencies f_1 each. Let other values of y be $-y_2$; y_2, $-y_3$; y_3 and so on. Let f_2, f_3......be the frequencies for $-y_2$; y_2, $-y_3$; y_3, ... Let N be the total frequency.

Now by def.,

$$\mu_{2r+1} = \frac{1}{N} \Sigma f (x - \bar{x})^{2r+1}$$

$$= \frac{1}{N} \Sigma f y^{2r+1}$$

$$= \frac{1}{N} \{ f_1(y_1^{2r+1} - y_1^{2r+1}) + f_2(y_2^{2r+1} - y_2^{2r+1}) + \ldots\ldots \} = 0.$$

Theorem 2.4-5. *Show that for a discrete dist* $\beta_2 > 1$.

Sol. Consider the frequency dist.

$$\begin{matrix} x \rightarrow \\ f \rightarrow \end{matrix} \begin{pmatrix} x_1 & x_2 & \ldots\ldots & x_n \\ f_1 & f_2 & \ldots\ldots & f_n \end{pmatrix}$$

where $\sum\limits_{i=1}^{n} f_i = N$

By def $\beta_2 = \dfrac{\mu_4}{\mu_2^2}$

$\therefore\ \beta_2 > 1$ if $\mu_4 > \mu_2^2$

i.e., $$\frac{1}{N} \sum_{i=1}^{n} f_i(x_i - \bar{x})^4 > \left\{ \frac{1}{N} \sum_{i=1}^{n} f_i(x_i - \bar{x})^2 \right\}^2 \text{where } \bar{x} = A.M.$$

i.e., $$N \sum_{i=1}^{n} f_i y_i^2 > \left(\sum_{i=1}^{n} f_i y_i \right)^2 \qquad \text{where } y_i = (x_i - \bar{x})^2$$

i.e., $(f_1+f_2+\ldots+f_n)(f_1y_1^2+f_1y_2^2+\ldots+f_ny_n^2)>(f_1y_1+f_2y_2+\ldots\ldots+f_ny_n)^2$

i.e., $f_1(f_1y_1^2+f_2y_2^2+\ldots\ldots+f_ny_n^2)+f_2(f_1y_1^2+f_2y_2^2+\ldots+f_ny_n^2)$
$+\ldots\ldots+f_n(f_1y_1^2+f_2y_2^2+\ldots\ldots+f_ny_n^2) > f_1^2y_1^2+f_2^2y_2^2+\ldots\ldots+f_n^2y_n^2$
$+2f_1f_2y_1y_2+\ldots\ldots$

i.e., $f_1f_2(y_1-y_2)^2+\ldots\ldots > 0$

which is true as each term on the left is positive.

$$\therefore \qquad \beta_2 > 1.$$

Ex. 2-34. *Define Factorial moments about the origin. Express first four factorial moments in terms of ordinary moments about origin and conversely.*

Sol. *Def.* Factorial moment of order 'r' about the origin is defined by

$$\mu_{(r)}'=\frac{1}{N}\sum_{i=1} f_ix_i^{(r)}$$

where $x^{(r)}=x(x-1)\ldots\ldots(x-r+1)$

Now $$\mu_{(1)}'=\frac{1}{N}\sum_{i=1}^{n} f_ix_i^{(1)}=\frac{1}{N}\sum_{i=1}^{n} f_ix_i=\mu_1'(0)$$

$$\mu_{(2)}'=\frac{1}{N}\sum_{i=1}^{n} f_ix_i^{(2)}=\frac{1}{N}\sum_{i=1}^{n} f_ix_i(x_i-1)$$

$$=\frac{1}{N}\sum_{i=1}^{n} f_ix_i^2-\frac{1}{N}\sum_{i=1}^{n} f_ix_i=\mu_2'(0)-\mu_1'(0)$$

$$\mu_{(3)}'=\frac{1}{N}\sum_{i=1}^{n} f_ix_i(x_i-1)(x_i-2)$$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i\{x_i^3-3x_i^2+2x_i\}=\mu_3'(0)-3\mu_2'(0)+2\mu_1'(0)$$

$$\mu_{(4)}'=\frac{1}{N}\sum_{i=1}^{n} f_i x_i\,(x_i-1)(x_i-2)(x_i-3)$$

$$=\mu_4'(0)-6\mu_3'(0)+11\mu_2'(0)-6\mu_1'(0)$$

Conversely. $\mu_1'(0)=\mu_{(1)}'$

$$\mu_2'(0)=\frac{1}{N}\sum_{i=1}^{n} f_i x_i^2=\frac{1}{N}\sum_{i=1}^{n} f_i\{x_i(x_i-1)+x_i\}.$$

$$=\mu_{(2)}'+\mu_{(1)}'$$

$$\mu_3'(0)=\frac{1}{N}\sum_{i=1}^{n} f_i x_i^3$$

Let $\quad x^3\equiv x(x-1)(x-2)+Ax(x-1)+Bx$

Put $\quad x=1, 2$

$\therefore \quad B=1$ and $2A+2B=8$ or $A=3$.

$\therefore \quad x^3\equiv x(x-1)(x-2)+3x(x-1)+x$

$$\therefore \quad \mu_3'(0)=\frac{1}{N}\sum_{i=1}^{n} f_i\{x_i(x_i-1)(x_i-2)+3x_i(x_i-1)+x_i\}$$

$$=\mu_{(3)}'+3\mu_{(2)}'+\mu_{(1)}'$$

$$\mu_4'(0)=\frac{1}{N}\sum_{i=1}^{n} f_i x_i^4$$

Now $\quad x^4\equiv x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7x(x-1)+x$

$\therefore \quad \mu_4'(0)=\mu_{(4)}'+6\mu_{(3)}'+7\mu_{(2)}'+\mu_{(1)}'$

Ex. 2·35. *Define absolute moments. Show that*

$$A_r^{2r}\leqslant A_{r-1}^{r}.A_{r+1}^{r}$$

where A is the rth absolute moment. Deduce that

$$A_r^{\frac{1}{r}}\leqslant A_{r+1}^{\frac{1}{r+1}}\qquad r=1, 2\ldots\ldots$$

Sol. *Def.* Absolute moment of order r about the origin is defined by

$$\frac{1}{N}\sum_{i=1}^{n} f_i\,|\,x_i\,|^{\,r}$$

and the absolute moment of order r about an arbitrary pt 'a' is defined by

$$\frac{1}{N}\sum_{i=1}^{n} f_i \mid x_i - a \mid^r$$

Evidently mean deviation is the first order absolute moment.

Let a and b be any two numbers. Then

$$\sum_{i=1}^{n} f_i \left\{ a \mid y_i \mid^{\frac{r-1}{2}} + b \mid y_i \mid^{\frac{r+1}{2}} \right\}^2 \geqslant 0$$

i.e., $$\sum_{i=1}^{n} f_i \{a^2 \mid y_i \mid^{r-1} + b^2 \mid y_i \mid^{r+1} + 2ab \mid y_i \mid^r\} \geqslant 0$$

i.e., $$a^2 \frac{1}{N}\sum_{i=1}^{n} f_i \mid y_i \mid^{r-1} + b^2 . \frac{1}{N}\sum_{i=1}^{n} f_i \mid y_i \mid^{r+1}$$

$$+2ab\frac{1}{N}\sum_{i=1}^{n} f_i \mid y_i \mid^r \geqslant 0$$

Put $y_i = x_i - a$.

Then $$a^2 . \frac{1}{N}\sum_{i=1}^{n} f_i \mid x_i - a \mid^{r-1} + b^2 . \frac{1}{N}\sum_{i=1}^{n} f_i \mid x_i - a \mid^{r+1}$$

$$+2ab . \frac{1}{N}\sum_{i=1}^{n} f_i \mid x_i - a \mid^r \geqslant 0$$

i.e., $$a^2 A_{r-1} + b^2 A_{r+1} + 2ab A_r \geqslant 0$$

i.e., $$A_{r-1}\left\{ a + \frac{A_r}{A_{r-1}} b \right\}^2 + \left\{ A_{r+1} - \frac{A_r^2}{A_{r-1}} \right\} b^2 \geqslant 0 \text{ (if } A_{r-1} \neq 0)$$

$$\therefore \quad A_{r+1} - \frac{A_r^2}{A_{r-1}} \geqslant 0 \text{ or } A_{r+1} A_{r-1} \geqslant A_r^2$$

$$\therefore \quad A_r^{2r} \leqslant A_{r+1}^r . A_{r-1}^r .$$

Put $$r=1, 2, 3\ldots\ldots r$$
$$A_1^2 \leqslant A_2A_0$$
$$A_2^4 \leqslant A_3^2A_1^2$$
$$A_3^6 \leqslant A_4^3A_2^3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_r^{2r} \leqslant A_{r+1}^r A_{r-1}^r.$$

Multiplying and using $A_0=\dfrac{1}{N}\sum\limits_{i=1}^{n} f_i \mid x_i-a \mid ^0=1$

$$A_r^{r+1} \leqslant A_{r+1}^r \qquad \text{for } r=1, 2\ldots\ldots$$
$$A_r^{\frac{1}{r}} \leqslant A_{r+1}^{\frac{1}{r+1}}.$$

Ex. 2-36. *Show that if the class interval of a grouped dist is less than one-third of the calculated s.d. Sheppard's adjustment makes a difference of less than ½% in the estimate of s.d.*

Sol. From sheppard's correction

$$\mu_2(\text{corrected})=\mu_2-\frac{h^2}{12}$$

Let μ_2 (corrected)$=\sigma_1^2$ and $\mu_2=\sigma^2$

Then $$\sigma_1=\left(\sigma^2-\frac{h^2}{12}\right)^{\frac{1}{2}}$$
$$=\sigma\left(1-\frac{h^2}{12\sigma^2}\right)^{\frac{1}{2}}=\sigma\left(1-\frac{h^2}{24\sigma^2}+\ldots\ldots\right)$$

$\therefore$ $$\sigma-\sigma_1=\frac{h^2}{24\sigma}$$

Now $$h<\frac{1}{3}\sigma$$

$\therefore$ $$\sigma-\sigma_1<\frac{\sigma}{216}<\frac{\sigma}{200}$$

$\therefore$ $$\frac{\sigma-\sigma_1}{\sigma}<\frac{1}{200}=\frac{1}{2}\%.$$

2.5. Skewness. *It means lack of symmetry. It is measured by either of the formulae :*

Skewness = Mean — Mode
Skewness = 3 (Mean – Median)
Skewness = Q_3+Q_1-2 Median

Co-efficient of Skewness.

Bowley's co-efficient of Skewness

$$=\frac{Q_3+Q_1-2\,Median}{Q_3-Q_1}.$$

Karl Pearson's co-efficient of Skewness

$$=\frac{Mean-Mode}{S.D.}$$

$$=\frac{3\ (Mean-Median)}{S.D.}$$

2nd formula is used when mode is ill-defined.

Ex. 2-37. *Compute Q.D. and coefficient of skewness from the following data.*

Size	*Freq.*	*Size*	*Freq.*
4— 8	6	24—28	12
8—12	10	28—32	10
12—16	18	32—36	6
16—20	30	36—40	2
20—24	15		

Sol. We have

$$Q_1=14{\cdot}5$$
$$Q_3=24{\cdot}92.$$

Median has $\frac{109}{2}=54{\cdot}5$ items below it.

$\therefore$ It lies in $16-20$

$$\therefore \qquad \text{Median}=16+\frac{4}{30}(54{\cdot}5-34)$$
$$=16+\frac{4}{30}(20{\cdot}5)$$
$$=16+\frac{82}{30}=16+2{\cdot}73$$
$$=18{\cdot}73.$$

$\therefore$ Bowley's coefficient of skewness

$$=\frac{Q_3+Q_1-2M}{Q_3-Q_1}$$
$$=\frac{24{\cdot}92+14{\cdot}5-37{\cdot}46}{24{\cdot}92-14{\cdot}5}$$
$$=\frac{39{\cdot}42-37{\cdot}46}{10{\cdot}42}=\frac{1{\cdot}96}{10{\cdot}42}$$
$$=0{\cdot}19.$$

Ex. 2-38. *From the data given below calculate Karl Pearson's co-efficient of skewness :*

Marks less than 10	*No. of students 5*
,, ,, ,, *20*	,, ,, ,, *12*
,, ,, ,, *30*	,, ,, ,, *32*
,, ,, ,, *40*	,, ,, ,, *44*
,, ,, ,, *50*	,, ,, ,, *50*

Sol.

Determination of Model class **Analysis Table**

Class intervals	*Given Frequency c. Freq.*	*Frequency (f)* (1)	(2)	(3)	(4)	(5)	(6)	*Columns* (1)	(2)	(3)	(4)	(5)	(6)	
0—10	5	5	12								√			1
10—20	12	7		√ 27	√ 32					√	√	√		3
20—30	32	√ 20	√ 32			√ 39		√	√	√	√	√	√	6
30—40	44	12		18			√ 38		√			√	√	3
40—50	50	6											√	1

∴ Modal class is 20—30

∴ $$\text{Mode} = 20 + \frac{20-7}{40-7-12} \times 10 = 26{\cdot}19$$

Calculation of S.D. and A.M.

Class intervals	Freq. (f)	Mid points (x)	$X=\frac{x-25}{10}$	fx	fx^2
0—10	5	5	−2	−10	20
10—20	7	15	−1	−7	7
20—30	20	25	0	0	0
30—40	12	35	1	12	12
40—50	6	45	2	12	24
	50			7	63

$$\text{A.M.}=25+10\left(\frac{7}{50}\right)=26{\cdot}4$$

$$\text{S.D.}=10\sqrt{\frac{63}{50}-\left(\frac{7}{50}\right)^2}=\frac{1}{5}\sqrt{3101}=11{\cdot}14.$$

$\therefore$ Co-efficient of skewness $=\dfrac{\text{Mean}-\text{Mode}}{\text{S.D.}}$

$$=\frac{26{\cdot}4-26{\cdot}19}{11{\cdot}14}=0{\cdot}02$$

Ex. 2-39. (*a*) *Find co-efficient of variation if S.D.* $=3{\cdot}5$, $N=10$, $\Sigma m=145$.

(*b*) *Find the co-efficient of skewness if*

Difference of the two quartiles $=8$

Sum of the two quartiles $=22$

Median $=10{\cdot}5$

(*c*) *For a series the value of M.D. is 15. Find the most likely value of its Q.D.*

Sol. (*a*) Let $\bar{x}$ be the A.M.

Then $\bar{x}=\dfrac{\Sigma m}{N}=\dfrac{145}{10}=14{\cdot}5$

$\therefore$ Co-efficient of variation

$$=\left(\frac{\text{S.D.}}{\text{A.M.}}\right)(100)$$

$$=\left(\frac{3{\cdot}5}{14{\cdot}5}\right)(100)$$

$$=\frac{3500}{145}=24{\cdot}14\%$$

(*b*) Here $\quad Q_3-Q_1=8$

$Q_3+Q_1=22$

Median$=10.5$

$$\therefore \quad \text{Co-efficient of skewness}=\frac{Q_3+Q_1-2\,(\text{Median})}{Q_3-Q_1}$$

$$=\frac{22-21}{8}=\frac{1}{8}=0{\cdot}125$$

(*c*) Let σ be the S.D.

Then $\quad \text{M.D.}=\frac{4}{5}\sigma$

and $\quad \text{Q.D.}=\frac{2}{3}\sigma$

$$\therefore \quad \frac{\text{M.D.}}{\text{Q.D.}}=\frac{12}{10}=\frac{6}{5}$$

$$\therefore \quad \text{Q.D.}=\frac{5}{6}\,(\text{M.D.})$$

$$=\frac{5}{6}\,(15)=\frac{25}{2}=12{\cdot}5$$

$\therefore$ Most likely value of Q.D.$=12{\cdot}5$.

Ex. 2·40. *Compute quartile deviation and the co-efficient of skewness, given the following values :*

Median$=18{\cdot}8''$, $Q_1=14{\cdot}6''$, $Q_3=25{\cdot}2''$.

Sol. $\quad \text{Q.D.}=\frac{Q_3-Q_1}{2}=\frac{25{\cdot}2-14{\cdot}6}{2}=5{\cdot}3$

Co-efficient of skewness

$$=\frac{Q_3+Q_1-2\,(\text{Median})}{Q_3-Q_1}$$

$$=\frac{39{\cdot}8-37{\cdot}6}{10{\cdot}6}=\frac{2{\cdot}2}{10{\cdot}6}=0{\cdot}2.$$

Ex. 2·41. (*a*) *Karl Pearson's coefficient of skewness of a distribution is 0·32. Its s.d. is 6·5 and mean is 29·6. Find the mode and median of the distribution.*

(*b*) *If the mode of the above distribution is 24·8, what will be the standard deviation.*

Sol. (*a*) We have

Karl Pearson's co-efficient of skewness

$$=\frac{\text{Mean}-\text{Mode}}{\text{S.D.}}$$

$$\therefore \qquad 0{\cdot}32=\frac{29{\cdot}6-\text{Mode}}{6{\cdot}5}$$

$$\therefore \qquad \text{Mode}=29{\cdot}6-(0{\cdot}32)(6{\cdot}5)=27{\cdot}52$$

Also

Karl Pearson's co-efficient of skewness

$$=\frac{3\,(\text{Mean}-\text{Median})}{\text{S.D.}}$$

$\therefore$ Median is given by

$$0{\cdot}32=\frac{3(29{\cdot}6-\text{Median})}{6{\cdot}5}$$

$$\therefore \qquad \text{Median}=29{\cdot}6-\frac{1}{3}(0{\cdot}32)(6{\cdot}5)$$

$$=29{\cdot}6-\frac{1}{3}(2{\cdot}08)$$

$$=29{\cdot}6-0{\cdot}69=28{\cdot}91$$

(*b*) Mean $=29{\cdot}6$, Mode $=24{\cdot}8$.

Karl Pearson's co-efficient of skewness

$$=0{\cdot}32$$

$\therefore$ S.D. is given by

$$0{\cdot}32=\frac{29{\cdot}6-24{\cdot}8}{\text{S.D.}}$$

i.e., $$\text{S.D.}=\frac{4{\cdot}8}{0{\cdot}32}=\frac{480}{32}=15.$$

EXERCISE 2.1

1. The following table gives the dist. of farms according to their sizes in a given region. Calculate the quartile deviation. (Size of the form is rounded to the nearest acre).

Farm size (in acres)	*No. of farms*	*Farm size (in acres)*	*No. of farms*
0—40	394	161—200	169
41—80	461	201—240	113
81—120	391	241 and over	148
121—160	334		

Also calculate the quartile co-efficient of dispersion.

[Ans. 50·96 ; 0·51]

2. The following table gives the frequency dist. of 290 workers of a factory according to their average monthly income in 1984—85.

Income group	*No. of workers*	*Income group*	*No. of workers*
Below 50	1	150—170	22
50— 70	16	170—190	15
70— 90	39	190—210	15
90—110	58	210—230	9
110—130	60	230 and above	10
130—150	46		

Locate the quartiles and hence calculate co-efficient of dispersion.

[**Ans.** 95·78 ; 149·24 ; 0·22]

3. Calculate the S.D. of each of the following sequences of binomial co-efficients :

   (*i*) 1, 5, 10, 5, 1.

   (*ii*) 1, 6, 15, 20, 15, 6, 1.

   (*iii*) 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1.

[**Ans.** 3·323 ; 6·958 ; 90·17]

4. Calculate the second moment about the mean and co-efficient of variation of the following dist. :

x :	0	1	2	3	4	5	6	7	8
f :	1	9	26	59	72	52	29	7	1

[**Ans.** 1·98 ; 35·4%]

5. Calculate the S.D. for the following data relating to the weekly wage dist. of 5000 employees of a big factory :

Wage (in Rs.)	50—55	45—50	40—45	35—40	30—35	25—30	20—25
No. of employees	250	300	400	450	800	1100	1700

[**Ans.** 9·024]

6. Calculate the S.D. of the following dist. :

Marks	*No. of students*	*Marks*	*No. of students*
More than 0	100	More than 40	25
,, ,, 10	90	,, ,, 50	15
,, ,, 20	75	,, ,, 60	5
,, ,, 30	50	,, ,, 70	0

[**Ans.** 15·94]

7. Calculate the variance of the following dist. :

Marks	*No. of students*	*Marks*	*No. of students*
30—35	5	50—55	16
35—40	7	55—60	12
40—45	8	60—65	7
45—50	20	65—70	5
			80

[Ans. 82·44]

8. Calculate the variance of the following data :

Marks	*No. of students*	*Marks*	*No. of students*
10—14	2	34—38	10
14—18	4	38—42	8
18—22	4	42—46	4
22—26	8	46—50	6
26—30	12	50—54	2
30—34	16	54—58	4

[Ans. 110·15]

9. Find the standard deviation and co-efficient of variation from the following data :

Wages	*No. of persons*	*Wages*	*No. of persons*
Up to Rs. 10	12	Up to Rs. 50	157
" " Rs. 20	30	" " Rs. 60	202
" " Rs. 30	65	" " Rs. 70	222
" " Rs. 40	107	" " Rs. 80	230

[Ans. 17·26 ; 42·69%]

10. Find out (*a*) median co-efficient of dispersion and (*b*) mean co-efficient of dispersion from the following data :

Age-group (in years)	*No. of men*	*Age-group (in years)*	*No. of men*
15—20	5	35—40	49
20—25	9	40—45	28
25—30	82	45—50	6
30—35	58	50—55	3

[Ans. 0·17 ; 0·16]

11. Calculate co-efficient of variation of the marks of 40 students given below :

Marks	*Students*	*Marks*	*Students*
80—84	1	50—54	6
75—79	1	45—49	6
70—74	1	40—44	6
65—69	4	35—39	3
60—64	4	30—34	0
55 - 59	7	25—29	1
			40

[Ans. 21·79%]

12. A sample of 5 items is taken from the production of a firm. Length and weight of the 5 items are given below :

Length (in inches) :	3	4	6	7	10
Weight (in ounces) :	9	11	14	15	16

By comparing the co-efficients of variation of two characters, conclude which of them is more variable. [Ans. Length]

13. During the first 10 weeks of a session the marks of two students X and Y taking the course were .

X:	58	59	60	54	65	66	52	75	69	52
Y:	56	87	89	78	71	73	84	65	66	46

Which of the two is more consistent ? [Ans. [illegible]]

14. The scores of two golfers for 24 rounds each are :

A: 74 75 78 78 72 77 79 78 81 76 72 72 77
74 70 78 79 80 81 74 80 75 71 73

B: 86 84 80 88 89 85 86 82 82 79 86 80 82
76 86 89 87 83 80 88 86 81 84 87

Which may be considered to be more consistent ? [Ans. B]

15. Goals scored by two teams A and B in a foot ball season were as follows :

No. of Goals scored in a match :	0	1	2	3	4
No. of matches by A :	24	18	16	10	8
No. of matches by B :	34	18	12	10	6

Which team is more consistent and why ? [Ans. B]

16. The following table gives the dist. of house-holds according to size in two cities A and B.

Size of house-hold	*City A*	*City B*
1	24	14
2	10	10
3	12	12
4	15	13
5	13	14
6	10	11
7	6	10
8	10	16
	100	100

Derive a measure to study the variability of the dist.
(**Ans.** *A* : 59·66% ; *B* : 51·47%)

17. The index numbers of prices of cotton and jute shares in a particular year were as follows :

Month	*Index no. of prices of cotton shares*	*Index no. of prices of jute share*
Jan.	188	131
Feb.	178	130
March	173	130
April	164	129
May	172	129
June	183	120
July	184	127
Aug.	185	127
Sep.	211	130
Oct.	217	137
Nov.	232	140
Dec.	240	142

Which of the two shares do you consider to be more variable in price ? (**Ans.** Cotton shares)

18. Calculate Karl Pearson's co-efficient of skewness from the following data :

Marks	*No. of students*	*Marks*	*No. of students*
Above 0	1500	Above 40	780
„ 10	1400	„ 50	700
„ 20	1000	„ 60	300
„ 30	780	„ 70	140
		„ 80	0

(**Ans.** 0·995)

19. The table below gives ages of children in two nursery schools. Compare their variability.

Age (in months)	*No. of children school A*	*No. of children school B*
15—16	—	1
17—18	1	2
19—20	2	2
21—22	5	5
23—24	7	10
25—26	9	7
27—28	8	6
29—30	5	3
31—32	3	3
33—34	1	1
35—36	1	—
37—38	—	2
	42	42

(**Ans.** Ages of children in school *B* are more variable.)

20. Find the co-efficient of variation and Pearson's co-efficient of skewness from the following data :

Years	*Price Index No. of wheat*	*Years*	*Price Index No. of wheat*
1910	83	1915	126
1911	87	1916	130
1912	93	1917	118
1913	100	1918	106
1914	124	1919	104

(**Ans.** 14·85 ; 0·396)

21. Find Pearson's measure of skewness for the data given below :

Weight (in lbs)	*No. of persons*	*Weight (in lbs)*	*No. of persons*
70— 79·99	12	110—119·99	50
80— 89·99	18	120—129·99	45
90— 99·99	35	130—139·99	20
100—109·99	42	140—149·99	8

(**Ans.** —5·719)

22. Find the co-efficient of dispersion and Pearsons's co-efficient of skewness for the following data :

Wage (in Rs.)	No. of persons	Wage (in Rs.)	No. of persons
70— 80	12	110—120	50
80— 90	18	120—130	45
90—100	35	130—14	20
100—110	42	140—150	8
			230

(Ans. 0·16 ; −0·33)

23. For data in Ex. 2 locate median, quartiles and hence co-efficient of skewness (Ans. 0·08)

24. Calculate the Pearson's co-efficient of skewness for the following dist. of weights of boys aged 3 years :

Weight (in lbs.)	Frequency	Weight (in lbs.)	Frequency
20·5—23·5	17	29·5—32·5	194
23·5—26·5	193	32·5—35·5	27
26·5—29·5	399	35·5—38·5	10

(Ans. 0·07)

25. Prove the following relation for Pearson's β-coefficients for skewness and kurtosis

(i) $\beta_2 > \beta_1$

(ii) $\beta_2 > \beta_1 + 1$

26. If in a series of measurements, there are m values of magnitude x_1, m_2 of magnitude x_2 etc. and if $\bar{x}$ is the mean of all the measurements, prove that standard deviation is

$$\sqrt{\frac{\Sigma m_r (k - x_r)^2}{\Sigma m_r} - \delta^2}$$

where $\bar{x} = k + \delta$ and k is any constant.

3

Theory and Association of Attributes

3.1. Introduction

Attribute means quality or property. The capital letters A, B, C,............are used to denote the several attributes. An object or individual possessing the attribute A is termed simply **A** *and the class of individual possessing A is termed* **the class A**. *The Greek letters* α, β, γ,.........*are used to denote the absence of attributes A, B, C,......respectively e.g. If a represents the attribute* **honesty**, α *represents* **dishonesty**. *The combination of attributes is represented by grouping the letters representing the attributes. e.g., if A represents deafness and B blindness, AB represents combination deafness and blindness.*

Any letter or combination of letters like A, AB by means of which the characters of the members of a class are specified are called **class-symbol.**

Class-frequencies. *The number of observations in any class is called the class-frequency. It is denoted by enclosing the corresponding class-symbol in brackets. e.g. (A) denotes the number of objects or individuals possessing the attribute A.*

Order of Classes and Class-frequencies. *A class specified by r attributes is known as a class of order r and its frequency as a frequency of rth order. e.g. A, AB are classes of first and second order and (A), (AB) are frequencies of order one and two respectively.*

The total frequency is denoted by N.

Ultimate Class-frequencies. *The frequencies of highest order are called ultimate class-frequencies.*

Positive and Negative Attributes. *The attributes denoted by capitals are termed as positive attributes and those denoted by Greek letters as negative attributes. If a class-symbol includes only capital letters, the corresponding class is termed as a positive class and if only Greek letters, a negative class e.g. the class AB is positive class and the class* $\alpha\beta$ *is negative class.*

Symbol. *A symbol 'A.N.' is used for the dichotomising N according to A and is written.*

$$A.N=(A)$$

which is the symbolic way of saying that if N is dichotomised according to A, class-frequency (A) is obtained.

Condition of Consistence. *The necessary and sufficient condition for the consistency of a set of independent class-frequencies is that no ultimate classs-frequency is negative.*

Ex. 3·1. *Show that if there are n attributes, the number of distinct classes is 3^n.*

Sol. There is only one class of order zero. If $A_1, A_2, \ldots\ldots A_n$ be n attributes the possible classes of order one are

$A_1, A_2, \ldots\ldots A_n, \alpha_1, \alpha_2 \ldots\ldots \alpha_n$ (α's denoting the absence of A's) which are $2n={}^nc_1.2$ in number.

To find classes of order 2, consider two attributes A_1 and A_2. These two attributes give the classes

$$A_1A_2,\ A_1\alpha_2,\ \alpha_1A_2 \text{ and } \alpha_1\alpha_2$$

of order two. These are $2^2=4$ in number. Since out of n two attributes can be chosen in ${}^nc_2$ ways, total number of classes of order two

$$={}^nc_2.2^2$$

Similarly the number of classes of order 3

$$={}^nc_3.2^3$$

and in general the number of classes of order r

$$={}^nc_r.2^r.$$

$\therefore$ Total number of classes

$$=1+{}^nc_1.2+{}^nc_2.2^2+\ldots\ldots+{}^nc_r.2^r+\ldots\ldots{}^nc_n.2^n$$
$$=(1+2)^n=3^n$$

Ex. 3·2. *A number of school-children were examined for the presence or absence of certain defects of which three chief descriptions were noted : A, development defects ; B, nerve signs ; C, low nutrition. Given the following ultimate frequencies, find the frequences of the classes defined by the presence of the defects i.e., those involving the Roman letters. A, B, C but not the Greek letters α, β, γ including the whole number of observations N.*

(ABC)	57	(αBC)	78
$(AB\gamma)$	281	$(\alpha B\gamma)$	670
$(A\beta C)$	86	$(\alpha\beta C)$	65
$(A\beta\gamma)$	453	$(\alpha\beta\gamma)$	8310

Sol. N=Total no. of observation

$=$Sum of frequencies given above$=10{,}000$

Now $(A)=(AB)+(A\beta)$

$(AB)=(ABC)+(AB\gamma)$

$(A\beta)=(A\beta C)+(A\beta\gamma)$

$\therefore \quad (A)=(ABC)+(AB\gamma)+(A\beta C)+(A\beta\gamma)$

$=57+281+86+453=877$

Similarly $(B)=(ABC)+(AB\gamma)+(\alpha BC)+(\alpha B\gamma)$

$=57+281+78+670=1{,}086$

$(C)=(ABC)+(A\beta C)+(\alpha BC)+(\alpha\beta C)$

$=57+86+78+65=286$

$(AB)=(ABC)+(AB\gamma)=57+281=338$

$(BC)=(ABC)+(\alpha BC)=57+78=135$

$(AC)=(ABC)+(A\beta C)=57+86=143$

Also $(ABC)=57.$

Ex. 3-3. *From the frequencies given below find all the class-frequencies.*

$N=10{,}000, \quad (A)=877, \quad (B)=1086, \quad (C)=286$

$(AB)=338, \quad (AC)=143, \quad (BC)=135, \quad (ABC)=57$

Sol. Now $(AB)=(ABC)+(AB\gamma)$

$\therefore \quad (AB\gamma)=(AB)-(ABC)=338-57=281$

Similarly $(A\beta C)=-(ABC)+(AC)=143-57=86$

$(\alpha BC)=(BC)-(ABC)=135-57=78$

Now $(A\beta\gamma)=(A\beta)-(A\beta C)$

$=(A)-(AB)-(A\beta C)$

$=877-338-86=453$

$(\alpha\beta C)=(\beta C)-(A\beta C)$

$=(C)-(BC)-(A\beta C)=286-135-86=65$

$(\alpha\beta\gamma)=(\alpha\beta)-(\alpha\beta C)$

$=(\alpha)-(\alpha B)-(\alpha\beta C)$

$=\{N-(A)\}-\{(B)-(AB)\}-(\alpha\beta C)$

$=10{,}000-877-1086+338-65=8310$

$(\alpha B\gamma)=(\alpha B)-(\alpha BC)$

$=(B)-(AB)-(\alpha BC)=1086-338-78=670.$

Thus all the ultimate frequencies are known. From these remaining frequencies can be obtained as in last example.

Ex. 3·4. *Show that $A+\alpha=1$ where A, α and 1 are operators defined by $A.N=(A)$, $\alpha.N=(\alpha)$ and $1.N=N$ Deduce that*

$$(\alpha\beta\gamma)=N-(A)-(B)-(C)+(AB)+(AC)+(BC)-(ABC)$$

Sol. By def. $A.N=(A)$ and $\alpha.N=(\alpha)$

$\therefore \quad A.N+\alpha.N=(A)+(\alpha)=N$

or $\quad (A+\alpha).N=1.N$

$\therefore \quad A+\alpha=1$

$\therefore \quad \alpha=1-A.$

Similarly $\quad \beta=1-B$ and $\gamma=1-C$

$$\therefore (\alpha\beta\gamma)=\alpha\beta\gamma.N=(1-A)(1-B)(1-C).N$$
$$=(1-A-B-C+AB+AC+BC-ABC).N$$
$$=1.N-A.N-B.N-C.N+AB.N+AC.N+BC.N-ABC.N$$
$$=N-(A)-(B)-(C)+(AB)+(AC)+(BC)-(ABC)$$

Ex. 3·5. *Given that*

$$(A)=(\alpha)=(B)=(\beta)=\frac{N}{2}$$

Show that $(AB)=(\alpha\beta)$, $(A\beta)=(\alpha B)$

Sol.

$$(AB)=AB.N=(1-\alpha)(1-\beta).N$$
$$=\{1-\alpha-\beta+\alpha\beta\}.N$$
$$=N-(\alpha)-(\beta)+(\alpha\beta)$$
$$=N-\frac{N}{2}-\frac{N}{2}+(\alpha\beta)=(\alpha\beta)$$

$$(A\beta)=A\beta.N=\{1-\alpha\}\{1-B\}.N$$
$$=\{1-\alpha-B+\alpha B\}.N$$
$$=N-(\alpha)-(B)+(\alpha B)=(\alpha B).$$

Ex. 3·6. *Given that*

$$(A)=(\alpha)=(B)=(\beta)=(C)=(\gamma)=\frac{N}{2}$$

and also that $\quad (ABC)=(\alpha\beta\gamma)$

show that $\quad 2(ABC)=(AB)+(AC)+(BC)-\dfrac{N}{2}$

Sol.

$$(ABC)=(\alpha\beta\gamma)=N-(A)-(B)-(C)+(AB)+(AC)+(BC)-(ABC)$$

$$\therefore \quad 2(ABC)=(AB)+(AC)+(BC)-\frac{N}{2}.$$

Ex. 3·7. *Measurements are made on a thousand husbands and a thousand wives. If the measurements of the husbands exceed the measurements of the wives in 800 cases for one measurements, in 700 cases for another and in 660 cases for both measurements, in how many cases will both measurements on the wife exceed the measurements on the husband?*

Sol. Let A and B denote the husbands exceeding wives in first and second measurements respectively. Then

N$=1000$, $(A)=800$, $(B)=700$ and $(AB)=660$

$$\therefore \quad (\alpha\beta)=\alpha\beta.N=\{1-A\}\{1-B\}.N$$
$$=\{1-A-B+AB\}.N=N-(A)-(B)+(AB)$$
$$=1000-800-700+660=160.$$

Ex. 3·8. *100 children took three examinations. 40 passed the first, 39 passed the second and 48 passed the third. 10 passed all three, 21 failed all three, 9 passed the first two and failed the third, 19 failed the first two and passed the third. Find how many children passed at least two examinations.*

Show that for the question asked certain of the given frequencies are not necessary. Which are they?

Sol. Let A, B, C denote passing first, second and third examinations respectively. Then

$$N=100,\ (A)=40,\ (B)=39,\ (C)=48,\ (ABC)=10$$
$$(\alpha\beta\gamma)=21,\ (AB\gamma)=9 \text{ and } (\alpha\beta C)=19$$

Now $(\alpha BC)=\alpha BC.N=\{1-A\}BCN=\{BC-ABC\}.N$
$$=(BC)-(ABC)$$

and $\quad (A\beta C)=A\beta C.N=\{1-\alpha\}\beta C\ N=\{\beta C-\alpha\beta C\}.N$
$$=(\beta C)-(\alpha\beta C)$$

$\therefore$ No. of children who passed at least two examinations
$$=(\alpha BC)+(A\beta C)+(AB\gamma)+(ABC)$$
$$=(BC)-(ABC)+(\beta C)-(\alpha\beta C)+(AB\gamma)+(ABC)$$
$$=(C)-(\alpha\beta C)+(AB\gamma)=48-19+9=38.$$

Evidently three frequencies have been used and hence others are not necessary.

Ex. 3·9. *Show that if A occurs in a larger proportion of the cases where B is than where B is not, then B will occur in a larger proportion of the cases where A is than where A is not.*

Sol. It is given that

$$\frac{(AB)}{(B)}>\frac{(A\beta)}{(\beta)}$$

and it is to be shown that

$$\frac{(AB)}{(A)} > \frac{(\alpha B)}{(\alpha)}$$

From given $$\frac{(\beta)}{(B)} > \frac{(A\beta)}{(AB)}$$

$\therefore$ $$1+\frac{(\beta)}{(B)} > 1+\frac{(A\beta)}{(AB)}$$

or $$\frac{(B)+(\beta)}{(B)} > \frac{(AB)+(A\beta)}{(AB)}$$

or $$\frac{N}{(B)} > \frac{(A)}{(AB)}$$

or $$\frac{N}{(A)} > \frac{(B)}{(AB)}$$

or $$\frac{(A)+(\alpha)}{(A)} > \frac{(AB)+(\alpha B)}{(AB)}$$

or $$1+\frac{(\alpha)}{(A)} > 1+\frac{(\alpha B)}{(AB)}$$

or $$\frac{(\alpha)}{(A)} > \frac{(\alpha B)}{(AB)}$$

or $$\frac{(AB)}{(A)} > \frac{(\alpha B)}{(\alpha)}.$$

Ex. 3·10. *At a competitive examination at which 600 graduates appeared, boys outnumbered girls by 96. Those qualifying for interview exceeded in number those failing to qualify by 310. The number of Science graduate boys interviewed was 300 while among the Arts graduate girls there were 25 who failed to qualify for interview. All together there were only 135 Arts graduates and 33 among them failed to qualify. Boys who failed to qualify numbered 18. Find out.*

(*i*) *the number of boys who qualified for interview.*

(*ii*) *the total number of Science graduate boys appearing.*

(*iii*) *the number of Science graduate girls who qualified.*

Sol. Let A, B and C denote the attributes of being a boy qualified for interview and science candidate. Then

$N=600$, $(A)-(\alpha)=96$, $(B)-(\beta)=310$, $(ABC)=300$, $(\alpha\beta\gamma)=25$, $(\gamma)=135$, $(\beta\gamma)=33$ and $(A\beta)=18$.

Since $N=(A)+(\alpha)=600=(B)+(\beta)=(C)+(\gamma)$

$$(A)=348,\ (\alpha)=252,\ (B)=455,\ (\beta)=145 \text{ and } (C)=465;$$

$\therefore$ (*i*) $(AB)=(A)-(A\beta)=348-18=330$

(*ii*) $(AC)=(ABC)+(A\beta C)=(ABC)+(A\beta)-(A\beta\gamma)$
$=(ABC)+(A\beta)-(\beta\gamma)+(\alpha\beta\gamma)$
$=300+18-33+25=310$

(*iii*) $(\alpha BC)=(BC)-(ABC)=\{1-\beta\}\{1-\gamma\}.N-(ABC)$
$=N-(\gamma)-(\beta)+(\beta\gamma)-(ABC)$
$-600-135-145+33-300=53.$

Ex. 3·11. *In a free vote in the house of commons, 600 members voted. 300 Government members representing English constituencies (including welsh) voted in favour of the motion. 25 Opposition members representing Scottish constituencies voted against the motion. The Government majority among those who voted was 96. 135 of the members voting represented Scottish constituencies. 18 Government members voted against the motion. 102 Scottish members voted in favour of the motion. The motion was carried by 310 votes. Analyse the voting according to the nationality of the constituencies and party.*

Sol. Let A, B, C denote the attributes of being Government members, voting for the motion and being members of English constituencies respectively.

Then $N=600$, $(ABC)=300$, $(\alpha\beta\gamma)=25$, $(A)-(\alpha)=96$, $(\gamma)=135$ $(A\beta)=18$, $(B\gamma)=102$ and $(B)-(\beta)=310$.

It is required to find all ultimate frequencies.

Now $N=(A)+(\alpha)=600=(B)+(\beta)$

$\therefore$ $(A)=348$, $(\alpha)=252$, $(B)=455$ and $(\beta)=145$.

$\therefore$ $(C)=N-(\gamma)=465$.

Now $(AB)=AB\,N=A\{1-\beta\}.N=(A)-(A\beta)$
$=348-18=330$

$(BC)=(B)-(B\gamma)=455-102=353$

$(\alpha\beta\gamma)=N-(A)-(B)-(C)+(AB)+(AC)+(BC)-(ABC)$

$\therefore$ $(AC)=(\alpha\beta\gamma)+(ABC)+(A)+(B)+(C)-(BC)-(AB)-N$
$=25+300+348+455+465-353-330-600$
$=310.$

Now $(AB\gamma)=(AB)-(ABC)=330-300=30$

$(A\beta C)=(AC)-(ABC)=310-300=10$

$(\alpha BC)=(BC)-(ABC)=353-300=53$

$(A\beta\gamma)=(A\beta)-(A\beta C)=18-10=8$

$(\alpha\beta C)=\{1-A\}\{1-B\}C.N=\{C-AC-BC+ABC\}.N$
$=(C)-(AC)-(BC)+(ABC)$
$=465-310-353+300=102$

$$(\alpha B\gamma) = (B) - (AB) - (BC) + (ABC)$$
$$= 455 - 330 - 353 + 300 = 72.$$

Ex. 3·12. *Prove that in case of two attributes A and B, the conditions of consistency are*

(i) $(AB) \geq 0$ (ii) $(AB) \leq (A)$ (iii) $(AB) \leq (B)$ and (iv) $(AB) \geq (A)+(B)-N$.

Sol. The necessary and sufficient condition for the consistency is that no ultimate frequency is negative *i.e.* $(AB) \geq 0$, $(A\beta) \geq 0$, $(\alpha B) \geq 0$ and $(\alpha\beta) \geq 0$.

Now $(A\beta) = \{A\beta\}.N = A\{1-B\}.N = A.N - \{AB\}.N$
$$= (A) - (AB)$$
$\therefore$ $(A\beta) \geq 0$ implies $(A) \geq (AB)$

Similarly $(\alpha B) \geq 0$ implies $(B) \geq (AB)$

Now $(\alpha\beta) = \{\alpha\beta\}.N = \{(1-A)(1-B)\}.N$
$$= \{1 - A - B + AB\}.N = N - (A) - (B) + (AB)$$
$\therefore$ $(\alpha\beta) \geq 0$ implies $(AB) \geq (A)+(B)-N$.

Ex. 3·13. *Prove that in case of three attributes A, B and C, the conditions of consistency are*

(i) $(ABC) \geq 0$ (ii) $(ABC) \geq (AB)+(AC)-(A)$

(iii) $(ABC) \geq (AB)+(BC)-(B)$ (iv) $(ABC) \geq (AC)+(BC)-(C)$

(v) $(ABC) \leq (AB)$ (vi) $(ABC) \leq (AC)$ (vii) $(ABC) \leq (BC)$

and (viii) $(ABC) \leq (AB)+(AC)+(BC)-(A)-(B)-(C)+N$

Hence deduce

$$(AB)+(AC)+(BC) \geq (A)+(B)+(C)-N$$
$$(AB)+(AC)-(BC) \leq (A)$$
$$(AB)-(AC)+(BC) \leq (B)$$
$$-(AB)+(AC)+(BC) \leq (C)$$

Sol. For consistency it is necessary and sufficient that all ultimate frequencies are non-negative *i e.*

(i) $(ABC) \geq 0$

(ii) $(A\beta\gamma) \geq 0$ *i.e.* $\{A\beta\gamma\}.N \geq 0$

i.e. $\{A(1-B)(1-C)\}.N \geq 0$

i.e., $\{A - AB - AC + ABC\}.N \geq 0$

i.e., $(A)-(AB)-(AC)+(ABC) \geq 0$.

i.e., $(ABC) \geq (AB)+(AC)-(A)$

Similarly $(\alpha B\gamma) \geq 0$ and $(\alpha\beta C) \geq 0$ implies (*iii*) and (*iv*).

(v) $(AB\gamma) \geqslant 0$

i.e., $\{AB\gamma\}.N \geqslant 0$

i.e., $AB(1-C).N \geqslant 0$

i.e., $\{AB-ABC\}.N \geqslant 0$

i.e., $(AB) \geqslant (ABC)$.

Similarly (vi) and (vii) follow from $(A\beta C) \geqslant 0$ and $(\alpha BC) \geqslant 0$.

(viii) $(\alpha\beta\gamma) \geqslant 0$

i.e., $\{1-A\}\{1-B\}\{1-C\}.N \geqslant 0$

i.e., $\{1-A-B-C+AB+AC+BC-ABC\}.N \geqslant 0$

i.e., $N-(A)-(B)-(C)+(AB)+(AC)+(BC)-(ABC) \geqslant 0$

i.e. $(ABC) \leqslant (AB)+(AC)+(BC)-(A)-(B)-(C)+N$.

From (i) and (viii)

$$(AB)+(AC)+(BC)-(A)-(B)-(C)+N \geqslant 0$$

i.e., $(AB)+(AC)+(BC) \geqslant (A)+(B)+(C)-N$

From (ii) and (vii)

$$(AB)+(AC)-(A) \leqslant (BC)$$

i.e., $(AB)+(AC)-(BC) \leqslant (A)$.

Similarly from (iii) and (vi), (iv) and (v)

$$(AB)+(BC)-(AC) \leqslant (B)$$

and $$(AC)+(BC)-(AB) \leqslant (C).$$

Ex. 3·14. *If a report gives the following frequencies as actually observed, show that there must be a misprint or mistake of some sort :*

$N=1000$, $(A)=510$, $(B)=490$, $(C)=427$ $(AB)=189$,

$(AC)=140$, $(BC)=85$

Sol. Now $(AB)+(AC)+(BC)=189+140+85=414$

and $(A)+(B)+(C)-N=510+490+427-1000=427$

$\therefore \quad (AB)+(AC)+(BC)<(A)+(B)+(C)-N$

$\therefore$ Data is not consistent and hence there must be a misprint or mistake of some sort.

Ex. 3·15. *If in an urban district 817 per thousand of the women between 20 and 25 years of age were returned as "occupied" at a census and 263 per thousand as married or widowed, what is the lowest proportion per thousand of the married or widowed that must have been occupied ?*

Sol. Let A and B denote the attributes of being occupied and being married or widowed respectively. Then

$N=1000$, $(A)=817$ and $(B)=2{\cdot}3$

$\therefore (AB) \geqslant (A)+(B)-N=817+263-1000=80$

$\therefore$ Lowest proportion per thousand of the married or widowed that must have been occupied

$$=\frac{80}{263}\times 1000=304$$

Ex. 3-16. *A market investigator returns the following data. Of 1000 people consulted, 811 liked chocolates, 752 liked toffee and 418 liked boiled sweets ; 570 liked chocolates and toffee, 356 liked chocolates and boiled sweets and 348 liked toffee and boiled sweets : 297 liked all three. Show that this information as it stands must be incorrect.*

Sol. Let A, B and C denote the attributes of having liking for chocolates, toffee and boiled sweets respectively. Then

$N=1000$, $(A)=811$, $(B)=752$, $(C)=418$. $(AB)=570$,
$(AC)=356$, $(BC)=348$ and $(ABC)=297$.

$$\begin{aligned}\text{Now } (\alpha\beta\gamma)&=N-(A)-B)-(C)+(AB)+(AC)+(BC)-(ABC)\\&=1000-811-752-418+570+356+348-297\\&=-4<0\end{aligned}$$

$\therefore$ Information is incorrect.

Ex. 3-17. *In a war between White and Red forces there are more Red soldiers than White ; there are more armed Whites than unarmed Reds : there are fewer armed Reds with ammunition than unarmed Whites withount ammunition. Show that there are more armed Reds without ammunition than unarmed Whites with ammunition.*

Sol. Let A, B and C denote the attributes of being white, armed and possessed with ammunition, Then

$$(A)<(\alpha),\ (AB)>(\alpha\beta),\ (\alpha BC)<(A\beta\gamma)$$

and it is to be shown that

$$(\alpha B\gamma)>(A\beta C)$$

$$\begin{aligned}\text{Now } (\alpha B\gamma)&=(\alpha B)-(\alpha BC)\\&>(\alpha B)-(A\beta\gamma)=(\alpha)-(\alpha\beta)-(A\beta\gamma)\\&>(A)-(AB)-(A\beta\gamma)=(A\beta)-(A\beta\gamma)\\&=(A\beta C).\end{aligned}$$

Ex. 3-18. *If, in a series of houses actually invaded by small-pox 70% of the inhabitants are attacked and 85% have been vaccinated, what is the lowest percentage of the vaccinated that must have been attacked ?*

Sol. Let A and B denote the attributes of being attacked and vaccinated. Then

$N=100$, $(A)=70$ and $(B)=85$

Now $(AB) > (A)+(B)-N = 70+85-100 = 55$

$\therefore$ Lowest percentage of the vaccinated that have been attacked

$$= \frac{55}{85} \times 100 = 65\% \text{ (approx.)}$$

Ex. 3-19. *If all A's are B's and all B's are C's, show that all A's are C's.*

Sol. It is given that $(AB)=(A)$, $(BC)=(B)$
and it is to be shown that

$$(AC)=(A)$$

From Ex. 3-13, $(AB)-(AC)+(BC) \leqslant (B)$

$\therefore \quad (A)-(AC)+(B) \leqslant (B)$

$\therefore \quad (A) \leqslant (AC)$

But $\quad (AC) \leqslant (A)$

$\therefore \quad (AC)=(A)$

Ex. 3-20. *If all A's are B's and no B's are C's, show that no A's are C's.*

Sol. It is given that $(AB)=(A)$ and $(BC)=0$
and it is to be proved that $(AC)=0$

Now from Ex. 3-13, $(AB)+(AC)-(BC) \leqslant (A)$

$\therefore \quad (A)+(AC) \leqslant (A)$

$\therefore \quad (AC) \leqslant 0$

$\therefore \quad (AC)=0 \qquad (\because (AC) \geqslant 0)$

Ex. 3-21. *Given that* $(A)=(B)=(C)=\frac{N}{2}$ *and 80% of the A's are B's, 75% of A's are C's, find the limits to the percentage of B's that are C's.*

Sol. It is given that

$$(AB)=0 \cdot 8\,(A)=0 \cdot 4\,N$$
$$(AC)=0 \cdot 75\,(A)=0 \cdot 375\,N$$

and the limits of (BC) are to be obtained. From Ex. 3-13,

$$(AB)+(AC)-(BC) \leqslant (A)$$

$\therefore \quad (BC) \geqslant (AB)+(AC)-(A)=0 \cdot 4N+0 \cdot 375N-0 \cdot 5N$

$\therefore \quad (BC) \geqslant 0 \cdot 275N=0 \cdot 55(B)$

and $\quad (AB)-(AC)+(BC) \leqslant (B)$

$\therefore \quad (BC) \leqslant (B)-(AB)+(AC)=0 \cdot 5N-0 \cdot 4N+0 \cdot 375N$

$\therefore \quad (BC) \leqslant 0 \cdot 475N=0 \cdot 95(B)$

$$\therefore \quad 0.55 < \frac{(BC)}{(B)} < 0.95$$

$\therefore$ Reqd. limits are 55% and 95%.

Ex. 3·22 *Among the adult population of a certain town 50% of the population are male, 60% are wage-earners and 50% are 45 years of age or over. 10% of the males are not wage earners and 40% of the males are under 45. Can you infer anything about what percentage of the population of 45 or over are wage-earners?*

Sol. Let A, B, C denote the attributes of being male, wage-earner and 45 years old or more. Then

$$N=100, (A)=50, (B)=60, (C)=50, (A\beta)=\frac{10}{100}\times 50=5$$

$$\text{and } (A\gamma)=\frac{40}{100}\times 50=20$$

The limits of (BC) are to be obtained

Now $\quad (AB)=(A)-(A\beta)=50-5=45$

and $\quad (AC)=(A)-(A\gamma)=50-20=30$

From Ex. 3·13, $(AB)+(AC)-(BC)\leqslant(A)$

$\therefore \quad (BC)\geqslant(AB)+(AC)-(A)=45+30-50$

$\therefore \quad (BC)\geqslant 25$

and $\quad (AB)-(AC)+(BC)\leqslant(B)$

$\therefore \quad (BC)\leqslant(B)+(AC)-(AB)=60+30-45$

$\therefore \quad (BC)\leqslant 45$

$\therefore \quad 25\leqslant(BC)\leqslant 45$

$\therefore$ Percentage of the population of 45 years old or more who are wage-earners lies between $\frac{25}{50}\times 100=50\%$ and $\frac{45}{50}\times 100=90\%$.

Ex. 3·23. (*a*) *The following are the proportions of boys observed for certain classes of defects amongst a number of school-children.*

A=development defects, B=nerve signs, C=mental dullness.

$N=10{,}000$, $(A)=877$, $(B)=1{,}086$, $(C)=789$

$(AB)=338$, $(BC)=455$.

Show that some dull boys do not exhibit development defects and state how many at least must be so.

(*b*) *The following are the corresponding figures for girls:*

$N=10{,}000$, $(A)=682$, $(B)=850$, $(C)=689$

$(AB)=248$, $(BC)=363$.

Show that some defectively developed girls are not dull and state how many at least must be so.

Sol. (*a*) It is required to find the lower limit of (αC)

Now $(\alpha C)=(C)-(AC)$

Now $(AC)+(AB)-(BC) \leqslant (A)$

and $-(AB)+(AC)+(BC) \leqslant (C)$

$\therefore \quad (AC) \leqslant (A)+(BC)-(AB)=877+455-338.$

$\therefore \quad (AC) \leqslant 994$

and $\quad (AC) \leqslant (C)+(AB)-(BC)=789+338-455$

$\therefore \quad (AC) \leqslant 672$

$\therefore \quad (\alpha C) \geqslant (C)-672=789-672=117$

(*b*) Left as an exercise.

Ex. 3·24. *Given that 50% of the inmates of an institution are men, 60% are aged (over 60), 80% non-able-bodied, 35% aged men, 45% non-able-bodied men and 42% non-able-bodied and aged, find the greatest and least possible proportions of non-able-bodied aged men.*

Sol. Let A, B and C denote the attributes of being man, aged and non-able-bodied.

Then $N=100$, $(A)=50$, $(B)=60$, $(C)=80$.

$(AB)=35$, $(AC)=45$ and $(BC)=42$

and the limits of (ABC) are to be obtained.

From Ex. 3·13, $(ABC) \geqslant (AB)+(AC)-(A)=30$

$(ABC) \geqslant (AB)+(BC)-(B)=17$

and $\quad (ABC) \geqslant (AC)+(BC)-(C)=7$

$\therefore \quad (ABC) \geqslant 30$ satisfies all the three inequalities.

Also $(ABC) \leqslant (AB)=35$, $(ABC) \leqslant (AC)=45$, $(ABC) \leqslant (BC)=42$

and $\quad (ABC) \leqslant (AB)+(AC)+(BC)-(A)-(B)-(C)+N=32$

$\therefore \quad (ABC) \leqslant 32$

$\therefore \quad 30 \leqslant (ABC) \leqslant 32.$

Ex. 3·25. *50% of the imports of barley into a country come from the Dominions : 80% of the total imports go to brewing ; 75% of the imports are grown in Northern Hemisphere ; 80% of Northern-grown barley goes to brewing ; 100% of foreign Southern-grown barley goes to stock-feeding. Show that the foreign Northern grown barley which goes to brewing cannot be less than 30% nor more than 50% of the total imports. (It is assumed that brewing and stock-feeding are the only two uses to which imported barley is put).*

Sol. Let A, B and C denote the attributes of the barley coming from dominions, being used in brewing and growing in Northern Hemisphere respectively. Then

$N=100$, $(A)=50$, $(B)=80$, $(C)=75$

$$(BC)=\frac{80}{100}\times 75=60 \text{ and } (\alpha\gamma)=(\alpha\beta\gamma)$$

and it is to be shown that

$$30\leqslant(\alpha BC)\leqslant 50$$

Now $\quad(\alpha\beta\gamma)=(\alpha\gamma)=(\alpha B\gamma)+(\alpha\beta\gamma)$

$\therefore \quad (\alpha B\gamma)=0$

or $\quad (\alpha B)-(\alpha BC)=0$ *i e.*, $(\alpha B)=(\alpha BC)$

From Ex. 3-12, $(\alpha B)\geqslant(\alpha)+(B)-N=30$

and $\quad (\alpha B)\leqslant(\alpha)=N-(A)=50$

$\therefore \quad 30\leqslant(\alpha BC)\leqslant 50.$

Ex. 3-26. *A penny is tossed three times and the results, heads and tails, noted. The process is continued until there are 100 sets of threes. In 69 cases heads fell first, in 49 cases heads fell second and in 53 cases heads fell third. In 33 cases heads fell both first and second and in 21 cases heads fell both second and third. Show that there must have been at least 5 occasions on which heads fell three times. and that there could not have been more than 15 occasions on which tails fell three times, though there need not have been any.*

Sol. Let A, B and C denote the attributes of getting head in first, second and third trial respectively. Then

$N=100, (A)=69, (B)=49, (C)=53, (AB)=33, (BC)=21$

From Ex. 3-13, $(ABC)\geqslant(AB)+(BC)-(B)=5$

$$(\alpha\beta\gamma)\leqslant(\alpha\beta)=\alpha\beta.N=\{1-A\}\{1-B\}.N$$
$$=N-(A)-(B)+(AB)=15$$

and $\quad (\alpha\beta\gamma)\leqslant(\beta\gamma)=N-(B)-(C)+(BC)=19$

$(\alpha\beta\gamma)\leqslant 15.$

Ex. 3-27. *Given that* $(A)=(B)=(C)=\frac{N}{2}$ *and that*

$$\frac{(AB)}{N}=\frac{(AC)}{N}=p,$$

find what must be the greatest and least values of p *in order that we may infer that* $\frac{(BC)}{N}$ *exceeds any given value, say* q.

Sol. From Ex. 3-13, $(AB)+(BC)+(AC)\geqslant(A)+(B)+(C)-N$

$$\therefore \quad 2Np+(BC)\geqslant\frac{N}{2}$$

$$\therefore \quad \frac{(BC)}{N}\geqslant\frac{1}{2}-2p \qquad ...(1)$$

and $\quad (AB)+(AC)-(BC)\leqslant(A)$

$$\therefore \qquad 2pN-(BC) \nless \frac{N}{2}$$

$$\therefore \qquad \frac{(BC)}{N} > 2p-\frac{1}{2} \qquad \ldots(2)$$

Since $\frac{(BC)}{N}$ is to exceed q, from (1) and (2)

$$\frac{1}{2}-2p > q \text{ and } 2p-\frac{1}{2} > q$$

i.e., $p < \frac{1}{4}(1-2q)$ and $p > \frac{1}{4}(2q+1)$ ...(3)

From Ex. 3-12, $(AB) \leqslant (A)$

$$\therefore \qquad p < \frac{1}{2}$$

Since $(AB) \nless 0,\ p \geqslant 0$

$\therefore$ p must lie between 0 and $\frac{1}{4}(1-2q)$ or between $\frac{1}{4}(1+2q)$ and $\frac{1}{2}$.

Ex. 3-28. *Show that if* $\frac{(A)}{N}=x=\frac{(B)}{2N}=\frac{(C)}{3N}$

and
$$\frac{(AB)}{N}=\frac{(AC)}{N}=\frac{(BC)}{N}=y$$

the value of neither x nor y can exceed $\frac{1}{4}$.

Sol. From Ex. 3-12, $(BC) > (B)+(C)-N$ *i.e.*, $y > 5x-1$

and $(AB) \leqslant (A)$ *i.e.*, $y \leqslant x$,

$$\therefore \qquad x > 5x-1 \text{ i.e., } x \leqslant \frac{1}{4}$$

$$\therefore \qquad y \leqslant x \leqslant \frac{1}{4}.$$

Ex. 3-29. *Show that for n attributes A, B, C, ...M*

$$(ABC\ldots\ldots M) > \{(A)+(B)+(C)+\ldots\ldots+M)\}-(n-1)N$$

where N is the total frequency.

Sol. From Ex. 3-12,

$$(AB) > (A)+(B)-N$$

Replacing B by BC

$$(ABC) \geqslant (A)+(BC)-N$$

Also $\qquad (BC) > (B)+(C)-N$

$\therefore \qquad (ABC) > (A)+(B)+(C)-2N$

In general for m attributes $A, B, C, \ldots P$, let

$(ABC\ldots\ldots P) > (A)+(B)+(C)+\ldots\ldots+(P)-(m-1)N$

Replacing P by RS and using

$$(RS) > (R)+(S)-N,$$

for $(m+1)$ attributes

$$(ABC\ldots\ldots RS) > (A)+(B)+\ldots\ldots+(S)-mN$$

$\therefore$ If the inequality holds for m attributes it also holds for $(m+1)$ attributes. Since the inequality holds for three attributes, it also holds for four and hence five and any number of attributes.

Ex. 3-30. *In a very hotly fought battle 70% at least of the combatants lost an eye, 75% at least lost an ear, 80% at least lost an arm and 85% at least lost a leg. How mony at least must have lost all four?*

Sol. Let A, B, C, D denote losing an eye, an ear, an arm and a leg respectively. Then $N=100$, $(A) \geq 70$, $(B) > 75$, $(C) > 80$ and $(D) > 85$. Now from Ex. 3-29.

$$\begin{aligned} (ABCD) &> (A)+(B)+(C)+(D)-3N \\ &> 70+75+80+85-300 \\ &=10 \end{aligned}$$

$\therefore$ 10% at least have lost all four.

3.2. Association of Attributes

Independence. *If there is no relationship of any kind between two attributes A and B, it is expected to have the same proportion of A's among B's as among not B's i.e., β's. Two such attributes are termed as independent and the criterion of independence of two attributes A and B is*

$$\frac{(AB)}{(B)}=\frac{(A\beta)}{(\beta)}$$

Association. *If A and B are not independent, these are related in some way or other, however complicated. These are said to be positively associated or simply associated if*

$$(AB) > \frac{(A)(B)}{N}$$

negatively associated if

$$(AB) < \frac{(A)(B)}{N}$$

In statistics A and B are said to be associated only if th appear together in a greater number of cases than is to be expecte these are independent.

Complete Association. *Two attributes are said to be com tely associated if one of them cannot occur without the other tho the other may occur without the one i.e., all A's are B's or all B's A's according as whether A's or B's are in minority.*

Complete Disassociation. *It may be taken either as the c when no A's are B's or the case when no α's are β's.*

Coefficient of Association. *It is given by*

$$Q=\frac{(AB)(\alpha\beta)-(A\beta)(\alpha B)}{(AB)(\alpha\beta)+(A\beta)(\alpha B)}$$

Coefficient of Colligation. *It is defined by*

$$R=\frac{\sqrt{\{(AB)(\alpha\beta)\}}-\sqrt{\{(A\beta)(\alpha B)\}}}{\sqrt{\{(AB)(\alpha\beta)\}}+\sqrt{\{(A\beta)(\alpha B)\}}}$$

Symbols $\qquad (AB)_0=\dfrac{(A)(B)}{N}$

$$\delta=(AB)-(AB)_0.$$

Ex. 3-31. *If A and B be two independent attributes, prove th*

(*i*) $\dfrac{(\alpha B)}{(B)}=\dfrac{(\alpha\beta)}{(\beta)}$ (*ii*) $\dfrac{(\alpha\beta)}{(\alpha)}=\dfrac{(A\beta)}{(A)}$ (*iii*) $\dfrac{(AB)}{(A)}=\dfrac{(\alpha B)}{(\alpha)}$

and (*iv*) $(AB)=\dfrac{(A)(B)}{N}$

Sol. Since A and B are independent,

$$\frac{(AB)}{(B)}=\frac{(A\beta)}{(\beta)} \qquad \ldots($$

(*i*) From (1), $\dfrac{(AB)}{(A\beta)}=\dfrac{(B)}{(\beta)}=\dfrac{(AB)+(\alpha B)}{(A\beta)+(\alpha\beta)}$

$\therefore$ each ratio$=\dfrac{(\alpha B)}{(\alpha\beta)}$

$$\therefore \qquad \frac{(B)}{(\beta)}=\frac{(\alpha B)}{(\alpha\beta)} \quad \text{or} \quad \frac{(\alpha B)}{(B)}=\frac{(\alpha\beta)}{(\beta)}$$

(*ii*) From (1), $\dfrac{(AB)}{(B)}=\dfrac{(A\beta)}{(\beta)}=\dfrac{(AB)+(A\beta)}{(B)+(\beta)}=\dfrac{(A)}{N}$ $\qquad \ldots($

$$\therefore \qquad \frac{(A\beta)}{(\beta)}=\frac{(A)}{N}$$

$$\therefore \qquad \frac{(A\beta)}{(\beta)-(A\beta)}=\frac{(A)}{N-(A)} \quad \text{or} \quad \frac{(A\beta)}{(\alpha\beta)}=\frac{(A)}{(\alpha)}$$

$$\therefore \quad \frac{(A\beta)}{(A)}=\frac{(\alpha\beta)}{(\alpha)}$$

(*iii*) From (2), $\frac{(AB)}{(B)}=\frac{(A)}{N}$

$$\therefore \quad \frac{(AB)}{(B)-(AB)}=\frac{(A)}{N-(A)} \quad \text{or} \quad \frac{(AB)}{(\alpha B)}=\frac{(A)}{(\alpha)}$$

$$\therefore \quad \frac{(AB)}{(A)}=\frac{(\alpha B)}{(\alpha)}$$

(*iv*) From (2), $(AB)=\frac{(A)(B)}{N}$

Ex. 3-32. *Show, whether A and B are independent, positively associated or negatively associated in each of the following cases :*

(*i*) $N=100, (A)=47, (B)=62, (AB)=32.$

(*ii*) $(A)=490, (AB)=294, (\alpha)=570, (\alpha B)=380.$

(*iii*) $(AB)=256, (\alpha B)=768, (A\beta)=48, (\alpha\beta)=144.$

Sol. (*i*) $\frac{(A)(B)}{N}=\frac{(47)(62)}{100}=29{\cdot}14=(AB)_0$

Since $(AB)>(AB)_0$, attributes are positively associated.

(*ii*) $N=(\alpha)+(A)=1060, (B)=(AB)+(\alpha B)=674$

$$\therefore \quad (AB)_0=\frac{(A)(B)}{N}=\frac{(490)(674)}{1060}=311{\cdot}6 \text{ (nearly)}$$

$$\therefore \quad (AB)<(AB)_0$$

$\therefore$ Attributes are negatively associated.

(*iii*) $(A)=(AB)+(A\beta)=256+48=304$

$(B)=(AB)+(\alpha B)=256+768=1024$

$(\alpha)=(\alpha B)+(\alpha\beta)=768+144=912$

$\therefore \quad N=(A)+(\alpha)=304+912=1216$

$$\therefore (AB)_0=\frac{(A)(B)}{N}=\frac{(304)(1024)}{1216}=256=(AB)$$

$\therefore$ A and B are independent.

Ex. 3-33. *The male population of U.P. is 250 lakhs. The number of literate males is 20 lakhs and the total number of male criminals is 26 thousands. The number of literate male criminals is two thousands. Do you find any association between literary and criminality ?*

Sol. Let A and B denote the attributes of being literate and criminal respectively. Then

$(A)=20$ lakhs, $(B)=26$ thousands$=0{\cdot}26$ lakhs

$(AB)=2$ thousands$=0.02$ lakhs and $N=250$ lakhs

$$\therefore \quad (AB)_0=\frac{(A)(B)}{N}=\frac{(20)(0.26)}{250}=0.0208>(AB)$$

$\therefore$ A and B are negatively associated.

Ex. 3-34. *Show that (i)* $Q=\dfrac{N\delta}{(AB)(\alpha\beta)+(A\beta)(\alpha B)}$ *and deduc that $Q=0$ when A and B are independent.*

(ii) $Q=1$ for complete association.

(iii) $Q=-1$ for complete disassociation.

(iv) $Q=\dfrac{2R}{1+R^2}$ *where R is the co-efficient of colligation.*

Sol. *(i)* $\delta=(AB)-(AB)_0=(AB)-\dfrac{(A)(B)}{N}$

$$\begin{aligned}\therefore \quad N\delta&=(AB)\{(A)+(\alpha)\}-(A)\{(AB)+(\alpha B)\}\\&=(AB)\{(\alpha B)+(\alpha\beta)\}-\{(AB)+(A\beta)\}(\alpha B)\\&=(AB)(\alpha\beta)-(A\beta)(\alpha B)\end{aligned}$$

$$\therefore \quad Q=\frac{N\delta}{(AB)(\alpha\beta)+(A\beta)(\alpha B)}$$

If A and B are independent, $\delta=0$

$\therefore \quad Q=0$

(ii) If there is complete association, all A's are B's or all B are A's according as A's or B's are in minority.

$\therefore$ Either $(A\beta)=0$ or $(\alpha B)=0$

$\therefore \quad Q=1$

(iii) If there is complete disassociation, either no A's are B's o no α's are β's.

$\therefore$ Either $(AB)=0$ or $(\alpha\beta)=0$

$\therefore \quad Q=-1.$

(iv) By def., $R=\dfrac{\sqrt{\{(AB)(\alpha\beta)\}}-\sqrt{\{(A\beta)(\alpha B)\}}}{\sqrt{\{(AB)(\alpha\beta)\}}+\sqrt{\{(A\beta)(\alpha B)\}}}$

$$\therefore \quad \frac{1+R}{1-R}=\frac{\sqrt{(AB)(\alpha\beta)}}{\sqrt{(A\beta)(\alpha B)}}$$

$$\therefore \quad Q=\frac{\left\{\dfrac{(1+R)^2}{(1-R)^2}-1\right\}}{\left\{\dfrac{(1+R)^2}{(1-R)^2}+1\right\}}=\frac{2R}{1+R^2}$$

Note. From (*i*) $Q>0$ for positive association

<0 for negative association

$=0$ for independence.

Ex. 3·35. *If A and B are independent, find the* (AB), $(A\beta)$, (αB) *and* $(\alpha\beta)$.

Sol. Since A and B are independent, $(AB)=\dfrac{(A)(B)}{N}$

$$\therefore\quad (A\beta)=(A)-(AB)=(A)\left\{1-\frac{(B)}{N}\right\}=\frac{(A)\{(N-(B)\}}{N}$$

$$=\frac{(A)(\beta)}{N}$$

$$(\alpha B)=(B)-(AB)=(B)\left\{1-\frac{(A)}{N}\right\}=\frac{(\alpha)(B)}{N}$$

$$(\alpha\beta)=N-(A)-(B)+(AB)$$

$$=N-(A)-(B)+\frac{(A)(B)}{N}$$

$$=(\alpha)-\frac{(B)}{N}\{N-(A)\}$$

$$=(\alpha)\left\{1-\frac{(B)}{N}\right\}=\frac{(\alpha)(\beta)}{N}$$

Ex. 3·36. *Show that* $\delta=\dfrac{(B)(\beta)}{N}\left\{\dfrac{(AB)}{B}-\dfrac{(A\beta)}{(\beta)}\right\}$.

Sol. $\delta=(AB)-\dfrac{(A)(B)}{N}=\dfrac{1}{N}\{N(AB)-(A)(B)\}$

$$=\frac{1}{N}[(AB)\{(B)+(\beta)\}-\{(AB)+(A\beta)\}(B)]$$

$$=\frac{1}{N}[(AB)(\beta)-(A\beta)(B)]$$

$$=\frac{(B)(\beta)}{N}\left[\frac{(AB)}{(B)}-\frac{(A\beta)}{(\beta)}\right]$$

Interchanging A and B

$$\delta=\frac{(A)(\alpha)}{N}\left\{\frac{(AB)}{(A)}-\frac{(\alpha B)}{(\alpha)}\right\}$$

Ex. 3·37. *Show that*

$$(AB)^2+(\alpha\beta)^2-(\alpha B)^2-(AB)^2=[(A)-(\alpha)][(B)-(\beta)]+2N\delta$$

Sol. R.H.S. $=\{(A)-(\alpha)\}\{(B)-(\beta)\}+2N\left\{(AB)-\dfrac{(A)(B)}{N}\right\}$

$=[\{(AB)+(A\beta)\}-\{(\alpha B)+(\alpha\beta)\}][\{(AB)+(\alpha B)\}$
$\qquad -\{(A\beta)+(\alpha\beta)\}]+2N(AB)-2(A)(B)$

$=\{(AB)-(\alpha\beta)\}^2-\{(A\beta)-(\alpha B)\}^2+2\{(A)+(\alpha)\}(AB)-2(A)(B)$

$=\{(AB)-(\alpha\beta)\}^2-\{(A\beta)-(\alpha B)\}^2+2(A)\{(AB)-(B)\}+2(\alpha)(AB)$

$=\{(AB)-(\alpha\beta)\}^2-\{(A\beta)-(aB)\}^2-2\{(AB)+(A\beta)\}(\alpha B)$
$\qquad +2\{(\alpha B)+(\alpha\beta)\}(AB)$

$=(AB)^2+(\alpha\beta)^2-(A\beta)^2-(\alpha B)^2$

Ex. 3·38. *Show that if*

$$(AB)_1,\ (\alpha B)_1,\ (A\beta)_1,\ (\alpha\beta)_1$$
$$(AB)_2,\ (\alpha B)_2,\ (A\beta)_2,\ (\alpha\beta)_2$$

be two aggregates corresponding to the same values of (A), (B), (α) *and* (β)

$$(AB)_1-(AB)_2=(\alpha B)_2-(\alpha B)_1=(A\beta)_2-(A\beta)_1=(\alpha\beta)_1-(\alpha\beta)_2.$$

Sol. $(A)=(AB)_1+(A\beta)_1=(AB)_2+(A\beta)_2$

$\therefore \qquad (AB)_1-(AB)_2=(A\beta)_2-(A\beta)_1 \qquad \ldots(1)$

$(B)=(AB)_1+(\alpha B)_1=(AB)_2+(\alpha B)_2$

$\therefore \qquad (AB)_1-(AB)_2=(\alpha B)_2-(\alpha B)_1 \qquad \ldots(2)$

$(\alpha)=(\alpha B)_1+(\alpha\beta)_1=(\alpha B)_2+(\alpha\beta)_2$

$\therefore \qquad (\alpha B)_1-(\alpha B)_2=(\alpha\beta)_2-(\alpha\beta)_1 \qquad \ldots(3)$

From (1), (2) and (3) result follows.

Ex. 3·39. *Investigate the association between eye colour of husband and eye-colour of wife from the data given below :*

Husbands with light eyes and wives with light eyes $=309$
" " " " " " not " " $=214$
" " " not " " " " " $=132$
" " " " " " " " " " $=119$

Sol. Let A, B denote the attributes of husbands with light eyes and wives with light eyes respectively.

Then $(AB)=309$, $(A\beta)=214$, $(\alpha B)=132$ and $(\alpha\beta)=119$.

$$\therefore\quad Q=\frac{(AB)(\alpha\beta)-(A\beta)(\alpha B)}{(AB)(\alpha\beta)+(A\beta)(\alpha B)}=\frac{(309)(119)-(214)(132)}{(309)(119)+(214)(132)}$$

$$=\frac{36771-28248}{36771+28248}=0{\cdot}13$$

$\therefore$ There seems to be positive association of small degree. Working out percentages :

Percentages of light-eyed amongst the wives of light-eyed husbands$=\frac{309}{214+309}\times 100=59\%$ and percentage of light-eyed amongst the wives of not light-eyed husbands$=\frac{132}{132+119}\times 100=53\%$. Comparison brings out that the association is small, so small that no stress can be laid on it as indicating anything but a fluctuation of sampling.

Ex. 3·40. *Investigate the association between darkness of eye colour in father and son from the following data :*

Father with dark eyes and sons with dark eyes$=50$
" " " " " without " " $=79$
" without " " " " with " " $=89$
" " " " " without " " $=782$

What would have been the frequency of 'fathers with dark eyes and sons with dark eyes' for the same total number, had there been complete independence ?

Sol. Let A and B be the attributes of father and son to be with dark eyes respectively. Then

$(AB)=50$, $(A\beta)=79$, $(\alpha B)=89$ and $(\alpha\beta)=782$

$$\therefore \quad Q=\frac{(50)(782)-(79)(89)}{(50)(782)+(79)(89)}=\frac{32069}{46131}=0\cdot7$$

$\therefore$ There is a positive association of high degree between the darkness of eye-colour in father and son.

Now $\quad (A)=(AB)+(A\beta)=50+79=129$

$\quad\quad (B)=(AB)+(\alpha B)=50+89=139$

$$\therefore \quad (AB)_0=\frac{(A)(B)}{N}=\frac{(129)(139)}{1000}=18 \text{ (approx.)}$$

$\therefore$ Had A and B been independent.

$$(AB)=(AB)_0=18.$$

Ex. 3·41. *The following data relates to literacy and unemployment in a group of 500 persons. You are required to calculate Yule's co-efficient of association between literacy and unemployment and interpret it.*

Illiterate unemployed	220
Literate employed	20
Illiterate employed	180

Sol. Let A and B denote the attributes of being literate and unemployed respectively. Then

$$(AB)=500-(220+20+180)=80$$

$$(A\beta)=20 \quad (\alpha B)=220 \text{ and } (\alpha\beta)=180$$

$$\therefore \quad Q=\frac{(AB)(\alpha\beta)-(A\beta)(\alpha B)}{(AB)(\alpha\beta)+(A\beta)(\alpha B)}=\frac{(80)(180)-(20)(220)}{(80)(180)+(20)(220)}$$

$$=0{\cdot}5.$$

Thus there is a positive association of high degree between literacy and unemployment. Thus in general literate person is unemployed.

Ex. 3·42. *From the following, find whether blindness and baldness are associated.*

Total population=16,264,000, number of bald-headed=24,441, number of blind=7,623, number of bald-headed blind=221.

Sol. Let A and B denote the attributes of being bald-headed and blind respectively. Then

$$(AB)=221, (A)=24{,}441, (B)=7{,}623 \text{ and } N=16{,}264{,}000$$

$$\therefore \quad (AB)_0=\frac{(A)(B)}{N}=\frac{(24{,}441)(7{,}623)}{(16{,}264{,}000)}=11 \text{ (nearly)}$$

$$\therefore \quad (AB)>(AB)_0$$

$\therefore$ There is positive association between baldness and blindness.

Ex. 3·43. *Do you find any association between the tempers of brothers and sisters from the following data :*

Good-natured	*brothers*	*and*	*good-natured sisters*	*=1230*
,,	,,	,,	*sullen* ,,	*=850*
Sullen	,,	,,	*good-natured* ,,	*=530*
,,	,,	,,	*sullen* ,,	*=980*

Sol. Let A and B denote the attributes of being good-natured for brother and sister respectively. Then

$$(AB)=1230, (A\beta)=850, (\alpha B)=530 \text{ and } (\alpha\beta)=980$$

$$\therefore \quad Q=\frac{(AB)(\alpha\beta)-(A\beta)(\alpha B)}{(AB)(\alpha\beta)+(A\beta)(\alpha B)}=\frac{(1230)(980)-(850)(530)}{(1230)(980)+(850)(530)}$$

$$=0{\cdot}46.$$

$\therefore$ There is positive association.

Ex. 3·44 *Can vaccination be regarded as a preventive measure for small-pox from the data given below ?*

'Of 1482 persons in a locality exposed to small pox, 368 in all were attacked'.

'Of 1482 persons, 343 had been vaccinated and of these only 35 were attacked.'

Sol. Let A and B denote the attributes of being vaccinated and attacked respectively. Then

$$N=1482,\ (AB)=35,\ (A)=343,\ (B)=368,$$

$$(AB)_0=\frac{(A)(B)}{N}=\frac{(343)(368)}{1482}=85\cdot2$$

$$\therefore \qquad (AB)<(AB)_0$$

$\therefore$ **There is negative association and hence vaccination can be regarded as a preventive measure for small-pox.**

Ex. 3·45. *In an antimalarial campaign in a certain area, quinine was administered to 812 persons out of a total population of 3248. The number of fever cases is shown below :*

Treatment	*Fever*	*No. Fever*
Quinine	*20*	*792*
No. quinine	*220*	*2,216*

Discuss the usefulness of quinine in checking malaria.

Sol. Let A and B correspond to fever and quinine. Then $(AB)=20$, $(\alpha B)=792$, $(A\beta)=220$ and $(\alpha\beta)=2{,}216$

$$\therefore \qquad Q=\frac{(20)(2216)-(792)(220)}{(20)(2216)+(792)(220)}=-0\cdot6$$

$\therefore$ **There is a negative association of high degree. Hence quinine may be taken as preventive malaria.**

Ex. 3·46. *A group of 1000 fathers was studied and it was found that 12·9% had dark eyes. Among them the ratio of those having sons with dark eyes to those having sons with not dark eyes was 1 : 1·58. The number of cases where fathers and sons both did not have dark eyes was 782. Calculate a co-efficient of association between darkness of eye-colour in father and son. Give the frequencies that would have been observed had there been completely no heredity.*

Sol. Let A and B denote the attributes of having dark eyes for father and son respectively. Then

$$(A)=\frac{12\cdot9}{100}\times1000=129,\ N=1000,\ \frac{(AB)}{(A\beta)}=\frac{1}{1\cdot58}\text{ and, }(\alpha\beta)=782$$

Now $$\frac{(AB)}{1}=\frac{(A\beta)}{1\cdot58}=\frac{(AB)+(A\beta)}{2\cdot58}=\frac{(A)}{2\cdot58}=\frac{129}{2\cdot58}=50$$

$$\therefore \qquad (AB)=50 \text{ and } (A\beta)=79$$

$$(\beta)=(A\beta)+(\alpha\beta)=79+782=861$$

$\therefore \quad (B)=N-(\beta)=1000-861=139$

$\therefore \quad (AB)+(\alpha B)=139$

$\therefore \quad (\alpha B)=89$

Now see Ex. 3-40.

EXERCISE 3.1

1. If a collection contains N items, each of which is characterized by one or more of the attributes A, B, C and D, show that with the usual notation

   (i) $(ABCD) \geq (A)+(B)+(C)+(D)-3N$

   and (ii) $(ABCD)=(ABD)+(ACD)-(AD)+(AD\beta\gamma)$.

   where β and γ represent the characteristics of the absence of B and C respectively.

2. Three aptitude tests A, B, C were given to 200 apprentice trainees. From amongst them 80 passed test A, 78 passed test B and 96 passed the third test. While 20 passed all three tests, 42 failed all the three, 18 passed A and B but failed C and 38 failed A and B but passed the third. Determine (i) how many trainees passed at least two of the three tests and (ii) whether the performances in tests A and B are associated.

   [**Ans.** 76, $Q=0{\cdot}3$]

3. In a college 50% of the students are boys, 60% of the students are above 18 years, and 80% receive scholarships. 35% of the students are boys above 18 years of age, 45% are boys receiving scholarships and 42% are above 18 years and receive scholarships. Determine the limits to the proportion of boys above 18 years who are in receipt of scholarships.

   [**Ans.** Lies between 30 and 32]

4. A study was made about the studying habits of the students of a certain university and the following summary is given at one place in the report.

   'Of the students surveyed 75% were from well-to-do families, 55% were boys and 60% were irregular in their studies. Out of the irregular ones 50% were boys and two-thirds were from well-to-do families. The percentage of irregular boys from well-to-do families were 8. Is there any inconsistency in the data ? [**Ans.** Yes]

5. The following data relate to flexibility and selling ability of 20 salesmen. Test whether the two attributes are independent.

	Flexibility	
Selling ability	*Good*	*Poor*
Good	7	3
Poor	2	8

[**Ans.** $Q=0{\cdot}8$]

6. Calculate the co-efficient of association between illiteracy and criminality from the data given below and interpret it.

The total population of a city is 244,000 out of which 40,000 are literates. The number of criminals in the group of literates is 300 and in the group of illiterates 4,000. [**Ans.** 0·5]

7. The following data were observed for hybrids of Datura.

Flowers violet, fruits prickly $(AB)=47$
" " " smooth $(A\beta)=12$
" white " prickly $(\alpha B)=21$
" " " smooth $(\alpha\beta)=3$

Investigate the association between colour of flower and character of fruit.
[**Ans.** (−0·28)]

8. From the data given below, compare the association between literacy and unemployment in the rural and urban areas.

	Rural	*Urban*
Total number of adult males	25 lakhs	200 lakhs
Literate males	10 "	40 "
Unemployed "	5 "	4 "
Literate and unemployed males	3 "	4 "

9. In an assortative mating study to find whether tall husbands tend to marry tall wives the following information about the wives of 125 tall and 125 short-statured husbands was published.

	Tall husbands (percent)	*Short husbands (percent)*
Tall wives	56	13
Short wives	11	48

Find the co-efficient of association between the stature of wives and husbands, ignoring medium-sized wives. [**Ans.** 0·9]

10. From the figures in the following table compare the association between literacy and unemployment in rural and urban areas.

	Urban	*Rural*
Total adult males	25 lakhs	20 lakhs
Literate males	10 "	10 "
Unemployed males	5 "	12 "
Literate and unemployed males	4 "	4 "

11. In a state with a total population of 70,000 adults, 34,000 are males and out of a total of 6,000 graduates 700 are females. Out of 1200 graduate employees of the state, 200 are females. Is there any sex bias in education among the people? The

state holds that no distinction is made in appointments in respect of sex. How far is their claim substantiated by the data given above ?

12. A census revealed the following figures of the blind and the insane in two age groups in a certain population.

	Age—group (15—25 years)	*Age—group* (over 25 years)
Total population	270,000	160,200
No. of blind	1,000	2,000
,, ,, insane	6,000	1,000
,, ,, insane among the blind	19	9

(*a*) Obtain a measure of the association between blindness and insanity in each of the two age groups (*b*) Do you consider that blindness and insanity are associated or disassociated with each other in the two age groups or more in one age group than the other ?

13. Obtain the co-efficient of association between un-employment and educational attainments from the following results of an urban survey.

	Employed	*Unemployed*
Illiterate or below matric	5997	432
Matric and above	572	96

[Ans. 0·4]

14. The following table gives the number of literates and criminals in three cities of U.P.

	Kanpur	*Allahabad*	*Agra*
Total number (in thousands)	244	184	230
Literates (,, ,,)	30	47	33
Literate criminals (in ,,)	3	2	2
Illiterate criminals (in ,,)	40	20	24

Compare the degree of association between criminality and illiteracy in each of the above three cities.

4

Difference Operators and Interpolation

4.1. Divided Differences. Let the values of $f(x)$ for $x=x_0$, $x_1, \ldots x_n$ be known. Then

$$f(x_0, x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0},$$

$$f(x_0, x_1, x_2)=\frac{f(x_1, x_2)-f(x_0, x_1)}{x_2-x_0}$$

$$f(x_0, x_1, x_2, x_3)=\frac{f(x_1, x_2, x_3)-f(x_0, x_1, x_2)}{x_3-x_0}$$

and so on, are called divided differences of first order, second order, third order etc.

Ex. 4-1. *Compute the divided differences of $f(x)$ from the following table :*

x :	1	2	3	4	7
$f(x)$:	2	4	8	16	128

[redacted]

x	$f(x)$	*1st order*	*2nd order*	*3rd order*	*4th order*
1	2	$\frac{4-2}{2-1}=2$	$\frac{4-2}{3-1}=1$	$\frac{2-1}{4-1}=\frac{1}{3}$	$\frac{\frac{16}{15}-\frac{1}{3}}{7-1}=\frac{11}{90}$
2	4	$\frac{8-4}{3-2}=4$	$\frac{8-4}{4-2}=2$	$\frac{\frac{22}{3}-2}{7-2}=\frac{16}{15}$	
3	8	$\frac{16-8}{4-3}=8$	$\frac{\frac{112}{3}-8}{7-3}=\frac{22}{3}$		
4	16	$\frac{128-16}{7-4}=\frac{112}{3}$			
7	128				

Ex. 4-2. *Obtain the divided differences of $f(x)=x^2$.*

Sol. Let $x_0, x_1, \ldots x_n$ be the values of x. Then

$$f(x_0, x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{x_1^2-x_0^2}{x_1-x_0}=x_1+x_0$$

$$f(x_0, x_1, x_2)=\frac{f(x_1, x_2)-f(x_0, x_1)}{x_2-x_0}$$

$$=\frac{(x_2+x_1)-(x_1+x_0)}{x_2-x_0}=1$$

$$f(x_0, x_1, x_2, x_3)=\frac{f(x_1, x_2, x_3)-f(x_0, x_1, x_2)}{x_3-x_0}=0$$

Evidently all higher order divided differences will be zero.

Ex. 4-3. *Prove that the divided differences are symmetrical in their arguments.*

Sol. Let $x_0, x_1, \ldots x_n$ be the arguments.

Then $f(x_0, x_1)=\dfrac{f(x_1)-f(x_0)}{x_1-x_0}=\dfrac{f(x_1)}{x_1-x_0}+\dfrac{f(x_0)}{x_0-x_1}$

$$f(x_0, x_1, x_2)=\frac{f(x_1, x_2)-f(x_0, x_1)}{x_2-x_0}$$

$$=\frac{1}{x_2-x_0}\left[\left\{\frac{f(x_2)}{x_2-x_1}+\frac{f(x_1)}{x_1-x_2}\right\}-\left\{\frac{f(x_1)}{x_1-x_0}+\frac{f(x_0)}{x_0-x_1}\right\}\right]$$

$$=\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}+\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$$

Let in general

$$f(x_0, x_1, \ldots x_m)=\frac{f(x_m)}{(x_m-x_0)(x_m-x_1)\ldots\ldots(x_m-x_{m-1})}$$

$$+\frac{f(x_{m-1})}{(x_{m-1}-x_0)(x_{m-1}-x_1)\ldots(x_{m-1}-x_{m-2})(x_{m-1}-x_m)}$$

$$+\ldots\ldots+\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_m)}$$

Then $f(x_0, x_1, \ldots x_{m+1})=\dfrac{f(x_1, x_2 \ldots x_{m+1})-f(x_0, x_1, \ldots x_m)}{x_{m+1}-x_0}$

$$=\frac{1}{x_{m+1}-x_0}\left[\frac{f(x_{m+1})}{(x_{m+1}-x_1)(x_{m+1}-x_2)\ldots(x_{m+1}-x_m)}\right.$$

$$+\frac{f(x_m)}{(x_m-x_1)(x_m-x_2)\ldots\ldots(x_m-x_{m-1})(x_m-x_{m+1})}$$

$$\left.+\ldots\ldots+\frac{f(x_1)}{(x_1-x_2)\ldots\ldots(x_1-x_{m+1})}\right]$$

$$-\frac{1}{x_{m+1}-x_0}\left[\frac{f(x_m)}{(x_m-x_0)\ldots\ldots(x_m-x_{m-1})}\right.$$

$$+\frac{f(x_{m-1})}{(x_{m-1}-x_0)\ldots\ldots(x_{m-1}-x_{m-2})(x_{m-1}-x_m)}$$

$$\left.+\ldots\ldots+\frac{f(x_0)}{(x_0-x_1)\ldots\ldots(x_0-x_m)}\right]$$

$$=\frac{f(x_{m+1})}{(x_{m+1}-x_0)\ldots\ldots(x_{m+1}-x_m)}$$

$$+\frac{f(x_m)}{(x_m-x_0)\ldots\ldots(x_m-x_{m-1})(x_m-x_{m+1})}+\ldots\ldots$$

$$+\frac{f(x_0)}{(x_0-x_1)\ldots\ldots(x_0-x_{m+1})}$$

$\therefore$ By induction,

$$f(x_0, x_1, \ldots\ldots x_n)=\frac{f(x_n)}{(x_n-x_0)\ldots\ldots(x_n-x_{n-1})}$$

$$+\frac{f(x_{n-1})}{(x_{n-1}-x_0)\ldots\ldots(x_{n-1}-x_{n-2})(x_{n-1}-x_n)}+\ldots\ldots$$

$$+\frac{f(x_0)}{(x_0-x_1)\ldots\ldots(x_0-x_n)}$$

Evidently $f(x_0, x_1, \ldots x_n)$ remains unchanged on interchanging the arguments. Hence $f(x_0, x_1, \ldots x_n)$ is symmetrical in its arguments.

Ex. 4·4. *Show that divided differences of the sum of two functions are equal to the sum of the divided differences of two f^ns.*

Sol. Let $f(x)$ and $g(x)$ be two $f^n s$ and

$$h(x)=f(x)+g(x)$$

Now $$h(x_0, x_1, \ldots x_n)=\frac{h(x_n)}{\prod\limits_{i\neq n}(x_n-x_i)}+\frac{h(x_{n-1})}{\prod\limits_{i\neq n-1}(x_{n-1}-x_i)}+\ldots\ldots+\frac{h(x_0)}{\prod\limits_{i\neq 0}(x_0-x_i)}$$

$$=\frac{1}{\prod\limits_{i\neq n}(x_n-x_i)}\{f(x_n)+g(x_n)\}+\frac{1}{\prod\limits_{i\neq n-1}(x_{n-1}-x_i)}\{f(x_{n-1})+g(x_{n-1})\}$$

$$+\ldots\ldots+\frac{1}{\prod\limits_{i\neq 0}(x_0-x_i)}\{f(x_0)+g(x_0)\}$$

$$=\left\{\frac{f(x_n)}{\prod\limits_{i\neq n}(x_n-x_i)}+\frac{f(x_{n-1})}{\prod\limits_{i\neq n-1}(x_{n-1}-x_i)}+\ldots\ldots+\frac{f(x_0)}{\prod\limits_{i\neq 0}(x_0-x_i)}\right\}$$

$$+\left\{\frac{g(x_n)}{\prod\limits_{i\neq n}(x_n-x_i)}+\frac{g(x_{n-1})}{\prod\limits_{i\neq n-1}(x_{n-1}-x_i)}+\ldots\ldots+\frac{g(x_0)}{\prod\limits_{i\neq 0}(x_0-x_i)}\right\}$$

$$=f(x_0, x_1, \ldots x_n)+g(x_0, x_1, \ldots\ldots x_n)$$

Ex. 4·5. *Show that the divided differences of '$cf(x)$', where 'c' is constant, are 'c' times the divided differences of $f(x)$.*

Sol. Let $\phi(x)=cf(x)$

Then $$\phi(x_0, x_1, \ldots\ldots x_n)=\frac{\phi(x_n)}{\prod\limits_{i\neq n}(x_n-x_i)}+\frac{\phi(x_{n-1})}{\prod\limits_{i\neq n-1}(x_{n-1}-x_i)}+\ldots\ldots+\frac{\phi(x_0)}{\prod\limits_{i\neq 0}(x_0-x_i)}$$

$$=c\left\{\frac{f(x_n)}{\prod\limits_{i\neq n}(x_n-x_i)}+\frac{f(x_{n-1})}{\prod\limits_{i\neq n-1}(x_{n-1}-x_i)}+\ldots\ldots+\frac{f(x_0)}{\prod\limits_{i\neq 0}(x_0-x_i)}\right\}$$

$$=cf(x_0, x_1, \ldots x_n)$$

Ex. 4·6. *Show that nth order divided differences of x^n are constant.*

Sol. Let $f(x)=x^n$

$$f(x_0, x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{x_1^n-x_0^n}{x_1-x_0}=x_1^{n-1}+x_1^{n-2}x_0+\ldots+x_0^{n-1}$$

$$f(x_0, x_1, x_2)=\frac{f(x_1, x_2)-f(x_0, x_1)}{x_2-x_0}$$

$$=\frac{(x_2^{n-1}+x_2^{n-2}x_1+\ldots\ldots+x_1^{n-1})-(x_0^{n-1}+x_0^{n-2}x_1+\ldots+x_1^{n-1})}{x_2-x_0}$$

$$=\frac{(x_2^{n-1}-x_0^{n-1})+x_1(x_2^{n-2}-x_0^{n-2})+\ldots\ldots+x_1^{n-2}(x_2-x_0)}{x_2-x_0}$$

$$=(x_2^{n-2}+x_2^{n-3}x_0+\ldots\ldots+x_0^{n-2})+x_1(x_2^{n-3}+x_2^{n-4}x_0+\ldots\ldots+x_0^{n-3})+\ldots\ldots+x_1^{n-2}$$

Thus if $f(x)=x^n$, $f(x_0, x_1)$ is a homogeneous function of degree $(n-1)$ in x_0, x_1; $f(x_0, x_1, x_2)$ is a homogeneous function of degree $(n-2)$ in x_0, x_1, x_2, and so on. Thus the operation of taking the divided difference lowers the degree by unity. Hence finally, $f(x_0, x_1, \ldots x_n)$ will be a homogeneous function of degree $n-n=0$ *i.e.*, a constant.

Ex. 4·7. *Prove that the third order divided difference with arguments a, b, c, d of the function $\frac{1}{x}$ is equal to $-\frac{1}{abcd}$*

Sol.

x	$f(x)=\frac{1}{x}$	*1st order*	*2nd order*	*3rd order*
a	$\frac{1}{a}$			
		$-\frac{1}{ab}$		
b	$\frac{1}{b}$		$\frac{1}{abc}$	
		$-\frac{1}{bc}$		$-\frac{1}{abcd}$
c	$\frac{1}{c}$		$\frac{1}{bcd}$	
		$-\frac{1}{cd}$		
d	$\frac{1}{d}$			

4-2 Descending and Ascending Differences

Descending Differences. The first descending difference of $f(x)$ is defined by

$$\Delta f(x)=f(x+h)-f(x)$$

where h is the increment in x. The operator 'Δ' is called descending or forward difference operator. The second, third etc., differences are defined by $\Delta\{\Delta f(x)\}$, $\Delta[\Delta\{\Delta f(x)\}]$ etc.

Operator E. The extension or shift operator 'E' is defined by

$$Ef(x)=f(x+h)$$

Ascending Differences. The first ascending difference of $f(x)$ is defined by

$$\nabla f(x)=f(x)-f(x-h)$$

The operator '∇' is called ascending or backward difference operator. The second, third etc., differences are defined by $\nabla\{\nabla f(x)\}$, $\nabla[\nabla\{\nabla f(x)\}]$ etc.

Central Differences. The first central difference of $f(x)$ is defined by

$$\delta f(x)=f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)$$

The operator 'δ' is called central difference operator. The second, third etc., differences are defined by $\delta\{\delta f(x)\}$, $\delta[\delta\{\delta f(x)\}]$ etc.

Central Mean Operator. It is defined by

$$\mu=\frac{1}{2}\{E^{1/2}+E^{-1/2}\}$$

Relations between Operators. (*i*) $\Delta\equiv E-I$.

(*ii*) $\nabla\equiv\frac{E-1}{E}\equiv\frac{\Delta}{E}\equiv\frac{\Delta}{1+\Delta}$

(iii) $\Delta \equiv \dfrac{\nabla}{1-\nabla}$.

(iv) $\delta \equiv E^{1/2} - E^{-1/2} \equiv \Delta E^{-1/2} \equiv \nabla E^{1/2}$.

Relation between divided differences and ordinary differences.

$$f(x_0, x_1, \ldots\ldots x_n) = \frac{\Delta^n f(x_0)}{n\,!\,h^n}$$

Factorial Notation

$$x^{(mh)} = x(x-h)(x-2h)\ldots\ldots(x-\overline{m-1}\;h)$$
$$x^{(-mh)} = (x+h)^{-1}(x+2h)^{-1}\ldots\ldots(x+mh)^{-1}$$

Ex. 4·8. *Given $U_0=3$, $U_1=12$, $U_2=81$, $U_3=200$, $U_4=100$ and $U_5=8$. Find $\Delta^5 U_0$.*

Sol. Difference table is

x	$U(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	3					
		9				
1	12		60			
		69		−10		
2	81		50		−259	
		119		−269		755
3	200		−219		496	
		−100		227		
4	100		8			
		−92				
5	8					

$\therefore \quad \Delta^5 U_0 = 755.$

Ex. 4·9. *Show that $E \equiv 1 + \Delta$.*

Sol. By def. $\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x)$
$$= (E-1)f(x)$$
where $1\,f(x) = f(x)$

$\therefore \qquad \Delta \equiv E-1 \quad \text{or} \quad E \equiv 1+\Delta.$

Ex. 4·10. *Show that $E \equiv e^{hD}$ where D denotes the derivative operator and deduce that $\Delta \equiv e^{hD} - 1$.*

Sol. By def, $Ef(x) = f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2\,!} f''(x) + \ldots\ldots$

$$= \left(1 + hD + \frac{h^2}{2\,!} D^2 + \ldots\right) f(x)$$

$$= e^{hD} f(x)$$

$E \equiv e^{hD}$

$1 + \Delta \equiv e^{hD} \quad \text{or} \quad \Delta \equiv e^{hD} - 1.$

Ex. 4-11. *Show that*

(i) $Dy=\frac{1}{h}\left(\Delta y-\frac{1}{2}\Delta^2 y+\frac{1}{3}\Delta^3 y-\frac{1}{4}\Delta^4 y+\ldots\ldots\right)$

(ii) $D^2 y=\frac{1}{h^2}\left(\nabla^2 y+\nabla^3 y+\frac{11}{12}\nabla^4 y+\ldots\ldots\right)$

Sol. From Ex. 4-10, $e^{hD}\equiv 1+\Delta$

(i) $\therefore\quad D\equiv\frac{1}{h}\log(1+\Delta)$

$$\equiv\frac{1}{h}\left\{\Delta-\frac{1}{2}\Delta^2+\frac{1}{3}\Delta^3\ldots\right\}$$

$\therefore\quad Dy=\frac{1}{h}\left(\Delta y-\frac{1}{2}\Delta^2 y+\frac{1}{3}\Delta^3 y\ldots\ldots\right)$

(ii) $e^{hD}\equiv 1+\frac{\nabla}{1-\nabla}\equiv\frac{1}{1-\nabla}$

$\therefore\quad hD\equiv-\log(1-\nabla)\equiv\nabla+\frac{1}{2}\nabla^2+\frac{1}{3}\nabla^3+\frac{1}{4}\nabla^4+\ldots\ldots$

$\therefore h^2D^2\equiv\left(\nabla+\frac{1}{2}\nabla^2+\frac{1}{3}\nabla^3+\frac{1}{4}\nabla^4+\ldots\ldots\right)^2$

$$\equiv\nabla^2+\nabla^3+\frac{11}{12}\nabla^4+\ldots\ldots$$

$\therefore\quad D^2 y=\frac{1}{h^2}\left(\nabla^2 y+\nabla^3 y+\frac{11}{12}\nabla^4 y+\ldots\ldots\right)$

Ex. 4-12. *Show that*

(i) $\Delta\{f(x)\pm g(x)\}=\Delta f(x)\pm\Delta g(x)$

(ii) $E\{f(x)\pm g(x)\}=Ef(x)\pm Eg(x)$

(iii) $\Delta\{c\,f(x)\}=c\,\Delta f(x)$

(iv) $E\{c\,f(x)\}=cEf(x)$

(v) $\Delta E\equiv E\Delta$

(vi) $\Delta^m\Delta^n\equiv\Delta^n\Delta^m\equiv\Delta^{m+n}$

(vii) $E^mE^n\equiv E^nE^m\equiv E^{n+m}$

(viii) $\Delta\{f.g\}=f(x+h)\Delta g(x)+g(x)\Delta f(x)$

(ix) $\Delta\left\{\frac{f(x)}{g(x)}\right\}=\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x)\,g(x+h)}$

Sol. (i) $\Delta\{f(x)\pm g(x)\}=\{f(x+h)\pm g(x+h)\}-\{f(x)\pm g(x)\}$

$$=\{f(x+h)-f(x)\}\pm\{g(x+h)-g(x)\}$$
$$=\Delta f(x)\pm\Delta g(x)$$

(ii) $E\{f(x)\pm g(x)\}=f(x+h)\pm g(x+h)=Ef(x)\pm Eg(x)$

(iii) $\Delta\{c\,f(x)\}=cf(x+h)-cf(x)$
$$=c\{f(x+h)-f(x)\}=c\Delta f(x)$$

(iv) $E\{c.f(x)\}=cf(x+h)=cEf(x)$

(v) $\Delta Ef(x)=\Delta\{f(x+h)\}=f(x+2h)-f(x+h)$
$$=E\{f(x+h)-f(x)\}=E\Delta f(x)$$

$\therefore \qquad \Delta E\equiv E\Delta$

(vi) $\Delta^m\Delta^n f(x)=(\Delta\Delta\ldots m\text{ times})(\Delta\Delta\ldots n\text{ times})f(x)$
$$=(\Delta\Delta\ldots(m+n)\text{ times})f(x)=\Delta^{m+n}f(x)$$

$\therefore \qquad \Delta^m\Delta^n\equiv\Delta^{m+n}$

Similarly $\qquad \Delta^n\Delta^m\equiv\Delta^{m+n}$

(vii) $E^mE^nf(x)=(EE\ldots m\text{ times})(EE\ldots n\text{ times})f(x)$
$$=\{EE\ldots(m+n)\text{ times}\}f(x)=E^{m+n}f(x)$$

$\therefore \qquad E^mE^n\equiv E^{m+n}$

(viii) $\Delta\{f.g\}=f(x+h).g(x+h)-f(x).g(x)$
$$=f(x+h)\{g(x+h)-g(x)\}+g(x)\{f(x+h)-f(x)\}$$
$$=f(x+h)\Delta g(x)+g(x)\Delta f(x)$$

(ix) $\Delta\left\{\dfrac{f(x)}{g(x)}\right\}=\dfrac{f(x+h)}{g(x+h)}-\dfrac{f(x)}{g(x)}=\dfrac{g(x)f(x+h)-f(x)g(x+h)}{g(x).g(x+h)}$
$$=\frac{g(x)\{f(x+h)-f(x)\}-f(x)\{g(x+h)-g(x)\}}{g(x).g(x+h)}$$
$$=\frac{g(x)\,\Delta f(x)-f(x)\Delta g(x)}{g(x).g(x+h)}.$$

Ex. 4·13. *If (i) $f(E)$ is a polynomial in E, show that*
$$f(E)a^x=a^xf(a^h)$$

(ii) $f(\Delta)$ is a polynomial in Δ, show that
$$f(\Delta)a^x=a^xf(a^h-1).$$

Sol. (i) Let $f(E)\equiv p_0+p_1E+p_2E^2+\ldots\ldots+p_nE^n$

$\therefore\ f(E)a^x=(p_0+p_1E+p_2E^2+\ldots\ldots p_nE^n)a^x$
$$=p_0a^x+p_1a^{x+h}+p_2a^{x+2h}+\ldots\ldots p_na^{x+nh}$$
$$=a^x\{p_0+p_1a^h+p_2a^{2h}+\ldots\ldots+p_na^{nh}\}$$
$$=a^xf(a^h)$$

(ii) Let $f(\Delta)\equiv p_0+p_1\Delta+p_2\Delta^2+\ldots\ldots+p_n\Delta^n$

$\therefore\ f(\Delta)a^x=(p_0+p_1\Delta+p_2\Delta^2+\ldots\ldots+p_n\Delta^n)a^x$
$$=\{p_0a^x+p_1(E-1)a^x+p_2(E-1)^2a^x+\ldots+p_n(E-1)^na^x\}$$

$=a^x\{p_0+p_1(a^h-1)+p_2(a^h-1)^2+\ldots\ldots+p_n(a^h-1)^n\}$
$=a^x f(a^h-1)$.

Ex. 4-14. *Show that*

$$e^{-x}=\left(\frac{\Delta^2}{E}\right)e^{-x}\cdot\frac{Ee^{-x}}{\Delta^2 e^{-x}}$$

Sol. $\quad \text{R.H.S.}=\left(\frac{\Delta^2}{E}e^{-x}\right)\frac{Ee^{-x}}{\Delta^2 e^{-x}}$

$$=\Delta^2(E^{-1}e^{-x}).\frac{(Ee^{-x})}{\Delta^2 e^{-x}}=\Delta^2(e^{-x+h}).\ \frac{e^{-x-h}}{\Delta^2 e^{-x}}$$

$$=(e^h\Delta^2 e^{-x}).\ \frac{e^{-x}.e^{-h}}{\Delta^2 e^{-x}}=e^{-x}=\text{L.H.S.}$$

Ex. 4-15. *Show that* $\Delta(\tan^{-1}x)=\tan^{-1}\left\{\frac{h}{1+xh+x^2}\right\}$, *where h is the interval of differencing.*

Sol. $\quad \text{L.H.S.}=\tan^{-1}(x+h)-\tan^{-1}x$

$$=\tan^{-1}\left\{\frac{x+h-x}{1+x(x+h)}\right\}=\tan^{-1}\left\{\frac{h}{1+xh+x^2}\right\}.$$

Ex. 4-16. *Explain the difference between* $\left(\frac{\Delta^2}{E}\right)U_x$ *and* $\left(\frac{\Delta^2U_x}{EU_x}\right)$ *and find the values of these functions when $U_x=x^3$.*

Sol. $\left(\frac{\Delta^2}{E}\right)U_x=$Result of operating Δ^2E^{-1} on U_x

and $\quad \left(\frac{\Delta^2U_x}{EU_x}\right)=$Ratio of the results of operating Δ^2 and E on U_x.

$$\left(\frac{\Delta^2}{E}\right)x^3=\Delta^2E^{-1}(x^3)=\Delta^2(x-h)^3=\Delta\{x^3-(x-h)^3\}$$

$$=(x+h)^3-2x^3+(x-h)^3=6xh^2$$

$$\frac{\Delta^2x^3}{Ex^3}=\frac{(x+2h)^3-2(x+h)^3+(x^3)}{(x+h)^3}=\frac{6xh^2+6h^3}{(x+h)^3}=\frac{6h^2}{(x+h)^2}$$

Ex. 4-17. *If $f(x)=e^{ax}$, show that $f(x)$ and its leading differences are in G.P.*

Sol. $f(x)=e^{ax}$; $\Delta f(x)=e^{a(x+h)}-e^{ax}=e^{ax}\{e^{ah}-1\}$

$\Delta^2f(x)=\Delta\{e^{ax}(e^{ah}-1)\}=(e^{ah}-1)\Delta e^{ax}$

$=(e^{ah}-1)^2e^{ax}$.

Similarly $\Delta^3f(x)=(e^{ah}-1)^3e^{ax}$ and so on.

Evidently $f(x)$, $\Delta f(x)$, $\Delta^2f(x)$...are in G.P.

Ex. 4·23. *Show that* (*i*) $\nabla \equiv \dfrac{E-1}{E}$ (*ii*) $\nabla E \equiv \Delta \equiv E\nabla$.

Sol. (*i*) By def, $\nabla f(x) \equiv f(x)-f(x-h)$

$$\equiv f(x)-E^{-1}f(x) \equiv (1-E^{-1})f(x)$$

$$\therefore \qquad \nabla \equiv 1-\frac{1}{E} \equiv \frac{E-1}{E}$$

(*ii*) $\quad \nabla E \equiv (1-E^{-1})\, E \equiv E-1 \equiv \Delta$

$\quad E\nabla \equiv E(1-E^{-1}) \equiv E-1 \equiv \Delta.$

Ex. 4·24. *Show that* (*i*) $(1+\Delta)(1-\nabla) \equiv 1$

(*ii*) $\Delta\nabla \equiv \Delta-\nabla$

Sol. (*i*) $(1+\Delta)(1-\nabla) \equiv E(1-\nabla) \equiv E-E\nabla \equiv E-\Delta \equiv 1$

(*ii*) $\Delta\nabla \equiv (E-1)\nabla \equiv E\nabla-\nabla \equiv \Delta-\nabla.$

Ex. 4·25. *Show that* (*i*) $\Delta^n x^{(mh)} = (mh)^{(nh)} x^{(\overline{m-n}\,h)}$

(*ii*) $\Delta x^{n(-mh)} = (-mh)^{(nh)}\, x^{(-m+nh)}$

Sol. $\Delta x^{(mh)} = (x+h)^{(mh)} - x^{(mh)}$

$= (x+h)(x)(x-h)\ldots(x+h-\overline{m-1}h) - x(x-h)\ldots(x-\overline{m-1}\,h)$

$= x(x-h)\ldots\ldots(x-\overline{m-2}h)\{(x+h)-(x-\overline{m-1}h)\}$

$= (mh)x^{(\overline{m-1}h)}$

$\therefore \quad \Delta^2 x^{(mh)} = (mh)(\overline{m-1}h)x^{(\overline{m-2}h)} = (mh)^{(2h)}x^{(\overline{m-2}h)}$

Proceeding likewise finally

$$\Delta^n x^{(mh)} = [mh]^{(nh)} x^{(\overline{m-n}h)}$$

(*ii*) Let as an exercise.

Ex. 4·26. *Show that* (*i*) $\Delta ab^{cx} = (b^c-1)\, ab^{cx}$

(*ii*) $\Delta x^{(r)} = rx^{(r-1)}$

Sol. (*i*) $\Delta ab^{cx} = ab^{c(x+1)} - ab^{cx} = ab^{cx}(b^c-1)$

(*ii*) See Ex 4·25.

Ex. 4·27. *Show that*

(*i*) $n(n-1)+(n-1)(n-2)+\ldots\ldots+2.1 = \frac{1}{3}(n+1)n(n-1)$

(*ii*) $n(n-1)(n-2)+(n-1)(n-2)(n-3)+\ldots\ldots+3.2.1$

$$= \frac{1}{4}(n+1)n(n-1)(n-2)$$

ol. (*i*) Let $S=n(n-1)+(n-1)(n-2)+\ldots\ldots+2.1$

$$=n^{(2)}+(n-1)^{(2)}+\ldots\ldots+2^{(2)}$$

Now $\Delta n^{(3)}=3n^{(2)}$

$\therefore$ $$n^{(2)}=\frac{1}{3}\{(n+1)^{(3)}-n^{(3)}\}$$

Changing n to $n-1, n-2, \ldots\ldots 3$

$$(n-1)^{(2)}=\frac{1}{3}\{n^{(3)}-(n-1)^{(3)}\}$$

$$(n-2)^{(2)}=\frac{1}{3}\{(n-1)^{(3)}-(n-2)^{(3)}\}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$3^{(2)}=\frac{1}{3}\{4^{(3)}-3^{(3)}\}$$

Adding $$S=\frac{1}{3}\{(n+1)^{(3)}-3^{(3)}\}+2^{(1)}$$

$$=\frac{1}{3}\{(n+1)n(n-1)-3.2.1\}+2.1$$

$$=\frac{1}{3}(n+1)n(n-1)$$

(*ii*) Left as an exercise.

Ex. 4·28. *Express x^3-3x+1 in the factorial notation and use it to obtain its second difference.*

Sol. Let $\phi(x)=x^3-3x+1\equiv a_0+a_1x^{(1)}+a_2x^{(2)}+a_3x^{(3)}$

Then $$\Delta\phi(x)=a_1+2^{(1)}a_2x^{(1)}+3^{(1)}a_3x^{(2)}$$

$$\Delta^2\phi(x)=2^{(1)}a_2+3^{(1)}.\ 2^{(1)}a_3x^{(1)}$$

$$=2^{(2)}a_2+3^{(2)}a_3x^{(1)}$$

and $$\Delta^3\phi(x)=3^{(2)}a_3=3^{(3)}a_3=6a_3$$

$\therefore$ $$a_3=\frac{1}{6}\Delta^3\phi(0)$$

$$a_2=\frac{1}{2}\Delta^2\phi(0)$$

$$a_1=\Delta\phi(0)$$

$$a_0=\phi(0)$$

$$\therefore\ \phi(x)\equiv\phi(0)+\Delta\phi(0)x^{(1)}+\frac{1}{2}\Delta^2\phi(0)x^{(2)}+\frac{1}{6}\Delta^3\phi(0)x^{(3)}$$

Difference table of $\phi(x)$ is given below

x	$\phi(x)$	Δ	Δ^2	Δ^3
0	1			
		-2		
1	-1		6	
		4		6
2	3		12	
		16		
3	19			

$\therefore \quad \phi(x)=1-2x^{(1)}+3x^{(2)}+x^{(3)}$

$\therefore \quad \Delta^2\phi(x)=6+6x^{(1)}=6+6x$

Ex. 4·29. *A third degree polynomial $f(x)$ is passed through the points $(0,-1)$, $(1, 1)$, $(2, 1)$ and $(3,-2)$. Find the value at $x=1{\cdot}2$.*

Sol. Difference table of $f(x)$ is

x	$f(x)$	Δ	Δ^2	Δ^3
0	-1			
		2		
1	1		-2	
		0		-1
2	1		-3	
		-3		
3	-2			

$$\therefore \quad f(x)=f(0)+\Delta f(0)x^{(1)}+\frac{1}{2}\Delta^2 f(0)x^{(2)}+\frac{1}{6}\Delta^3 f(0)x^{(3)}$$

$$=-1+2x^{(1)}-x^{(2)}-\frac{1}{6}x^{(3)}$$

$$=-1+2x-x(x-1)-\frac{1}{6}x(x-1)(x-2)$$

$$=x\left[(x-1)\left\{-\frac{1}{6}(x-2)-1\right\}+2\right]-1$$

Let $g_0(x)=-\frac{1}{6}$, $\quad g_1(x)=g_0(x)(x-2)-1$,

$g_2(x)=(x-1)g_1(x)+2$, $\quad g_3(x)=xg_2(x)-1=f(x)$.

$$\therefore \quad g_0(1{\cdot}2)=-\frac{1}{6}, \quad g_1(1{\cdot}2)=\left(-\frac{1}{6}\right)(-0{\cdot}8)-1=-\frac{2{\cdot}6}{3}$$

$$g_2(1{\cdot}2)=(0{\cdot}2)\left(-\frac{2{\cdot}6}{3}\right)+2=\frac{5{\cdot}48}{3}$$

$$g_3(1{\cdot}2)=(1{\cdot}2)\left(\frac{5{\cdot}48}{3}\right)-1=1{\cdot}192=f(x)$$

The calculations are best carried out using the following computational scheme which is clearly related to synthetic division.

$$c_0 = g_0 \rightarrow (x-2) \rightarrow (x-2)g_0$$

$$(x-1)g_1 \leftarrow (x-1) \leftarrow \frac{c_1}{g_1}$$

$$\frac{c_2}{g_2} \rightarrow x \rightarrow \frac{\frac{xg_2}{c_3}}{g_3} = f(x)$$

For the above question

$$-\frac{1}{6} \rightarrow (1{\cdot}2-2) \rightarrow \frac{0{\cdot}4}{3}$$

$$-\frac{{\cdot}52}{3} \leftarrow (1{\cdot}2-1) \leftarrow -\frac{\frac{-1}{2{\cdot}6}}{3}$$

$$\frac{\frac{2}{5{\cdot}48}}{3} \rightarrow (1{\cdot}2) \rightarrow \frac{\frac{2{\cdot}192}{-1}}{1{\cdot}192} = f(x).$$

Ex. 4-30 *Show that*

$$y_0 + y_1\frac{x}{\underline{|1}} + y_2\frac{x^2}{\underline{|2}} + y_3\frac{x^3}{\underline{|3}} + \ldots\ldots = e^x\left\{ y_0 + x\Delta y_0 + \frac{x^2}{\underline{|2}}\Delta^2 y_0 + \ldots \right\}$$

Sol. $\text{L.H.S.} = \left(1 + \frac{x}{\underline{|1}}E + \frac{x^2}{\underline{|2}}E^2 + \ldots \right) y_0$

$$= e^{xE} y_0 = e^{x(1+\Delta)} y_0$$

$$= e^x . e^{x\Delta} y_0 = e^x \left\{ 1 + x\Delta + \frac{x^2\Delta^2}{\underline{|2}} + \ldots\ldots \right\} y_0$$

$$= e^x \left\{ y_0 + x\Delta y_0 + \frac{x^2}{\underline{|2}}\Delta^2 y_0 + \ldots\ldots \right\} = \text{R.H.S.}$$

Ex. 4-31. *Show that*

$$xy_1 + x^2y_2 + x^3y_3 + \ldots\ldots = \frac{x}{1-x}y_1 + \frac{x^2}{(1-x)^2}\Delta y_1 + \frac{x^3}{(1-x)^3}\Delta^2 y_1 + \ldots\ldots$$

Sol. $\text{L.H.S.} = (xE + x^2E^2 + x^3E^3 + \ldots\ldots) y_0$

$$= \frac{xE}{1-xE} y_0 = \frac{xE}{1-x-x\Delta} y_0$$

$$= \frac{xE}{1-x}\left[1 - \frac{x\Delta}{1-x} \right]^{-1} y_0$$

$$=\frac{xE}{1-x}\left\{1+\frac{x}{1-x}\Delta+\frac{x^2}{(1-x)^2}\Delta^2+\ldots\ldots\right\}y_0$$

$$=\frac{x}{1-x}y_1+\frac{x^2}{(1-x)^2}\Delta y_1+\frac{x^3}{(1-x)^3}\Delta^2 y_1+\ldots\ldots$$

Ex. 4-32. *Show that*

$y_n=y_{n-1}+\Delta y_{n-2}+\Delta^2 y_{n-3}+\ldots\ldots+\Delta^{n-1}y_{n-n}+\Delta^n y_{n-n}$

Sol. R.H.S. $=E^{-1}\left\{1+\frac{\Delta}{E}+\frac{\Delta^2}{E^2}+\ldots\ldots+\frac{\Delta^{n-1}}{E^{n-1}}\right\}y_n+\Delta^n y_{n-n}$

$$=E^{-1}\{1+\nabla+\nabla^2+\ldots+\nabla^{n-1}\}y_n+\Delta^n y_{n-n}$$

$$=E^{-1}\frac{(1-\nabla^n)}{1-\nabla}y_n+\nabla^n y_{n-n}$$

$$=\frac{(1-\nabla^n)}{E-E\nabla}y_n+\Delta^n y_{n-n}=\frac{1-\nabla^n}{E-\Delta}\;y_n+\frac{\Delta^n}{E^n}y_n$$

$$=(1-\nabla^n)y_n+\nabla^n y_n=y_n.$$

Ex. 4-33. *Show that*

$$\Delta x^m-\frac{1}{2}\;\Delta^2x^m+\frac{1.3}{2.4}\;\Delta^3x^m-\frac{1.3.5}{2.4.6}\;\Delta^4x^m+\ldots\ldots+m \text{ terms}$$

$$=\Delta E^{-1/2}x^m=\left(x+\frac{1}{2}\right)^m-\left(x-\frac{1}{2}\right)^m$$

where $h=1$.

Sol. Since $\Delta^r x^m=0$ for $r>m$,

L.H.S. $=\Delta x^m-\frac{1}{2}\Delta^2x^m+\frac{1.3}{2.4}\Delta^3x^m-\frac{1.3.5}{2.4.6}\Delta^4x^m+\ldots$up to ∞

$$=\left(\Delta-\frac{1}{2}\;\Delta^2+\frac{1.3}{2.4}\;\Delta^3\ldots\ldots\right)x^m$$

$$=\Delta(1+\Delta)^{-1/2}\,x^m=\Delta E^{-1/2}x^m$$

$$=\Delta\left(x-\frac{1}{2}\right)^m=\left(x+\frac{1}{2}\right)^m-\left(x-\frac{1}{2}\right)^m$$

Ex. 4-34. *Show that*

$y_{x+n}=y_n+{}^xc_1\Delta y_{n-1}+{}^{x+1}c_2\Delta^2 y_{n-2}+{}^{x+2}c_3\Delta^3 y_{n-3}+\ldots\ldots$

Sol. R.H.S. $=\left(1+{}^xc_1\frac{\Delta}{E}+{}^{x+1}c_2\frac{\Delta^2}{E^2}+{}^{x+2}c_3\;\frac{\Delta^3}{E^3}+\ldots\ldots\right)y_n$

$$=\left(1+x\,\frac{\Delta}{E}+\frac{(x+1)x}{2\,!}\frac{\Delta^2}{E^2}+\frac{(x+2)(x+1)x}{3\,!}\frac{\Delta^3}{E^3}+\ldots\ldots\right)y_n$$

$$=\left(1-\frac{\Delta}{E}\right)^{-x}y_n=E^x y_n=y_{x+n}$$

Ex. 4-35 *Show that* $y_x-y_{x+1}+y_{x+2}-y_{x+3}+\ldots\ldots$

$$=\frac{1}{2}\Big[\ y_{x-\frac{1}{2}}-\frac{1}{8}\ \Delta^2y_{x-\frac{3}{2}}+\frac{1.3}{2\,!}\Big(\frac{1}{8}\Big)^2\Delta^4y_{x-\frac{5}{2}}-\frac{1.3.5}{3\,!}\Big(\frac{1}{8}\Big)^3\Delta^6y_{x-\frac{7}{2}}+\ldots\ldots\Big]$$

Sol. $$\text{R.H.S.}=\frac{1}{2}E^{-1/2}\Big\{1-\frac{1}{8}\ \frac{\Delta^2}{E}+\frac{1.3}{2\,!}\Big(\frac{1}{8}\Big)^2\frac{\Delta^4}{E^2}-\frac{1.3.5}{3\,!}\Big(\frac{1}{8}\Big)^3\frac{\Delta^6}{E^3}+\ldots\ldots\Big\}y_x$$

$$=\frac{1}{2}\ E^{-1/2}\Big(1+\frac{1}{4}\ \frac{\Delta^2}{E}\Big)^{-1/2}y_x$$

$$=\frac{1}{2}\Big(E+\frac{1}{4}\Delta^2\Big)^{-1/2}y_x=\frac{1}{2}\Big\{E+\frac{1}{4}(E-1)^2\Big\}^{-1/2}y_x$$

$$=\frac{1}{2}\Big\{\frac{(E+1)^2}{4}\Big\}^{-1/2}y_x=(E+1)^{-1}\,y_x$$

$$=(1-E+E^2-E^3+\ldots\ldots)\,y_x$$

$$=y_x-y_{x+1}+y_{x+2}-y_{x+3}+\ldots\ldots=\text{L.H.S.}$$

Ex. 4-36. *Show that*

$$y_x-\frac{1}{8}\Delta^2y_{x-1}+\frac{1.3}{8.16}\Delta^4y_{x-2}-\frac{1.3.5}{8.16.24}\Delta^6y_{x-3}+\ldots\ldots$$

$$=y_{x+\frac{1}{2}}-\frac{1}{2}\ \Delta y_{x+\frac{1}{2}}+\frac{1}{4}\Delta^2y_{x+\frac{1}{2}}-\frac{1}{8}\Delta^3y_{x+\frac{1}{2}}+\ldots$$

Sol. $$\text{L.H.S.}=\Big(1-\frac{1}{8}\ \frac{\Delta^2}{E}+\frac{1.3}{8.16}\ \frac{\Delta^4}{E^2}-\frac{1.3.5}{8.16.24}\ \frac{\Delta^6}{E^3}+\ldots\ldots\Big)y_x$$

$$=\Big(1+\frac{1}{4}\frac{\Delta^2}{E}\Big)^{-1/2}y_x=\Big\{\frac{4E+(E-1)^2}{4E}\Big\}^{-1/2}y_x$$

$$=2E^{1/2}\,(1+E)^{-1}\,y_x$$

$$=2E^{1/2}\,\{2+\Delta\}^{-1}\,y_x$$

$$=E^{1/2}\Big\{1+\frac{\Delta}{2}\Big\}^{-1}y_x$$

$$=E^{1/2}\Big\{1-\frac{\Delta}{2}+\frac{1}{4}\Delta^2-\frac{1}{8}\ \Delta^3\ldots\ldots\Big\}y_x$$

$$=y_{x+\frac{1}{2}}-\frac{1}{2}\ \Delta y_{x+\frac{1}{2}}+\frac{1}{4}\Delta^2y_{x+\frac{1}{2}}\ldots\ldots\ldots$$

$$=\text{R.H.S.}$$

Ex. 4·37. *Show that* $y_0+{}^nc_1\,\Delta y_1+{}^nc_2\,\Delta^2 y_2+{}^nc_3\,\Delta^3 y_3+\ldots\ldots$
$=y_n+{}^nc_1\,\Delta^2 y_{n-1}+{}^nc_2\,\Delta^4 y_{n-2}+\ldots\ldots$

Sol. R.H.S. $=\left(1+{}^nc_1\frac{\Delta^2}{E}+{}^nc_2\frac{\Delta^4}{E^2}+\ldots\ldots\right)y_n$

$$=\left(1+\frac{\Delta^2}{E}\right)^n y_n$$
$$=E^{-n}\{E+\Delta(E-1)\}^n\,y_n$$
$$=E^{-n}\{E-\Delta+\Delta E\}^n\,y_n$$
$$=E^{-n}\{1+\Delta E\}^n\,y_n$$
$$=E^{-n}\{1+{}^nc_1\,\Delta E+{}^nc_2\,\Delta^2E^2+{}^nc_3\,\Delta^3E^3+\ldots\ldots\}\,y_n$$
$$=y_0+{}^nc_1\,\Delta y_1+{}^nc_2\,\Delta^2 y_2+{}^nc_3\,\Delta^3 y_3+\ldots\ldots$$
$$=\text{L.H.S.}$$

Ex. 4·38. *Show that*

$$\Delta^n y_{n-n}=y_n-{}^nc_1\,y_{n-1}+{}^nc_2\,y_{n-2}+\ldots\ldots+(-1)^n\,y_{n-n}$$

Sol. R.H.S. $=(1-{}^nc_1\,E^{-1}+{}^nc_2\,E^{-2}\ldots\ldots)\,y_n$

$$=(1-E^{-1})^n\,y_n=E^{-n}(E-1)^n\,y_n$$
$$=E^{-n}\,\Delta^n y_n=\Delta^n y_{n-n}.$$

Ex 4·39. *Show that*

$$U_0+U_1+\ldots\ldots+U_n={}^{n+1}c_1\,U_0+{}^{n+1}c_2\,\Delta U_0+\ldots\ldots+\Delta^n U_0$$

Sol. L.H.S. $=(1+E+E^2+\ldots\ldots+E^n)U_0$

$$=\frac{E^{n+1}-1}{E-1}\;U_0$$
$$=\frac{1}{\Delta}\,\{(1+\Delta)^{n+1}-1\}\,U_0$$
$$=\frac{1}{\Delta}\,\{{}^{n+1}c_1\,\Delta+{}^{n+1}c_2\,\Delta^2+\ldots\ldots+\Delta^{n+1}\}\,U_0$$
$$={}^{n+1}c_1U_0+{}^{(n+1)}c_2\,\Delta U_0+\ldots\ldots+\Delta^n U_0.$$

Ex. 4·40. *Sum the series*

$1^3+3^3+5^3+\ldots\ldots$ *up to n terms*

Sol. $U_n=n$th *term* $=(2n-1)^3$

Difference table is

x	U_n	Δ	Δ^2	Δ^3
1	1			
2	27	26		
3	125	98	72	
4	343	218	120	48
5	729	386	168	48

From Ex. 4-39

$$U_1+U_2+\ldots\ldots+U_n={}^nc_1\,U_1+{}^nc_2\,\Delta U_1+\ldots\ldots+\Delta^{n-1}\,U_1$$

$$\therefore \qquad S=n+\frac{n(n-1)}{2\,!}\,26+\frac{n(n-1)(n-2)}{3\,!}\,72$$
$$+\frac{n(n-1)(n-2)(n-3)}{4\,!}\,48$$
$$=n\{1+13(n-1)+12(n^2-3n+2)$$
$$+2(n^3-6n^2+11n-6)\}$$
$$=n\{2n^3-n\}=n^2(2n^2-1)$$

4.3. Interpolation Formulae.

(*I*) *When the values of the argument are not equidistant, the following two formulae are used :*

(*i*) *Newton's divided difference formula.*

$$f(x)=f(x_0)+(x-x_0)f(x_0,\,x_1)+(x-x_0)(x-x_1)f(x_0,\,x_1,\,x_2)$$
$$+\ldots\ldots+(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})f(x_0,\,x_1\ldots x_n)$$

Derivation. Let $f(x)$ be a function which takes n values $f(x_0), f(x_1),\ldots\ldots f(x_n)$ at the points $x_0,\,x_1,\,\ldots\ldots x_n$ which are not necessarily equidistant. Then by def.

$$f(x,\,x_0)=\frac{f(x)-f(x_0)}{x-x_0}$$

$$\therefore \qquad f(x)=f(x_0)+(x-x_0)f(x,\,x_0)$$

$$f(x,\,x_0,\,x_1)=\frac{f(x,\,x_0)-f(x_0,\,x_1)}{x-x_1}$$

$$\therefore \quad f(x,\,x_0)=f(x_0,\,x_1)+(x-x_1)f(x,\,x_0,\,x_1)$$

Similarly

$$f(x,\,x_0,\,x_1)=f(x_0,\,x_1,\,x_2)+(x-x_2)f(x,\,x_0,\,x_1,\,x_2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f(x,\,x_0,\,x_1,\ldots x_{n-1})=f(x_0,\,x_1,\ldots x_n)+(x-x_n)f(x,\,x_0,\ldots x_n)$$

Multiplying eqs. by $(x-x_0)$, $(x-x_0)(x-x_1)$,......$(x-x_0)(x-x_1)$ ......$(x-x_{n-1})$ and adding.

$$f(x)=f(x_0)+(x-x_0)f(x_0,\,x_1)+(x-x_0)(x-x_1)f(x_0,\,x_1,\,x_2)+\ldots$$
$$+(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})f(x_0,\,\ldots\,x_n)+R$$

where $\qquad R=(x-x_0)(x-x_1)\ldots\ldots(x-x_n)f(x,\,x_0,\,\ldots\,x_n).$

This formula, due to Newton, is called Newton's divided-difference interpolation formula. When the values of $f(x)$ for $x=x_0,\,x_1,\,\ldots\,x_n$ are known, the evaluation of $f(x)$ is reduced to the problem of evaluating R. If it is known or negligible, the required value of $f(x)$ can be calculated from above formula. In the case of a polynomial of nth degree, since $(n+1)$th order divided difference is zero, $R=0$.

$\therefore$ If $f(x)$ is polynomial of nth degree,

$$f(x)=f(x_0)+(x-x_0)f(x_0, x_1)+(x-x_0)(x-x_1)f(x_0, x_1, x_2)+\ldots\ldots +(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})f(x_0, \ldots x_n)$$

(*ii*) **Lagrange's formula**

$$f(x)=\sum_{i=0}^{n} f(x_i)\left\{\prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}\right\}$$

Drivation. Let $f(x)$ be a function which takes values $f(x_0)$, $f(x_1)$, $\ldots\ldots f(x_n)$ for $(n+1)$ distinct points x_0, $x_1, \ldots\ldots x_n$ and it is required to find a polynomial

$$P(x)=a_0+a_1x+\ldots\ldots+a_nx^n$$

with the property that

$$P(x_i)=f(x_i) \quad i=0, 1, \ldots\ldots\ldots n.$$

The resulting polynomial is called Lagrange's interpolation polynomial or formula.

Evidently the unique polynomial $P(x)$ (of degree $\leqslant n$) with the required property is

$$P(x)=L_0(x)f(x_0)+L_1(x)f(x_1)+\ldots\ldots+L_n(x)f(x_n)$$

where $L_i(x)$ is a polynomial of degree n in x with the property that

$$\begin{aligned} L_i(x_j)&=0 \quad & j\neq i\\ &=1 & j=i \end{aligned}$$

Evidently $L_i(x)$ has the form

$$L_i(x)\equiv A_i\,(x-x_0)(x-x_1)\ldots\ldots(x-x_{i-1})(x-x_{i+1})\ldots\ldots(x-x_n)$$

$$\therefore \quad A_i=\frac{L_i(x_i)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

$$=\frac{1}{\prod\limits_{j\neq i}^{n}(x_i-x_j)}$$

$$\therefore \quad L_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

$$\therefore \quad P(x)=\sum_{i=0}^{n} f(x_i)\left\{\prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}\right\}$$

Note. The polynomial $P(x)$ is unique because if $P(x)$ and $Q(x)$ be two such polynomials, then $\{P(x)-Q(x)\}$, a polynomial of degree $\leqslant n$, will have $(n+1)$ zeros $x_0, x_1 \ldots\ldots x_n$ which is possible only when $P(x) \equiv Q(x)$.

Ex. 4·41. *Use Newton's formula for unequal intervals to find $f(8)$ from the following set of values:*

x:	4	5	7	10	11	13
$f(x)$:	2	4	8	104	114	452

Sol.

Divided difference table

x	$f(x)$	*1st order*	*2nd order*	*3rd order*	*4th order*	*5th order*
4	2					
		2				
5	4		0			
		2		1		
7	8		6		$-\frac{5}{12}$	
		32		$-\frac{23}{12}$		$\frac{5}{24}$
10	104		$-\frac{11}{2}$		$\frac{35}{24}$	
		10		$\frac{117}{12}$		
11	114		53			
		169				
13	452					

$$\therefore \quad f(8)=2+(8-4)(2)+(8-4)(8-5)(0)$$

$$+(8-4)(8-5)(8-7)(1)+(8-4)(8-5)(8-7)(8-10)\left(\frac{-5}{12}\right)$$

$$+(8-4)(8-5)(8-7)(8-10)(8-11)\left(\frac{5}{24}\right)=47$$

Ex. 4·42. *Given the values:*

x:	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

form the table of divided differences and use it to obtain $f(2)$ and $f(15)$.

Sol.

Divided difference table

x	$f(x)$	*1st order*	*2nd order*	*3rd order*	*4th order*	*5th order*
4	48					
		52				
5	100		15			
		97		1		
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		
11	1210		33			
		409				
13	2028					

$$\therefore \quad f(2)=48+(2-4)(52)+(2-4)(2-5)(15)$$

$$+(2-4)(2-5)(2-7)(1)=4$$

$$f(15)=48+(15-4)(52)+(15-4)(15-5)(15)$$

$$+(15-4)(15-5)(15-7)(1)=3150$$

Ex. 4·43. *The function 3^x tables, as it should, the values 1, 3, 9 and 81 when x equals 0, 1, 2, and 4 respectively. Obtain the value corresponding to $x=3$ and explain why the resulting value differ from $3^3=27$.*

Sol.

Divided difference table

x	3^x	1st order	2nd order	3rd order
0	1			
		2		
1	3		2	
		6		2
2	9		10	
		36		
4	81			

$$f(3)=1+(3-0)(2)+(3-0)(3-1)(2)+(3-0)(3-1)(3-2)(2)=31$$

Interpolating value differs from actual value because $f(x)=3^x$ is not a polynomial.

Ex. 4·44. *The mode of a certain frequency curve $y=f(x)$ is very near $x=9$ and the values of the frequency density for $x=8·9$, 9·0 and 9·3 are respectively equal to 0·30, 0·35 and 0·25. Calculate the approximate value of the mode.*

Sol.

Divided difference table

x	$f(x)$	1st order	2nd order
8·9	0·30		
		0·5	
9·0	0·35		−2·08
		−0·33	
9·3	0·25		

$\therefore \quad f(x)=0·30+(x-8·9)(0·5)+(x-8·9)(x-9·0)(-2·08)$

$\therefore \quad f'(x)=0·5-(2·08)\{(x-9·0)+(x-8·9)\}$

For modal value of x, $f'(x)=0$

$\therefore \quad 0·5-(2·08)\{2x-17·9\}=0$

$\therefore \quad 4·16\,x=0·5+37·232=37·732$

$\therefore \quad x=9·07.$

Ex. 4·45. *Use Lagrange's formula for interpolation to derive the form of the function $y=f(x)$, given*

x :	0	2	3	6
$f(x)$:	659	705	729	804

Sol. $$f(x)=\frac{(x-2)(x-3)(x-6)}{(0-2)(0-3)(0-6)}(659)+\frac{(x-0)(x-3)(x-6)}{(2-0)(2-3)(2-6)}(705)$$

$$+\frac{(x-0)(x-2)(x-6)}{(3-0)(3-2)(3-6)}(729)+\frac{(x-0)(x-2)(x-3)}{(6-0)(6-2)(6-3)}(804)$$

$$=-\frac{1}{72}x^3+\frac{29}{72}x^2+\frac{89}{4}x+659.$$

Ex. 4·46. *Use Lagrange's formula to find f(5) from the following data :*

x :	2	3	4	6	7
$f(x)$:	1	5	13	61	125

Sol.

$$f(5)=\frac{(5-3)(5-4)(5-6)(5-7)}{(2-3)(2-4)(2-6)(2-7)}(1)+\frac{(5-2)(5-4)(5-6)(5-7)}{(3-2)(3-4)(3-6)(3-7)}(5)$$

$$+\frac{(5-2)(5-3)(5-6)(5-7)}{(4-2)(4-3)(4-6)(4-7)}(13)+\frac{(5-2)(5-3)(5-4)(5-7)}{(6-2)(6-3)(6-4)(6-7)}(61)$$

$$+\frac{(5-2)(5-3)(5-4)(5-6)}{(7-2)(7-3)(7-4)(7-6)}(125)=28{\cdot}6$$

Ex. 4·47. *The following values of the function f(x) for values of x are given :*

$$f(1)=4, f(2)=5, f(7)=5, f(8)=4$$

Find the value of f(6) and also the value of x for which f(x) is maximum.

Sol. $$f(x)=\frac{(x-2)(x-7)(x-8)}{(1-2)(1-7)(1-8)}.4+\frac{(x-1)(x-7)(x-8)}{(2-1)(2-7)(2-8)}.5$$

$$+\frac{(x-1)(x-2)(x-8)}{(7-1)(7-2)(7-8)}.5+\frac{(x-1)(x-2)(x-7)}{(8-1)(8-2)(8-7)}.4$$

$$=-\frac{1}{6}x^2+\frac{3}{2}x+\frac{8}{3}.$$

$\therefore$ $$f(6)=\frac{17}{3}$$

$$f'(x)=-\frac{1}{3}x+\frac{3}{2}$$

Put $f'(x)=0$

$\therefore$ $$x=\frac{9}{2}.$$

Since $f''(x)=-\frac{1}{3}<0$, $f(x)$ is max for $x=\frac{9}{2}$.

Ex. 4·48. *The following table gives the normal weights of babies during the first 12 months of life.*

Age (in months)	0	2	5	8	10	12
Weight (in lbs)	7·5	10·25	15	16	18	21

Estimate the weight of the baby at the age of 7 months.

Sol. $$f(7)=\frac{(7-2)(7-5)(7-8)(7-10)(7-12)}{(0-2)(0-5)(0-8)(0-10)(0-12)}(7\cdot5)$$
$$+\frac{(7-0)(7-5)(7-8)(7-10)(7-12)}{(2-0)(2-5)(2-8)(2-10)(2-12)}(10\cdot25)$$
$$+\frac{(7-0)(7-2)(7-8)(7-10)(7-12)}{(5-0)(5-2)(5-8)(5-10)(5-12)}(15)$$
$$+\frac{(7-0)(7-2)(7-5)(7-10)(7-12)}{(8-0)(8-2)(8-5)(8-10)(8-12)}(16)$$
$$+\frac{(7-0)(7-2)(7-5)(7-8)(7-12)}{(10-0)(10-2)(10-5)(10-8)(10-12)}(18)$$
$$+\frac{(7-0)(7-2)(7-5)(7-8)(7-10)}{(12-0)(12-2)(12-5)(12-8)(12-10)}(21)$$
$$=15\cdot67.$$

Ex. 4·49. *The observed values of a function are respectively 168 120, 72 and 63 at four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable ?*

Sol. $$f(6)=\frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)}(168)$$
$$+\frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)}(120)+\frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}(72)$$
$$+\frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}(63)$$
$$=147.$$

Ex. 4·50. *Given the following table, find* $\log_{10} 656$.

x :	654	658	659	661
$f(x)=\log_{10} x$:	2·8156	2·8182	2·8189	2·8202

Sol.

$$f(656)=\frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}(2\cdot8156)$$
$$+\frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)}(2\cdot8182)$$
$$+\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)}(2\cdot8189)$$
$$+\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}(2\cdot8202)$$
$$=2\cdot81681=2\cdot8168.$$

Ex. 4·51. *Four equidistant values U_{-1}, U_0, U_1 and U_2 being given, a value is interpolated by Lagrange's formula. Show that it may be written in the form*

$$U_x = yU_0 + xU_1 + \frac{y(y^2-1)}{3!}\Delta^2 U_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 U_0$$

where $x+y=1$.

Sol. $$\text{R.H.S.} = (1-x)U_0 + xU_1 + \frac{(1-x)\{(1-x)^2-1\}}{3!}(E-1)^2 U_{-1} + \frac{x(x^2-1)}{3!}(E-1)^2 U_0$$

$$= (1-x)U_0 + xU_1 + \frac{(1-x)(x^2-2x)}{3!}(U_1 - 2U_0 + U_{-1}) + \frac{x(x^2-1)}{3!}(U_2 - 2U_1 + U_0)$$

$$= \frac{x(1-x)(x-2)}{3!}U_{-1} + U_0\left\{(1-x) - \frac{1}{3}x(1-x)(x-2) + \frac{x(x^2-1)}{6}\right\}$$
$$+ U_1\left\{x + \frac{x(1-x)(x-2)}{6} - \frac{1}{3}x(x^2-1)\right\} + U_2\left\{\frac{x(x^2-1)}{3!}\right\}$$

$$= \frac{x(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}U_{-1} + \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}U_0$$
$$+ \frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)}U_1 + \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}U_2$$

$= U_x$.

Ex. 4·52. *Given $\log 100 = 2$, $\log 101 = 2{\cdot}0043$, $\log 103 = 2{\cdot}0128$, $\log 104 = 2{\cdot}0170$, find $\log 102$.*

Sol. $$\log 102 = \frac{(102-101)(102-103)(102-104)}{(100-101)(100-103)(100-104)}(2)$$
$$+ \frac{(102-100)(102-103)(102-104)}{(101-100)(101-103)(101-104)}(2{\cdot}0043)$$
$$+ \frac{(102-100)(102-101)(102-104)}{(103-100)(103-101)(103-104)}(2{\cdot}0128)$$
$$+ \frac{(102-100)(102-101)(102-103)}{(104-100)(104-101)(104-103)}(2{\cdot}0170)$$
$$= 2{\cdot}00857 \simeq 2{\cdot}0086.$$

(2) *When the values of the argument are equidistant the following formulae are used:*

(*i*) $$f(x) = y_0 + U^{(1)}\Delta y_0 + \frac{U^{(2)}}{2!}\Delta^2 y_0 + \ldots\ldots + U^{(n)}\frac{\Delta^n y_0}{n!}$$

(Newton's forward interpolation formula)

It is used to interpolate near the beginning of the table.

(ii) $$f(x)=y_n+U^{(1)}\nabla y_n+\frac{U^{(2)}}{2!}\nabla^2 y_n+\ldots\ldots+\frac{U^{(n)}}{n!}\nabla^n y_n$$

(Newton's backward interpolation formula)

It is used to interpolate near the end of the table.

(iii) **Central Difference formulae.**

(a) $$f(x)=y_0+U\mu\delta y_0+\frac{U^2}{2!}\delta^2 y_0+\frac{U(U^2-1)}{3!}\mu\delta^3 y_0$$
$$+\frac{U^2(U^2-1)}{4!}\delta^4 y_0+\ldots\ldots+\frac{U^2(U^2-1^2)\ldots\{U^2-(r-1)^2\}}{2r!}\delta^{2r} y_0$$
$$+\frac{U(U^2-1^2)\ldots\ldots(U^2-r^2)}{(2r+1)!}\mu\delta^{2r+1} y_0+\ldots\ldots$$ *(Stirling's formula)*

(b) $$f(x)=\mu y_{\frac{1}{2}}+V\delta y_{\frac{1}{2}}+\frac{1}{2!}\left(V^2-\frac{1}{4}\right)\mu\delta^2 y_{\frac{1}{2}}$$
$$+\frac{1}{3!}V\left(V^2-\frac{1}{4}\right)\delta^3 y_{\frac{1}{2}}+\ldots\ldots+\frac{1}{2r!}\left(V^2-\frac{1}{4}\right)\left(V^2-\frac{9}{4}\right)\ldots\ldots$$
$$\left\{V^2-\frac{(2r-1)^2}{4}\right\}\mu\delta^{2r} y_{\frac{1}{2}}+\frac{1}{(2r+1)!}V\left(V^2-\frac{1}{4}\right)\left(V^2-\frac{9}{4}\right)\ldots\ldots$$
$$\left\{V^2-\frac{(2r-1)^2}{4}\right\}\delta^{2r+1} y_{\frac{1}{2}}+\ldots\ldots$$ *(Bessel's formula)*

where $V=U-\frac{1}{2}$.

(c) $$f(x)=y_0+U^{(1)}\delta y_{-\frac{1}{2}}+\frac{(U+1)^{(2)}}{2!}\delta^2 y_0+\ldots\frac{(U+r)^{(2r)}}{2r!}\delta^{2r} y_0$$
$$+\frac{(U+r)^{(2r+1)}}{(2r+1)!}\delta^{2r+1} y_{-\frac{1}{2}}+\ldots$$ *(Gauss backward formula)*

It is used when U is negative.

(d) $$f(x)=y_0+U^{(1)}\delta y_{\frac{1}{2}}+\frac{U^{(2)}}{2!}\delta^2 y_0+\ldots\ldots$$
$$+\frac{(U+\overline{r-1})^{(2r-1)}}{(2r-1)!}\delta^{2r-1} y_{\frac{1}{2}}+\frac{(U+\overline{r-1})^{(2r)}}{2r!}\delta^{2r} y_0+\ldots\ldots$$

(Gauss forward formula)

It is used when U is positive.

Central difference formulae are used to interpolate near the middle of the table.

Ex. 4-53. *Derive Newton-Gregory forward interpolation formula.*

Sol. Let $y=f(x)$ be a function which assumes the values y_0, $y_1, \ldots\ldots y_n$ for equidistant values $x_0, x_1 \ldots\ldots x_n$ of x. Let

$$I(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\ldots\ldots$$
$$+\ldots\ldots+a_n(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})$$

where the coefficients $a_0, a_1, \ldots\ldots a_n$ are to be determined s.t. $I(x)$ takes the values $y_0, y_1 \ldots\ldots y_n$ for $x=x_0, x_1 \ldots\ldots x_n$ respectively. Since the values x_i are equidistant,

$$x_1-x_0=x_2-x_1=\ldots\ldots=x_n-x_{n-1}=h \text{ (say)}$$

$$\therefore \quad x_i-x_0=ih \qquad i=1,\ldots\ldots n$$

Now $\quad I(x_0)=a_0=y_0$

$$I(x_1)=a_0+a_1(x_1-x_0)=y_1$$

$$\therefore \quad a_1=\frac{y_1-y_0}{h}=\frac{1}{h}\Delta y_0$$

$$I(x_2)=a_0+a_1(x_2-x_0)+a_2(x_2-x_0)(x_2-x_1)=y_2$$

$$\therefore \quad a_2=\frac{1}{2h^2}\{y_2-2y_1+y_0\}=\frac{1}{2\,!\,h^2}\Delta^2 y_0$$

Similarly $\quad a_3=\dfrac{\Delta^3 y_0}{3\,!\,h^3},\ldots\ldots,a_n=\dfrac{1}{n\,!\,h^n}\Delta^n y_0$

$$\therefore \quad I(x)=y_0+\frac{(x-x_0)}{h}\Delta y_0+\frac{(x-x_0)(x-x_1)}{2\,!\,h^2}\Delta^2 y_0$$
$$+\ldots\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{n\,!\,h^n}\Delta^n y_0$$

Let $\quad U=\dfrac{x-x_0}{h}$

Then $\quad \dfrac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{h^n}$

$$=\left(\frac{x-x_0}{h}\right)\left(\frac{x-x_0}{h}-1\right)\ldots\ldots\left(\frac{x-x_0}{h}-\overline{n-1}\right)$$
$$=U(U-1)\ldots\ldots(U-n-1)=U^{(n)}$$

$$\therefore \quad I(x)=y_0+U^{(1)}\Delta y_0+\frac{U^{(2)}}{2\,!}\Delta^2 y_0+\ldots\ldots+\frac{U^{(n)}}{n\,!}\Delta^n y_0$$

Ex. 4·54. *Given the following pairs of corresponding values of x and y.*

x :	20	25	30	35
y :	73	198	573	1198

Find the estimated value af y for $x=22$.

Sol.

Difference table

x	y	Δy	$\Delta^2 y$	
20	73			$U=\dfrac{x-20}{5}$
		125		
25	198		250	$=\dfrac{22-20}{5}=0{\cdot}4$
		375		
30	573		250	
		625		
35	1198			

$$\therefore \quad U_{22}=73+(0{\cdot}4)(125)+\frac{(0{\cdot}4)(-0{\cdot}6)}{2\,!}(250)$$

$$=93.$$

Ex. 4·55. *Given $\sin 45°=0{\cdot}7071$, $\sin 50°=0{\cdot}7660$, $\sin 55°=0{\cdot}8192$, $\sin 60°=0{\cdot}8660$. Find $\sin 48°$.*

Sol.

Difference table

x^0	$y=\sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	
45	0·7071				$U=\dfrac{48-45}{5}$
		0·0589			
50	0·7660		−0·0057		$=0{\cdot}6$
		0·0532		−0·0007	
55	0·8192		−0·0064		
		0·0468			
60	0·8660				

$$\therefore \quad \sin 48°=0{\cdot}7071+(0{\cdot}6)(0{\cdot}0589)+\frac{(0{\cdot}6)(-0{\cdot}4)}{2\,!}(-0{\cdot}0057)$$

$$+\frac{(0{\cdot}6)(-0{\cdot}4)(-1{\cdot}4)}{3\,!}(-0{\cdot}0007)$$

$$=0{\cdot}7431.$$

Ex. 4·56. *Find the number of men getting wages between Rs. 10 and Rs. 15 from the following table :*

Wages per week (in Rs.)	0—10	10—20	20—30	30—40
Frequency	9	30	35	42

Sol. Rewriting data in cumulative frequency form and taking differences :

Difference table

No. of persons getting less than	*Freq.*	Δ	Δ^2	Δ^3	
10	9				
		30			
20	39		5		$U=\frac{15-10}{10}$
		35		2	$=0{\cdot}5$
30	74		7		
		42			
40	116				

$\therefore$ **No. of men getting less than Rs. 15**

$$=9+(0{\cdot}5)(30)+\frac{(0{\cdot}5)(-0{\cdot}5)}{2\,!}(5)$$

$$+\frac{(0{\cdot}5)(-0{\cdot}5)(-1{\cdot}5)}{3\,!}(2)=23{\cdot}5\simeq 24.$$

$\therefore$ **No. of men getting between Rs. 10 and Rs. 15**

$$=24-9=15.$$

Ex. 4·57. *Use Newton's formula for interpolation to find annual net premium at age 25 from the table given below :*

Age	*Annual Net premium*	*Age*	*Annual Net Premium*
20	0·01427	28	0·01772
24	0·01581	32	0·01996

Sol.

Difference table

Age	*Premium*	Δ	Δ^2	Δ^3	
20	0·01427				
		0·00154			
24	0·01581		0·00037		$U=\frac{25-20}{4}$
		0·00191		−0·00004	$=1{\cdot}25$
28	0·01772		0·00033		
		0·00224			
32	0 01996				

$\therefore$ **Premium at age 25** $=0{\cdot}01427+(1{\cdot}25)(0{\cdot}00154)$

$$+\frac{(1{\cdot}25)(0{\cdot}25)}{2\,!}(0{\cdot}00037)+\frac{(1{\cdot}25)(0{\cdot}25)(-0{\cdot}75)}{3\,!}(-0{\cdot}00004)$$

$$\simeq 0{\cdot}01625.$$

Ex. 4·58. *The following are the marks obtained by 492 candidates in a certain examination.*

Not more than...	*Candidates*	*Not more than...*	*Candidates*
40	212	60	460
45	296	65	481
50	368	70	490
55	429	75	492

Find out the number of candidates who secured more than 42 but not more than 45 marks.

Sol.

Difference table

x	*Freq.*	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
40	212	84	-12	1	-20	59	-120	210
45	296	72	-11	-19	39	-61	90	
50	368	61	-30	20	-22	29		
55	429	31	-10	-2	7			
60	460	21	-12	5				
65	481	9	-7					
70	490	2						
75	492							

$$U=\frac{42-40}{5}=0{\cdot}4$$

$\therefore$ Number of candidates getting marks less than 42

$$=212+(0{\cdot}4)(84)+\frac{(0{\cdot}4)(-0{\cdot}6)}{2\,!}(-12)+\frac{(0{\cdot}4)(-0{\cdot}6)(-1{\cdot}6)}{3\,!}(1)$$

$$+\frac{(0{\cdot}4)(-0{\cdot}6)(-1{\cdot}6)(-2{\cdot}6)}{4\,!}(-20)$$

$$+\frac{(0{\cdot}4)(-0{\cdot}6)(-1{\cdot}6)(-2{\cdot}6)(-3{\cdot}6)}{5\,!}(59)$$

$$+\frac{(0{\cdot}4)(-0{\cdot}6)(-1{\cdot}6)(-2{\cdot}6)(-3{\cdot}6)(-4{\cdot}6)}{6\,!}(-120)$$

$$+\frac{(0{\cdot}4)(-0{\cdot}6)(-1{\cdot}6)(-2{\cdot}6)(-3{\cdot}6)(-4{\cdot}6)(-5{\cdot}6)}{7\,!}(210)$$

$=256$ (approx.)

$\therefore$ Number of candidates getting marks more than 42 but not more than $45=296-256=40$.

Ex. 4·59. *Find $f(0{\cdot}0477)$ from the following data :*

x :	0	0·05	0·10	0·15	0·20
$f(x)$:	1·00000	0·99750	0·99005	0·97775	0·96079

Sol.

Difference table

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
0·00	1·00000				
		$-0·00250$			
0·05	0·99750		$-0·00495$		
		$-0·00745$		0·00010	
0·10	0·99005		$-0·00485$		0·00009
		$-0·01230$		0·00019	
0·15	0·97775		$-0·00466$		
		$-0·01696$			
0·20	0·96079				

$$U=\frac{0·0477-0}{0·05}=0·954$$

$$\therefore \quad f(0·0477)=1·00000+(0·954)(-0·00250)$$

$$+\frac{(0·954)(-0·046)}{2\,!}(-0·00495)$$

$$+\frac{(0·954)(-0·046)(-1·046)}{3\,!}(0·00010)$$

$$+\frac{(0·954)(-0·046)(-1·046)(-2·046)}{4\,!}(0·00009)$$

$$=0·9977240\simeq 0·99772$$

Ex. 4·60. *Derive Newton-Gregory backward interpolation formula.*

Sol. Let $y=f(x)$ be a function which assumes the values $y_0, y_1, \ldots y_n$ for equidistant values $x_0, x_1, \ldots x_n$ of x. Let

$$I(x)=a_0+a_1(x-x_n)+a_2(x-x_n)(x-x_{n-1})+\ldots\ldots$$
$$+a_n(x-x_n)(x-x_{n-1})\ldots\ldots(x-x_1)$$

where the co-efficients $a_0, a_1, \ldots\ldots a_n$ are to be determined s.t. $I(x)$ takes the values $y_0, y_1, \ldots y_n$ for $x=x_0, x_1, \ldots x_n$ respectively.

Let $x_1-x_0=x_2-x_1=\ldots\ldots=x_n-x_{n-1}=h.$

$\therefore \quad x_i-x_0=ih \qquad i=1, \ldots n$

Now $\qquad I(x_n)=a_0=y_n$

$$I(x_{n-1})=a_0+a_1(x_{n-1}-x_n)$$

$$\therefore \qquad a_1=\frac{1}{h}(y_n-y_{n-1})=\frac{1}{h}\nabla y_n$$

$$I(x_{n-2})=a_0+a_1(x_{n-2}-x_n)+a_2(x_{n-2}-x_n)(x_{n-2}-x_{n-1})$$

$$\therefore \qquad a_2=\frac{1}{2h^2}\{y_n-2y_{n-1}+y_{n-2}\}=\frac{1}{2\,!\,h^2}\nabla^2 y_n$$

Similarly

$$a_3=\frac{1}{3\,!\,h^3}\nabla^3 y_n, \ldots\ldots a_n=\frac{1}{n\,!\,h^n}\nabla^n y_n$$

$$\therefore \qquad I(x)=y_n+\frac{(x-x_n)}{h}\nabla y_n+\frac{(x-x_n)(x-x_{n-1})}{2\,!\,h^2}\nabla^2 y_n$$

$$+\ldots\ldots+\frac{(x-x_n)\ldots\ldots(x-x_1)}{n\,!\,h^n}\nabla^n y_n$$

Let $\quad U=\dfrac{x-x_n}{h}, \quad$ *i.e.,* $\quad x=x_n+Uh$

Then $(x-x_n)\ldots\ldots\ldots(x-x_1)=h^n U(U+1)\ldots\ldots\ldots(U+\overline{n-1})$

$$=h^n U^{(n)}$$

where $\quad U^{(n)}=U(U+1)\ldots\ldots(U+n-1)$

$$\therefore \qquad I(x)=y_n+U^{(1)}\nabla y_n+\frac{U^{(2)}}{2\,!}\nabla^2 y_n+\ldots\ldots+\frac{U^{(n)}}{n\,!}\nabla^n y_n$$

where $\quad U^{(n)}=U(U+1)\ldots\ldots(U+\overline{n-1})$.

Ex. 4·61. *Estimate the population in 1925 of a place having the following record.*

Year	*1891*	*1901*	*1911*	*1921*	*1931*
Population (in thousands)	*46*	*66*	*81*	*93*	*101*

Sol. Since 1925 is near the end of the table, Newton's backward formula will be used.

Difference table

Year	*Population*	Δ	Δ^2	Δ^3	Δ^4	
1891	46					
		20				
1901	66		−5			
		15		2		
1911	81		−3		−3	$U=\dfrac{1925-1931}{10}$
		12		−1		
1921	93		−4			$=-0{\cdot}6$
		8				
1931	101					

$$\therefore \quad \text{Population in } 1925=101+(-0{\cdot}6)(8)+\frac{(-0{\cdot}6)(0{\cdot}4)}{2\,!}(-4)$$

$$+\frac{(-0{\cdot}6)(0{\cdot}4)(1{\cdot}4)}{3\,!}(-1)+\frac{(-0{\cdot}6)(0{\cdot}4)(1{\cdot}4)(2{\cdot}4)}{4\,!}(-3)$$

$$=101-4{\cdot}8+0{\cdot}48+0{\cdot}056+0{\cdot}1008=96{\cdot}8368$$

$$\simeq 96{\cdot}84 \text{ thousands.}$$

Ex. 4.62. *Find the value of an annuity at* $5\frac{3}{8}\%$ *from the following table :*

Rate percent	*Annuity-value*	*Rate percent*	*Annuity-value*
4	17·29203	$5\frac{1}{2}$	14·53375
$4\frac{1}{2}$	16·288889	6	13·76483
5	15·37245		

Sol. Since $5\frac{3}{8}\%$ lies near the middle of the table, any central difference formula can be applied.

Difference table

Rate	*Annuity-value*	Δ	Δ^2	Δ^3	Δ^4
4·0	17·29203				
		—1·00314			
4·5	16·28889		0·08670		
		—0·91644		—0·00896	
5·0	15·37245		0·07774		0·00100
		—0·83870		—0·00796	
5·5	14·53375		0·06978		
		—0·76892			
6·0	13·76483				

Here $x_0=5{\cdot}0$; $h=0{\cdot}5$.

$$\therefore \qquad U=\frac{5\frac{3}{8}-5}{0{\cdot}5}=0{\cdot}75$$

(i) **Using Stirling's formula.**

$$y_u=y_0+U\mu\delta y_0+\frac{U^2}{2\,!}\delta^2 y_0+\frac{U(U^2-1)}{3\,!}\mu\delta^3 y_0+\frac{U^2(U^2-1)}{4\,!}\delta^4 y_0+\cdots$$

$$=y_0+U\mu\Delta y_{-1/2}+\frac{U^2}{2\,!}\Delta^2 y_{-1}+\frac{U(U^2-1)}{3\,!}\mu\Delta^3 y_{-3/2}$$
$$+\frac{U^2(U^2-1)}{4\,!}\Delta^4 y_{-2}\ldots\ldots$$

$$=y_0+U\left\{\frac{\Delta y_0+\Delta y_{-1}}{2}\right\}+\frac{U^2}{2\,!}\Delta^2 y_{-1}+\frac{U(U^2-1)}{3\,!}\left\{\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right\}$$
$$+\frac{U^2(U^2-1)}{4\,!}\Delta^4 y_{-2}+\ldots\ldots$$

$$=15{\cdot}37245+(0{\cdot}75)\left\{\frac{-0{\cdot}83870-0{\cdot}91644}{2}\right\}+\frac{(0{\cdot}75)^2}{2\,!}(0{\cdot}07774)$$
$$+\frac{(0{\cdot}75)\{(0{\cdot}75)^2-1\}}{3\,!}\left\{\frac{-0{\cdot}00796-0{\cdot}00896}{2}\right\}$$
$$+\frac{(0{\cdot}75)^2\{(0{\cdot}75)^2-1\}}{4\,!}(0{\cdot}00100)$$

$$=14{\cdot}736589\simeq 14{\cdot}73659$$

(ii) Using Bessel's formula.

$$y_u = \mu y_{1/2} + V\delta y_{1/2} + \frac{1}{2!}\left(V^2-\frac{1}{4}\right)\mu\delta^2 y_{1/2} + \frac{1}{3!}V\left(V^2-\frac{1}{4}\right)\delta^3 y_{1/2}$$
$$+\frac{1}{4!}\left(V^2-\frac{1}{4}\right)\left(V^2-\frac{9}{4}\right)\mu\delta^4 y_{1/2}+\ldots\ldots$$

where $V=U-\frac{1}{2}$

$$=\frac{y_0+y_1}{2}+V\Delta y_0+\frac{1}{2!}\left(V^2-\frac{1}{4}\right)\left(\frac{\Delta^2 y_0+\Delta^2 y_{-1}}{2}\right)$$
$$+\frac{1}{6}V\left(V^2-\frac{1}{4}\right)\Delta^3 y_{-1}$$
$$+\frac{1}{24}\left(V^2-\frac{1}{4}\right)\left(V^2-\frac{9}{4}\right)\left(\frac{\Delta^4 y_{-1}+\Delta^4 y_{-2}}{2}\right)+\ldots\ldots$$

Here $V=0{\cdot}75-0{\cdot}5=0{\cdot}25$

$$\therefore\quad y_u=\frac{15{\cdot}37245+14{\cdot}53\ 75}{2}+(0{\cdot}25)(-0{\cdot}83870)$$
$$+\frac{1}{2}\left\{(0{\cdot}25)^2-0{\cdot}25\right\}\left\{\frac{0{\cdot}06978+0{\cdot}07774}{2}\right\}$$
$$+\frac{1}{6}(0{\cdot}25)\{(0{\cdot}25)^2-0{\cdot}25\}(-0{\cdot}00796)$$
$$+\frac{1}{24}\{(0{\cdot}25)^2-0{\cdot}25\}\{(0{\cdot}25)^2-2{\cdot}25\}(0{\cdot}00100)$$
$$=14{\cdot}95310-0{\cdot}209675-0\ 0069150+0{\cdot}0000622+0{\cdot}0000171$$
$$=14{\cdot}736589\simeq 14{\cdot}73659$$

Ex. 4·63. *From the following data, find the annual premium at the age of 33.*

Age	*24*	*28*	*32*	*36*	*40*
Annual Premium (in Rs.)	*28·06*	*30·19*	*32·75*	*35·94*	*40·00*

Sol.

Age x	u	Premium y	Δ^1	Δ^2	Δ^3	Δ^4	
24	−2	28·06					
			2·13				
28	−1	30·19		0·43			
			2·56		0·20		
32	0	32·75		0·63		0·04	$U=\left(\frac{33-32}{4}\right)$
			3·19		0·24		$=0{\cdot}25$
36	1	35·94		0·87			
			4 06				
40	2	40·00					

By Stirling's formula, we have

$$y_{0.25}=32\cdot75+(0\cdot25)\left\{\frac{3\cdot19+2\cdot56}{2}\right\}+\frac{(0\cdot25)^2}{2\,!}\;(0\cdot63)$$

$$+\frac{(0\cdot25)\{(0\cdot25)^2-1\}}{3\,!}\left\{\frac{0\cdot24+0\cdot20}{2}\right\}+\frac{(0\cdot25)^2\{(0\cdot25)^2-1\}}{4\,!}\;(0\cdot04)$$

$$=32\cdot75+0\cdot719+0\cdot02-0\cdot009-0\cdot000$$

$$=33\cdot48$$

Ex. 4·64. *Show that*

$$y_{2\frac{1}{2}}=\frac{1}{2}\;c+\frac{25(c-b)+3(a-c)}{256}$$

where $a=y_0+y_5$, $b=y_1+y_4$, $c=y_2+y_3$ if $\Delta^5 y_x$ are constant.

Sol. From Bessel's formula

$$y_x=\frac{y_0+y_1}{2}+V\Delta y_0+\frac{1}{2\,!}\left(V^2-\frac{1}{4}\right)\left(\frac{\Delta^2 y_0+\Delta^2 y_{-1}}{2}\right)$$

$$+\frac{1}{6}\;V\left(V^2-\frac{1}{4}\right)\Delta^3 y_{-1}+\frac{1}{24}\left(V^2-\frac{1}{4}\right)\left(V^2-\frac{9}{4}\right)\left(\frac{\Delta^4 y_{-1}+\Delta^4 y_{-2}}{9}\right)$$

$$+\frac{1}{120}\;V\left(V^2-\frac{1}{4}\right)\left(V^2-\frac{9}{4}\right)\Delta^5 y_{-2}$$

Let $\qquad x_0=2.$

Then $y_0=y_2$, $y_{-1}=y_1$, $y_{-2}=y_0$, $y_1=y_3$, $y_2=y_4$ and $y_3=y_5$.

Also $V=U-\dfrac{1}{2}=(x-x_0)-\dfrac{1}{2}=\left(2\dfrac{1}{2}-2\right)-\dfrac{1}{2}=0$

$$\therefore\;\; y_{2\frac{1}{2}}=y_{1/2}=\frac{y_0+y_1}{2}-\frac{1}{16}\{(E-1)^2 y_0+(E-1)^2 y_{-1}\}$$

$$+\frac{9}{768}\;\{(E-1)^4 y_{-1}+(E-1)^4 y_{-2}\}$$

$$=\frac{y_0+y_1}{2}-\frac{1}{16}\;\{(y_2-2y_1+y_0)+(y_1-2y_0+y_{-1})\}$$

$$+\frac{9}{768}\{(y_3-4y_2+6y_1-4y_0+y_{-1})+(y_2-4y_1+6y_0-4y_{-1}+y_{-2})\}$$

$$=\frac{y_2+y_3}{2}-\frac{1}{16}\{y_4-y_3-y_2+y_1\}$$

$$+\frac{9}{768}\{y_5-3y_4+2y_3+2y_2-3y_1+y_0\}$$

$$=\frac{c}{2}-\frac{1}{16}(b-c)+\frac{3}{256}\;\{a-3b+2c\}$$

$$=\frac{c}{2}+\frac{1}{256}\;\{3(a-c)+25(c-b)\}$$

Ex. 4·65. *If third differences are constant, show that*

$$y_{x+\frac{1}{2}}=\frac{1}{2}(y_x+y_{x+1})-\frac{1}{16}(\Delta^2 y_{x-1}+\Delta^2 y_x).$$

Sol. From Bessel's formula

$$y_x=\frac{y_0+y_1}{2}+V\Delta y_0+\frac{1}{2!}\left(V^2-\frac{1}{4}\right)\left(\frac{\Delta^2 y_0+\Delta^2 y_{-1}}{2}\right)$$
$$+\frac{1}{3!}V\left(V^2-\frac{1}{4}\right)\Delta^3 y_{-1}$$

Let $x_0=x$

Then $y_0=y_x,\ y_1=y_{x+1},\ y_{-1}=y_{x-1}$, etc.

$$U=\left(x+\frac{1}{2}\right)-x=\frac{1}{2}$$

and $\therefore\quad V=U-\frac{1}{2}=0$

$$\therefore\quad y_{x+\frac{1}{2}}=\frac{y_x+y_{x+1}}{2}-\frac{1}{16}(\Delta^2 y_x+\Delta^2 y_{x-1}).$$

Ex. 4·66. *The following table relates to income earned per month by a certain number of workers in a big manufacturing concern :*

Earnings per month	*Freq.*	*Earnings per month*	*Freq.*
up to Rs. 10	*50*	*up to Rs. 40*	*500*
up to Rs. 20	*150*	*up to Rs. 50*	*700*
up to Rs. 30	*300*	*up to Rs. 60*	*800*

Find out the number of workers falling within the Rs. 25–35 earning group.

Sol.

x	u	y		Δ^1	Δ^2	Δ^3	Δ^4	Δ^5	
10	–2	50	y_{-2}						
				100					
20	–1	150	y_{-1}		50				
				150		0			
30	0	300	y_0		50		–50		
				200		–50		0	$U=\frac{x-30}{10}$
40	1	500	y_1		0		–50		
				200		–100			
50	2	700	y_2		–100				
				100					
60	3	800	y_3						

$$U(\text{for } x=25)=\frac{25-30}{10}=-0.5$$

$$U(\text{for } x=35)=\frac{35-30}{10}=0{\cdot}5.$$

For $x=25$ since U is negative, we apply Gauss's backward formula, by this formula.

$$\therefore\ y_{-0\cdot5}=300+(-0{\cdot}5)(150)+\frac{(-0{\cdot}5)(0{\cdot}5)}{2\,!}(50)$$

$$+\frac{(-0{\cdot}5)\{(-0{\cdot}5)^2-1\}}{3\,!}(0)+\frac{(-0{\cdot}5)\{(-0{\cdot}5)^2-1)\}\{1{\cdot}5\}}{4\,!}(-50)$$

$$=300-75-6{\cdot}25-1{\cdot}17=217{\cdot}58\simeq218.$$

For $x=35$, since U is positive, we apply Gauss's forward formula. By this formula we have

$$y_{0\cdot5}=300+(0{\cdot}5)(200)+\frac{(0.5)(-0{\cdot}5)}{2\,!}(50)$$

$$+\frac{(0{\cdot}5)\{0{\cdot}25-1\}}{3\,!}(-50)+\frac{(0{\cdot}5)(0{\cdot}25-1)\{-1{\cdot}5\}}{4\,!}(-50)$$

$$=300+100-6{\cdot}25+3{\cdot}125-1{\cdot}17$$

$$=395{\cdot}705\simeq396.$$

$\therefore$ No. of persons earning between Rs. 25 and Rs. $35=396-218$
$=178$.

Ex. 4·67. *If p, q, r, s be the successive entries corresponding to equidistant arguments in a table, show that when third differences are taken into account, the entry corresponding to the argument half way between the arguments of q and r is $A+\frac{1}{24}B$, where A is the A.M. of q and r and B is the A.M. of $3q-2p-s$ and $3r-2s-p$.*

Sol. In Ex. 4-65, let

$$y_{x-1}=p,\ y_x=q,\ y_{x+1}=r \text{ and } y_{x+2}=s$$

Then $$y_{x+\frac{1}{2}}=\frac{q+r}{2}-\frac{1}{16}\{(E-1)^2y_x+(E-1)^2y_{x-1}\}$$

$$=\frac{q+r}{2}-\frac{1}{16}\{y_{x+2}-y_{x+1}-y_x+y_{x-1}\}$$

$$=A-\frac{1}{16}\{s-r-q+p\}$$

Also $$B=\frac{1}{2}\{(3q-2p-s)+(3r-2s-p)\}$$

$$=\frac{3}{2}(q+r-p-s)$$

$$\therefore \quad y_{x+\frac{1}{2}} = A + \frac{1}{24}B.$$

To Find Missing Terms

Ex. 4·68. *Given* $U_0+U_8=1{\cdot}9243$, $U_1+U_7=1{\cdot}9590$, $U_2+U_6=1{\cdot}9823$ *and* $U_3+U_5=1{\cdot}9956$. *Find* U_4.

Sol. Since eight entries are given, from these values a polynomial of degree seven can be obtained and hence $\Delta^7 U_x$ is assumed to be constant and consequently $\Delta^8 U_x=0$ for all x.

$$\therefore \quad \Delta^8 U_0 = 0$$

i.e., $$(E-1)^8 U_0 = 0$$

i.e., $$(E^8-8E^7+28E^6-56E^5+70E^4-56E^3+28E^2-8E+1)U_0=0$$

i.e., $$(U_8+U_0)-8(U_7+U_1)+28(U_6+U_2)-56(U_5+U_3)+70U_4=0$$

$$\therefore \; 70U_4 = -1{\cdot}9243+8(1{\cdot}9590)-28(1{\cdot}9823)+56(1{\cdot}9956)$$
$$=69{\cdot}9969$$

$$\therefore \quad U_4 = \frac{69{\cdot}9969}{70} = 0{\cdot}999956 \simeq 1{\cdot}000.$$

Ex. 4·69. *Find the missing terms in the following data :*

x :	2·0	2·1	2·2	2·3	2·4	2·5	2·6
y :	13·5	?	11·1	10	?	8·2	7·4

Sol. Taking the missing entries as x and y the difference table is given below ;

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
2·0	13·5					
		$x-13{\cdot}5$				
2·1	x		$24{\cdot}6-2x$			
		$11{\cdot}1-x$		$3x-36{\cdot}8$		
2·2	11·1		$x-12{\cdot}2$		$y-4x+40{\cdot}1$	
		$-1{\cdot}1$		$y-x+3{\cdot}3$		$5x-5y\;16{\cdot}3$
2·3	10·0		$y-8{\cdot}9$		$x-4y+23{\cdot}8$	
		$y-10{\cdot}0$		$27{\cdot}1-3y$		$10y-x-78{\cdot}1$
2·4	y		$18{\cdot}2-2y$		$6y-54{\cdot}3$	
		$8{\cdot}2-y$		$3y-27{\cdot}2$		
2·5	8·2		$y-9$			
		$-0{\cdot}8$				
2·6	7·4					

Taking $$\Delta^5 y = 0$$

$$5x-5y-16{\cdot}3=0$$

$$-x+10y-78{\cdot}1=0$$

$$\therefore \quad x=12{\cdot}3 \text{ and } y=9{\cdot}04.$$

Ex. 6·70. *Find the value of* y *for* $x=5$ *from the set of values.*

x :	2	3	4	6	7
y :	1	5	13	61	125

Sol.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
2	1					
		4				
3	5		4			
		8		$x-25$		
4	13		$x-21$		$120-4x$	
		$x-13$		$95-3x$		$10x-286$
5	x		$74-2x$		$6x-166$	
		$61-x$		$3x-71$		
6	61		$x+3$			
		64				
7	125					

Assuming $\Delta^5 y=0$, $x=28{\cdot}6$.

Ex. 4-71. *The following table gives the age of mother and the average number of children born per mother. Find the average number of children born per mother age 30—34 years :*

Age of mothers :	15—19	20—24	25—29	30—34	35—39	40—44
No. of children :	0·7	2·1	3·5	?	5·7	5·8

Sol.

Age of mother in years		*No. of children* y	
15—19	0	0·7	y_0
20—24	1	2·1	y_1
25—29	2	3·5	y_2
30—34	3	?	y_3
35—39	4	5·7	y_4
40—44	5	5·8	y_5

$\Delta^5 y_0=0$ or $(E-1)^5 y_0=0$.

$\therefore\ y_5-5y_4+10y_3-10y_2+5y_1-y_0=0.$

$\therefore\ 5{\cdot}8-5(5.7)+10y_3-10(3{\cdot}5)+5(2{\cdot}1)-0{\cdot}7=0.$

or $\qquad y_3=4{\cdot}79.$

Ex. 4-72. *Interpolate the missing figures in the following table of rice cultivation :*

Year	*Acres (in Millions)*	*Years*	*Acres (in Millions)*
1911	76·6	1916	?
1912	78·7	1917	50·6
1913	?	1918	77·6
1914	77·7	1919	78·6
1915	78·7		

Sol.

Year		Acres		Year		Acres	
1911	0	76·6	y_0	1916	5	?	y_5
1912	1	78 7	y_1	1917	6	50·6	y_6
1913	2	?	y_2	1918	7	77·6	y_7
1914	3	77·7	y_3	1919	8	78·6	y_8
1915	4	78·7	y_4				

As two missing terms are to be determined, two equations are needed. We take them to be

$$\Delta^7 y_0 = 0 \quad \ldots(1)$$

and

$$\Delta^7 y_1 = 0 \quad \ldots(2)$$

Eq. (1) and (2) gives

$$y_7 - 7y_6 + 21y_5 - 35y_4 + 35y_3 - 21y_2 + 7y_1 - y_0 = 0$$

and

$$y_8 - 7y_7 + 21y_6 - 35y_5 + 35y_4 - 21y_3 + 7y_2 - y_1 = 0$$

$\therefore$

$$21y_5 - 21y_2 = -162{\cdot}7$$

or

$$35y_5 - 7y_2 = 1642{\cdot}1$$

$\therefore$

$$y_5 = 60{\cdot}58 \simeq 60{\cdot}6$$

and

$$y_2 = 68{\cdot}3.$$

Ex. 4·73. *Estimate the production for the years 1935 and 1945 with the help of the following table :*

Year :	1920,	1925,	1930,	1935,	1940,	1945,	1950
Production in : 000,000 tons	200,	220,	260,	?,	350,	?,	430

Sol.

x			
1920	0	200	y_0
1925	1	220	y_1
1930	2	260	y_2
1935	3	?	y_3
1940	4	350	y_4
1945	5	?	y_5
1950	6	430	y_6

We take $\Delta^5 y_0 = 0$

and $\Delta^5 y_1 = 0$

i.e., $y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

and $y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$

$\therefore$ $y_5 + 10y_3 = 3450$

and $5y_5 + 10y_3 = 5010$

$\therefore$ $y_5 = 390$ and $y_3 = 306$.

Ex. 4·74. *Interpolate U_2 from the following table :*

x :	1	2	3	4	5
U_x :	7	—	13	21	37

and explain why the value obtained is different from that obtained by putting $x=2$ in the expression 2^x+5.

Sol.

x			
1	0	7	y_0
2	1	?	y_1
3	2	13	y_2
4	3	21	y_3
5	4	37	y_4

we take $\Delta^4 y_0 = 0$

i.e., $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

$\therefore$ $y_1 = 9{\cdot}5$.

EXERCISE 4.1

1. If (*i*) $f(x) = x^{n+1}$, show that

$$f(x_0, x_1, \ldots x_n) = x_0 + x_1 + \ldots\ldots + x_n$$

(*ii*) $f(x) = \dfrac{1}{x}$, show that

$$f(x_0, x_1, \ldots\ldots x_n) = \frac{(-1)^n}{x_0 . x_1 \ldots\ldots x_n}$$

(*iii*) Show that *n*th order divided differences of a *n*th degree polynomial are constant and higher order divided differences are zero.

2. Show that

$$\Delta^n(ax^n + bx^{n-1}) = n!\,a \qquad (h=1)$$

3. Find the values of

   (1) $2x^{(4)}+3x^{(2)}+x^{(1)}-7$ at $x=5$. [Ans. 298]

   (2) $2x^3-3x^2+3x-10$ at $x=5$. [Ans. 180]

4. Represent the following polynomials in the factorial notations.

   (1) $x^4-12x^3+42x^2-30x+9$.
   [Ans. $x^{(4)}-6x^{(3)}+13x^{(2)}+x^{(1)}+9$]

   (2) x^4-3x^3+2x+6. [Ans. $x^{(4)}+6x^{(3)}+4x^{(2)}+6$]

5. Find a cubic function of x which has the values 1, -3, -1, 13 when $x=1$, 2, 3, 4 respectively.
   [Ans. $5-2x-3x^2+x^3$]

6. Sum the series

   (1) $2\cdot3+3\cdot6+4\cdot11+\ldots\ldots+(n+1)(n^2+2)$.

   $$\left[\text{Ans. } \frac{n}{12}(3n^3+10n^2+21n+38)\right]$$

   (2) $1^2.\,2^2+2^2.3^2+3^2.4^2+\ldots\ldots$upto n terms.

   $$\left[\text{Ans. } \frac{n}{15}(3n^4+15n^3+25n^2+15n+2)\right]$$

7. Apply Lagrange's formula to find $f(5)$ and $f(6)$ given that

x :	1	2	3	4	7
$f(x)$:	2	4	8	16	128

   and explain why the result differ from those obtained by completing the series of powers of 2. [Ans. 32·9 ; 66·7]

8. Use Newton's formula for interpolation to find the annual premium at the age of 33 from the table given below :

Age	24	28	32	36	40
Annual premium (in Rs.)	28·06	30·19	32·75	35·94	40·00

   [Ans. 33.48]

9. From the following table estimate, by using Newton's formula, the premium payable at the age of 22 years :

Age (in yrs.)	20	25	30	35	40	45
Premium (in Rs.)	25	28	32	37	43·5	52·25

   [Ans. 26·05]

10. Use Newton's formula to find the annual premium payable at the age of 26 years from the following table giving the annual premiums charged by an insurance company for a policy of Rs. 1,000 ;

Age next birthday :	20	25	30	35	40
Annual premium (in Rs.)	23	26	30	35	42

[Ans. 26·73]

11. Using Newton's formula for interpolation estimate the population for the year 1905 :

Year	*Population*
1891	98,754
1901	132,285
1911	168,076
1921	195,690
1931	246,050

[Ans. 1,47,841]

12. From the following information find the number of students who obtained less than 45 marks :

Marks	30–40	40–50	50–60	60—70	70–80
Frequency	31	42	51	35	31

[Ans. 48]

13. Determine the number of workers earning Rs. 124 or more but less than Rs. 125 from the following data :

Earnings less than Rs.	*No. of workers*	*Earnings less than Rs.*	*No. of workers*
120	276	135	918
125	599	140	966
130	804		

[Ans. 54]

14. From the following table estimate the number of persons earning wages between Rs. 60 and Rs. 70.

Wage (in Rs.)	*No. of persons (in thousands)*	*Wage (in Rs.)*	*No. of persons (in thousands)*
Below 40	250	80—100	70
40—60	120	100–120	60
60—80	100		

[Ans. 54]

15. The following table relates to income earned per month by a certain number of workers in a big manufacturing concern :

Earnings per month	*No. of workers*	*Earnings per month*	*No. of workers*
upto Rs. 10	50	upto Rs. 40	500
,, ,, 20	150	,, ,, 50	700
,, ,, 30	300	,, ,, 60	800

Find the number of workers falling within the Rs. 25—35 earning group. [Ans. 178]

16. Find $f(4)$ from table given below :

x :	1	2	3	5	6	7
$f(x)$:	2	4	8	32	64	128

[Ans. 16·1]

17. Find (0·6538) using the following data :

x	$f(x)$	x	$f(x)$	x	$f(x)$
0·62	0·6194114	0·64	0·6345857	0·67	0·6566275
0·63	0·6270463	0·65	0·6420292	0·68	0·6637820
		0·66	0·6493765		

[Ans. 0·6448325]

18. Find log 324 using the following data :

x	310	320	330	340	350
log x	2·491362	2·505150	2·518514	2·531479	2·544068

[Ans. 2·510545]

19. Use Stirling's formula to obtain $f(1·22)$ from the following data :

x :	0	0·5	1·0	1·5	2·0	2·5	3·0
$f(x)$:	0	0·19146	0·34134	0·43319	0·47725	0·49379	0·49865

[Ans. 0·38871]

20. If l_x represents the number living at age x in a life table, find, as accurately as the data will permit, l_x for $x=35$, 42 and 47. Given :

x :	20	30	40	50
l_x :	512	439	346	243

[Ans. 395, 326, 274]

21. Find $\sqrt{12516}$, using Gauss's backward formula, from the following data :

x :	12500	12510	12520	12530
$\sqrt{x}$:	111·803399	111·848111	111·892806	111·937483

[Ans. 111·874929]

5

Numerical Differentiation and Integration

5.1. Numerical Differentiation. *It is the process of finding the derivatives of a function which may not be given in explicit mathematical form but for which a certain set of values are given. The procedure is to represent the function by an interpolatian formula and then to differentiate this formula as many times as desired*

Rules of representing the function by an interpolation formula.

Argument Values		*Formula use*
Equidistant	*Newton's forward*	*(for differentiating near the beginning of the table)*
	Newton's backward	*(near the end)*
	Stirling's or Bessel's	*(near the middle)*
Non-equidistant	*Newton's divided difference or Lagrange's*	

Ex. 5·1. *Find the first and second order derivatives of the function tabulated below at the points $x=0$, $0{\cdot}03$ and $0{\cdot}06$.*

x :	0	0·01	0·02	0·03	0·04	0·05	0·06
$f(x)$:	0	0·0301	0·0604	0·0909	0·1216	0·1525	0·1836

Sol.

Difference table

x	$f(x)$	Δ	Δ^2
0·00	0·0000		
		0·0301	
0·01	0·0301		0·0002
		0·0303	
0·02	0·0604		0·0002
		0·0305	
0·03	0·0909		0·0002
		0·0307	
0·04	0·1216		0·0002
		0·0309	
5·05	0·1525		0·0002
		0·0311	
0·06	0·1836		

5.2

(*i*) *At* $x=0$.

Since the differentiation is to be done near the beginning of the table, Newton's forward formula will be used. By the said formula,

$$f(x)=f(x_0)+U\Delta f(x_0)+\frac{U(U-1)}{2\,!}\Delta^2 f(x_0)$$

Here $U=\dfrac{x-0}{0{\cdot}01}=100x.$

$$\therefore\quad f'(x)=\frac{df(x)}{dU}\frac{dU}{dx}=100\left\{\Delta f(x_0)+\frac{2U-1}{2}\Delta^2 f(x_0)\right\}$$

$$=100\{0{\cdot}0301+(U-0{\cdot}5)(0{\cdot}0002)\}$$

$$\therefore\quad f'(0)=100\{0\;0301+(U-0{\cdot}5)(0{\cdot}0002)\}U=0$$

$$=100\{0\;0301-0{\cdot}0001\}=3.$$

Also $f''(0)=\left\{\dfrac{df'(x)}{dU}\cdot\dfrac{dU}{dx}\right\}_{x=0}=(100)^2\{0{\cdot}0002\}=2$

(*ii*) *At* $x=0{\cdot}03$.

Since the differentiation is to be done near the middle of the table, any central difference formula can be used. By Stirling's formula.

$$f(x)=f(x_0)+U\mu\delta f(x_0)+\frac{U^2}{2\,!}\delta^2 f(x_0)+\frac{U(U^2-1)}{3\,!}\mu\delta^3 f(x_0)+\ldots\ldots$$

$$=f(x_0)+U\frac{E^{1/2}+E^{-1/2}}{2}\Delta f\left(x_0-\frac{h}{2}\right)+\frac{U^2}{2\,!}\Delta^2 f(x_0-h)$$

$$+\frac{U(U^2-1)}{3\,!}\frac{E^{1/2}+E^{1/2}}{2}\Delta^3 f\left(x_0-\frac{3h}{2}\right)+\ldots\ldots$$

$$=f(x_0)+\frac{U}{2}\{\Delta f(x_0)+\Delta f(x_0-h)\}+\frac{U^2}{2\,!}\Delta^2 f(x_0-h)+\ldots\ldots$$

Here $U=\dfrac{x-0{\cdot}03}{0{\cdot}01}=100x-3$

$$\therefore\quad f'(x)=100\left\{\frac{df}{dU}\right\}=100\left\{\frac{\Delta f(x_0)+\Delta f(x_0-h)}{2}+U\Delta^2 f(x_0-h)\right\}$$

$$=100\left\{\frac{0{\cdot}0307+0{\cdot}0305}{2}+U(0{\cdot}0002)\right\}$$

$$\therefore\quad f'(0{\cdot}03)=100\{0{\cdot}0306+(0{\cdot}0002(U)_{U=0}$$

$$=3{\cdot}06$$

$$f''(0{\cdot}03)=\left\{\frac{df'}{dU}\cdot\frac{dU}{dx}\right\}_{x=0{\cdot}03}=(100)^2\{0{\cdot}0002\}=2.$$

(iii) *At* $x=0{\cdot}06.$

Since differentiation is to be done near the end of the table, Newton's backward formula will be used. By the said formula,

$$\therefore \quad f(x)=f(x_n)+U\nabla f(x_n)+\frac{U(U+1)}{2\,!}\nabla^2 f(x_n)+\ldots\ldots$$

Here $\quad U=\dfrac{x-0{\cdot}06}{0{\cdot}01}=100x-6$

$$\therefore \quad f'(x)=100\left\{\nabla(fx_n)+\frac{2U+1}{2}\nabla^2 f(x_n)+\ldots\ldots\right\}$$

$$=100\{0{\cdot}0311+(U+0{\cdot}5)(0{\cdot}0002)\}$$

$$\therefore f'(0{\cdot}06)=100\{0{\cdot}0311+(U+0{\cdot}5)(0{\cdot}0002)\}_{U=0}$$

$$=100\{0{\cdot}0311+0{\cdot}0001\}$$

$$=3{\cdot}12$$

$$f''(0{\cdot}06)=\left\{\frac{df'(x)}{dU}\quad\frac{dU}{dx}\right\}_{x=0{\cdot}06}=(100)^2\{0{\cdot}0002\}=2$$

Ex. 5-2. *Find the value of* $\cosh x=\dfrac{d}{dx}\{\sinh x\}$ *at* $x=1{\cdot}52$ *from the following table :*

x	$\sinh x$	x	$\sinh x$
1·5	2·129279	1·8	2·942174
1·6	2·375568	1·9	3·268163
1·7	2·645632	2·0	3·626860

Sol.

Difference table

x	$f(x)=\sinh x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1·5	2·129279					
		0 246289				
1·6	2·375568		0 023775			
		0·270064		0·002703		
1·7	2·645632		0·026478		0·000266	
		0·296542		0·002969		0·000026
1·8	2·942174		0·029447		0·000292	
		0·325989		0·003261		
1·9	3·268163		0·032708			
		0·358697				
2·0	3·626860					

$$U=\frac{x-1{\cdot}5}{0{\cdot}1}=10x-15$$

For $\quad x=1{\cdot}52,\ U=0{\cdot}2.$

Now from Newton's forward formula.

$$f(x)=f(x_0)+U\Delta f(x_0)+\frac{U(U-1)}{2!}\Delta^2 f(x_0)+\frac{U(U-1)(U-2)}{3!}\Delta^3 f(x_0)$$
$$+\frac{U(U-1)(U-2)(U-3)}{4!}\Delta^4 f(x_0)$$
$$+\frac{U(U-1)(U-2)(U-3)(U-4)}{5!}\Delta^5 f(x_0)$$

(neglecting higher differences)

$$=f(x_0)+U\Delta f(x_0)+\frac{U^2-U}{2}\Delta^2 f(x_0)+\frac{U^3-3U^2+2U}{6}\Delta^3 f(x_0)$$
$$+\frac{U^4-6U^3+11U^2-6U}{24}\Delta^4 f(x_0)$$
$$+\frac{U^5-10U^4+35U^3-50U^2+24U}{120}\Delta^5 f(x_0)$$

$$\therefore\ f'(x)=10\Big[\Delta f(x_0)+\frac{2U-1}{2}\Delta^2 f(x_0)+\frac{3U^2-6U+2}{6}\Delta^3 f(x_0)$$
$$+\frac{4U^3-18U^2+22U-6}{24}\Delta^4 f(x_0)$$
$$+\frac{5U^4-40U^3-105U^2-100U+24}{120}\Delta^5 f(x_0)\Big]$$

$$\therefore\ f'(1{\cdot}52)=10\Big[0{\cdot}246289+(-0{\cdot}3)(0{\cdot}023775)$$
$$+\left(\frac{0{\cdot}46}{3}\right)(0{\cdot}002703)+\left(-\frac{0{\cdot}286}{3}\right)(0{\cdot}000266)+\left(\frac{0{\cdot}986}{15}\right)(0{\cdot}000026)\Big]$$
$$=10[0{\cdot}246289-0{\cdot}0071325+0{\cdot}00041446-0{\cdot}00002536+0{\cdot}00000171]$$
$$=2{\cdot}395473.$$

Ex. 5-3. *Assuming Newton's forward interpolation formula show that*

$$(i)\ f'(x_0)=\left\{\frac{df(x)}{dx}\right\}_{x=x_0}=\frac{1}{h}\left\{\Delta y_0-\frac{1}{2}\Delta^2 y_0+\frac{1}{3}\Delta^3 y_0-\frac{1}{4}\Delta^4 y_0+\cdots\right\}$$

$$(ii)\ f'(x_0)=\frac{1}{4}\left\{\Delta y_{-1}+\frac{1}{2}\Delta^2 y_{-1}-\frac{1}{6}\Delta^3 y_{-1}+\cdots\right\}$$

where $y_0=f(x_0)$ etc.

Sol. By the said formula,

$$f(x)=y_0+U\Delta y_0+\frac{U(U-1)}{2!}\Delta^2 y_0+\frac{U(U-1)(U-2)}{3!}\Delta^3 y_0+\cdots$$

$\therefore \quad \dfrac{df(x)}{dx}=\dfrac{1}{h} \quad \dfrac{d\{f(x)\}}{dU}$

$$=\frac{1}{h}\left\{\Delta y_0+\frac{2U-1}{2}\Delta^2 y_0+\frac{3U^2-6U+2}{6}\Delta^3 y_0+\dots\right\}$$

For $x=x_0$, $U=\dfrac{x_0-x_0}{h}=0$

$$\therefore \quad f'(x_0)=\frac{1}{h}\left\{\Delta y_0-\frac{1}{2}\Delta^2 y_0+\frac{1}{3}\Delta^3 y_0+\dots\right\}$$

To obtain (*ii*) put $x=x_1$ so that

$$U=\frac{x_1-x_0}{h}=\frac{h}{h}=1$$

$$\therefore \quad f'(x_1)=\frac{1}{h}\left\{\Delta y_0+\frac{1}{2}\Delta^2 y_0-\frac{1}{6}\Delta^3 y_0+\dots\right\}$$

Shifting the origin to x_1,

$$f'(x_0)=\frac{1}{h}\left\{\Delta y_{-1}+\frac{1}{2}\Delta^2 y_{-1}-\frac{1}{6}\Delta^3 y_{-1}+\dots\right\}$$

Ex. 5-4. *Assuming Newton's forward interpolation formula show that*

$$h^2 f''(x_0)=\Delta^2 y_0-\Delta^3 y_0+\dots$$

Sol. By the said formula

$$f(x)=y_0+U\Delta y_0+\frac{U(U-1)}{2\,!}\Delta^2 y_0+\frac{U(U-1)(U-2)}{3\,!}\Delta^3 y_0+\dots$$

Now $\quad U=\dfrac{x-x_0}{h}$

$\therefore \quad x=x_0+Uh$

$$\therefore \quad f(x_0+hU)=y_0+U\Delta y_0+\frac{U(U-1)}{2\,!}\Delta^2 y_0$$

$$+\frac{U(U-1)(U-2)}{3\,!}\Delta^3 y_0+\dots \quad \dots(1)$$

Now by Taylor's theorem

$$f(x_0+hU)=f(x_0)+hUf'(x_0)+\frac{h^2U^2}{2\,!}f''(x_0)+\dots$$

$\therefore$ Equating co-efficients of U^3 in (1)

$$\frac{h^2}{2\,!}f''(x_0)=\frac{\Delta^2 y_0}{2\,!}-\frac{1}{2}\Delta^3 y_0+\dots$$

$$\therefore \quad h^2 f''(x_0)=\Delta^2 y_0-\Delta^3 y_0+\dots$$

Note. Equating the co-efficients of various powers of U in (1) expressions for derivatives of various orders at $x=x_0$ can be obtained.

Ex. 5·5. *Assuming Newton's backward interpolation formula show that*

(*i*) $hf'(x_0)=\nabla y_0+\frac{1}{2}\nabla^2 y_0+\frac{1}{3}\nabla^3 y_0+\ldots$

(*ii*) $hf'(x_0)=\nabla y_1-\frac{1}{2}\nabla^2 y_1-\frac{1}{6}\nabla^3 y_1+\ldots$

Sol. $f(x)=y_0+U\nabla y_0+\frac{U(U+1)}{2\,!}\nabla^2 y_0$

$$+\frac{U(U+1)(U+2)}{3\,!}\nabla^3 y_0+\ldots$$

(where the origin is at x_0)

$$\therefore\quad hf'(x)=\frac{df(x)}{dU}=\nabla y_0+\frac{2U+1}{2}\nabla^2 y_0+\frac{3U^2+6U+2}{6}\nabla^3 y_0\ldots$$

(*i*) $\because\quad hf'(x_0)=\nabla y_0+\frac{1}{2}\Delta^2 y_0+\frac{1}{3}\nabla^3 y_0+\ldots$

(*ii*) Put $x=x_{-1}$ so that $U=-1$

$\therefore\quad hf'(x_{-1})=\nabla y_0-\frac{1}{2}\nabla^2 y_0-\frac{1}{6}\nabla^3 y_0+\ldots$

Shifting the origin to x_{-1}

$$hf'(x_0)=\nabla y_1-\frac{1}{2}\nabla^2 y_1-\frac{1}{6}\nabla^3 y_1+\ldots$$

Ex. 5·6. *Show that*

(*i*) $f'\left(x_0+\frac{h}{2}\right)=\frac{1}{h}\left\{\Delta y_0-\frac{1}{24}\Delta^3 y_0+\ldots\right\}$

(*ii*) $f'\left(x_0-\frac{h}{2}\right)=\frac{1}{h}\left\{\nabla y_0-\frac{1}{24}\nabla^3 y_0+\ldots\right\}$

Sol. (*i*) $f'\left(x_0+\frac{h}{2}\right)=E^{1/2}Df(x_0)$

$$=(1+\Delta)^{1/2}\,Df(x_0)$$

Now $\quad D\equiv\frac{1}{h}\left\{\Delta-\frac{1}{2}\Delta^2+\frac{1}{3}\Delta^3\ldots\right\}$

$$\therefore\quad f\left(x_0+\frac{h}{2}\right)=\frac{1}{h}(1+\Delta)^{1/2}\left(\Delta-\frac{1}{2}\Delta^2+\frac{1}{3}\Delta^3\ldots\right)f(x_0)$$

$$=\frac{1}{h}\left\{1+\frac{1}{2}\Delta-\frac{1}{8}\Delta^2\ldots\right\}\left\{\Delta-\frac{1}{2}\Delta^2+\frac{1}{3}\Delta^3\ldots\right\}f(x_0)$$

$$=\frac{1}{h}\left\{\Delta-\frac{1}{24}\Delta^3\ldots\right\}f(x_0)$$

$$=\frac{1}{h}\left\{\Delta y_0-\frac{1}{24}\Delta^3 y_0\ldots\right\}$$

where $y_0=f(x_0)$

Similarly (*ii*) can be proved.

Ex. 5·7. *Assuming Stirling's formula find the expressions for first six derivatives of $f(x)$ at $x=x_0$.*

Sol. From the said formula

$$f(x)=y_0+U\mu\delta y_0+\frac{U^2}{2\,!}\delta^2 y_0+\frac{U(U^2-1)}{3\,!}\mu\delta^3 y_0+\ldots$$

$$+\frac{U^2(U^2-1)\ldots\{U^2-(r-1)^2\}}{2r\,!}\delta^{2r}y_0$$

$$+\frac{U(U^2-1)\ldots(U^2-r^2)}{2r+1\,!}\mu\delta^{2r+1}y_0\ldots$$

$$=y_0+U\left\{\mu\delta y_0-\frac{1}{6}\mu\delta^3 y_0+\frac{1}{30}\mu\delta^5 y_0-\frac{1}{140}\mu\delta^7 y_0+\ldots\ldots\right\}$$

$$+\frac{U^2}{2\,!}\left\{\delta^2 y_0-\frac{1}{12}\delta^4 y_0+\frac{1}{90}\delta^6 y_0-\frac{1}{560}\delta^8 y_0+\ldots\ldots\right\}$$

$$+\frac{U^3}{3\,!}\left\{\mu\delta^3 y_0-\frac{1}{4}\mu\delta^5 y_0+\frac{7}{120}\mu\delta^7 y_0\ldots\ldots\right\}$$

$$+\frac{U^4}{4\,!}\left\{\delta^4 y_0-\frac{1}{6}\delta^6 y_0+\frac{7}{240}\delta^8 y_0\ldots\ldots\ldots\right\}$$

$$+\frac{U^5}{5\,!}\left\{\mu\delta^5 y_0-\frac{1}{3}\mu\delta^7 y_0\ldots\ldots\ldots\ldots\right\}$$

$$+\frac{U^6}{6\,!}\left\{\delta^6 y_0-\frac{1}{4}\delta^8 y_0+\ldots\ldots\right\}\ldots\ldots\ldots\ldots$$

By Taylor's theorem,

$$f(x)=f(x_0+Uh)=f(x_0)+Uhf^{(1)}(x_0)+\frac{U^2h^2}{2\,!}f^{(2)}(x_0)+\ldots\ldots$$

Equating co-efficients of different powers of U

$$hf^{(1)}(x_0)=\mu\delta y_0-\frac{1}{6}\mu\delta^3 y_0+\frac{1}{30}\mu\delta^5 y_0-\frac{1}{140}\mu\delta^7 y_0+\ldots\ldots$$

$$h^2f^{(2)}(x_0)=\delta^2 y_0-\frac{1}{12}\delta^4 y_0+\frac{1}{90}\delta^6 y_0-\frac{1}{560}\delta^8 y_0\ldots\ldots$$

$$h^3 f^{(3)}(x_0)=\mu\delta^3 y_0-\frac{1}{4}\mu\delta^5 y_0+\frac{7}{120}\mu\delta^7 y_0\cdots\cdots\cdots$$

$$h^4 f^{(4)}(x_0)=\delta^4 y_0-\frac{1}{6}\delta^6 y_0+\frac{7}{240}\delta^8 y_0\cdots\cdots\cdots$$

$$h^5 f^{(5)}(x_0)=\mu\delta^5 y_0-\frac{1}{3}\mu\delta^7 y_0\cdots\cdots$$

$$h^{(6)} f^{(6)}(x_0)=\delta^6 y_0\ \frac{1}{4}\delta^8 y_0\cdots\cdots\cdots$$

Ex. 5-8. *Show that*

$$\frac{dy_x}{dx}=\frac{2}{3}(y_{x+1}-y_{x-1})-\frac{1}{12}(y_{x+2}-y_{x-2})$$

Sol. From Ex. 5-7 (taking $h=1$)

$$\left(\frac{dy_x}{dx}\right)_{\text{at } x=x_0}=\mu\delta y_0-\frac{1}{6}\mu\delta^3 y_0 \text{ (neglecting higher differences)}$$

$$=\frac{1}{2}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})y_0$$

$$-\frac{1}{12}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})^3 y_0$$

$$=\frac{1}{2}(E-E^{-1})y_0-\frac{1}{12}(E+E^{-1}-2)(E-E^{-1})y_0$$

$$=\frac{1}{2}(y_1-y_{-1})-\frac{1}{12}\{E^2-E^{-2}-2(E-E^{-1})\}y_0$$

$$=\frac{1}{2}(y_1-y_{-1})-\frac{1}{12}(y_2-y_{-2})+\frac{1}{6}(y_1-y_{-1})$$

$$=\frac{2}{3}(y_1-y_{-1})-\frac{1}{12}(y_2-y_{-2})$$

Shifting the origin to x

$$\frac{dy_x}{dx}=\frac{2}{3}(y_{x+1}-y_{x-1})-\frac{1}{12}(y_{x+2}-y_{x-2})$$

Ex. 5-9. *Starting with Bessel's formula obtain the expressions of first four derivatives of $f(x)$ at $x=x_0$*

Sol. From the said formula

$$f(x)=\mu y_{1/2}+\left(U-\frac{1}{2}\right)\delta y_{1/2}+\frac{U(U-1)}{2\,!}\mu\delta^2 y_{1/2}$$

$$+\left(U-\frac{1}{2}\right)\frac{U(U-1)}{3\,!}\delta^3 y_{1/2}+\ldots$$

$$+\frac{(U+r-1)(U+r-2)\ldots\ldots(U-r)}{2r\,!}\mu\delta^{2r} y_{1/2}$$

$$+\frac{\left(U-\frac{1}{2}\right)(U+r-1)(U+r-2)\ldots\ldots(U-r)}{(2r+1)\,!}\delta^{2r+1} y_{1/2}+\ldots\ldots$$

Now $\mu y_{1/2}+\left(U-\frac{1}{2}\right)\delta y_{1/2}=\frac{y_1+y_0}{2}+U\delta y_{1/2}-\frac{1}{2}(y_1-y_0)$

$$=y_0+U\delta y_{1/2}$$

$$\therefore\ f(x)=y_0+U\delta y_{1/2}+\frac{U(U-1)}{2!}\mu\delta^2 y_{1/2}$$

$$+\left(U-\frac{1}{2}\right)\frac{U(U-1)}{3!}\delta^3 y_{1/2}$$

$$+\ldots\ldots+\frac{(U+r-1)(U+r-2)\ldots\ldots(U-r)}{2r!}\mu\delta^{2r}y_{1/2}$$

$$+\frac{\left(U-\frac{1}{2}\right)(U+r-1)(U+r-2)\ldots\ldots(U-r)}{(2r+1)!}\delta^{2r+1}y_{1/2}+\ldots\ldots$$

$$=y_0+U\left\{\delta y_{1/2}-\frac{1}{2}\mu\delta^2 y_{1/2}+\frac{1}{12}\delta^3 y_{1/2}+\frac{1}{12}\mu\delta^4 y_{1/2}-\frac{1}{120}\delta^5 y_{1/2}+\ldots\ldots\right\}$$

$$+\frac{U^2}{2!}\left\{\mu\delta^2 y_{1/2}-\frac{1}{2}\delta^3 y_{1/2}-\frac{1}{12}\mu\delta^4 y_{1/2}+\frac{1}{24}\delta^5 y_{1/2}\ldots\ldots\ldots\right\}$$

$$+\frac{U^3}{3!}\left\{\delta^3 y_{1/2}-\frac{1}{2}\mu\delta^4 y_{1/2}\ldots\ldots\ldots\right\}$$

$$+\frac{U^4}{4!}\left\{\mu\delta^4 y_{1/2}-\frac{1}{2}\delta^5 y_{1/2}\ldots\ldots\ldots\right\}+\ldots\ldots$$

$\therefore$ As in Ex. 5-7.

$$hf^{(1)}(x_0)=\delta y_{1/2}-\frac{1}{2}\mu\delta^2 y_{1/2}+\frac{1}{12}\delta^3 y_{1/2}+\frac{1}{12}\mu\delta^4 y_{1/2}-\frac{1}{120}\delta^5 y_{1/2}\ldots\ldots$$

$$hf^{(2)}(x_0)=\mu\delta^2 y_{1/2}-\frac{1}{2}\delta^3 y_{1/2}-\frac{1}{12}\mu\delta^4 y_{1/2}+\frac{1}{24}\delta^5 y_{1/2}\ldots\ldots$$

$$hf^{(3)}(x_0)=\delta^3 y_{1/2}-\frac{1}{2}\mu\delta^4 y_{1/2}\ldots\ldots\ldots$$

$$h^4 f^{(4)}(x_0)=\mu\delta^4 y_{1/2}-\frac{1}{2}\delta^5 y_{1/2}\ldots\ldots\ldots$$

Ex. 5.10 *Using Bessel's formula show that*

$$\frac{dy_x}{dx}\Delta y_{x-\frac{1}{2}}-\frac{1}{24}\Delta^3 y_{x-\frac{3}{2}}+\ldots\ldots$$

Sol. From the said formula,

$$y_x=\mu y_{1/2}+\left(U-\frac{1}{2}\right)\delta y_{1/2}+\frac{U(U-1)}{2!}\mu\delta^2 y_{1/2}$$

$$+\left(U-\frac{1}{2}\right)\frac{U(U-1)}{3!}\delta^3 y_{1/2}+\ldots\ldots$$

$$=\frac{y_1+y_0}{2}+\left(U-\frac{1}{2}\right)\Delta y_0+\frac{U(U-1)}{2\,!}\cdot\frac{\Delta^2 y_0+\Delta^2 y_{-1}}{2}$$

$$+\left(U-\frac{1}{2}\right)\frac{U(U-1)}{3\,!}\Delta^3 y_{-1}+\ldots\ldots$$

Here $\quad U=x-x_0 \quad (\because\ h=1)$

$\therefore$ Change x to $x+\frac{1}{2}$. Then U changes to $U+\frac{1}{2}$.

$$\therefore\ y_{x+1/2}=\frac{y_1+y_0}{2}+U\Delta y_0+\frac{\left(U+\frac{1}{2}\right)\left(U-\frac{1}{2}\right)}{2\,!}\cdot\frac{\Delta^2 y_0+\Delta^2 y_{-1}}{2}$$

$$+\frac{U\left(U+\frac{1}{2}\right)\left(U-\frac{1}{2}\right)}{3\,!}\Delta^3 y_{-1}+\ldots\ldots\ldots$$

$$\therefore\ \frac{dy_{x+\frac{1}{2}}}{dx}=\Delta y_0+\frac{2U}{2\,!}\ \frac{\Delta^2 y_0+\Delta^2 y_{-1}}{2}+\frac{3U^2-\frac{1}{4}}{3}\Delta^3 y_{-1}+\ldots$$

Put $x=x_0$ so that $U=0$

$$\therefore\quad \left(\frac{dy_{x+\frac{1}{2}}}{dx}\right)_{x=x_0}=\Delta y_0-\frac{1}{24}\Delta^3 y_{-1}+\ldots\ldots$$

Shifting the origin from x_0 to $x-\frac{1}{2}$

$$\frac{dy_x}{dx}=\Delta y_{x-\frac{1}{2}}-\frac{1}{24}\Delta^3 y_{x-\frac{3}{2}}+\ldots\ldots$$

5.2. Numerical Integration. *It is the process of computing the value of a definite integral from a set of numerical values of the integrand. When the function to be integrated is of single variable the process is called* **Mechanical Quadrature.** *The procedure is to represent the integrand by an interpolation formula and then to integrate this formula between the desired limits.*

Quadrature Formulae.

(1) **Trapezoidal rule.**

$$\int_{x_0}^{x_0+nh} y\,dx=h\left\{\frac{y_0+y_n}{2}+(y_1+y_2+\ldots\ldots+y_{n-1})\right\}$$

(2) **Simpson's one-third rule.**

$$\int_{x_0}^{x_0+nh} y\,dx=\frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1})+2(y_2+y_4+\ldots+y_{n-2})\Big]$$

(Can be applied only when 'n' is even)

(3) **Simpson's three-eighth rule.**

$$\int_{x_0}^{x_0+nh} y\,dx=\frac{3}{8}h[(y_0+y_n)+3\{(y_1+y_2)+(y_4+y_5)+\ldots\ldots+(y_{n-2}+y_{n-1})\}$$
$$+2(y_3+y_6+\ldots\ldots+y_{n-3})]$$

(Can be applied only when 'n' is a multiple of '3')

(4) **Weddle's rule.**

$$\int_{x_0}^{x_0+nh} y\,dx=\frac{3h}{10}[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6)$$
$$+(y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}+y_{12})+\ldots\ldots\ldots$$
$$+(y_{n-6}+5y_{n-5}+y_{n-4}+6y_{n-3}+y_{n-2}+5y_{n-1}+y_n)]$$

(Can be applied only when 'n' is a multiple of '6'.)

Ex. 5-11. *Derive general quadrature formula for equidistant ordinates and deduce from it.*

(1) *The Trapezoidal rule.*

(2) *Simpson's one-third rule.*

(3) *Simpson's three-eighth rule.*

(4) *Weddle's rule.*

Sol. Let $y=f(x)$ be the function and it is required to evaluate

$$I=\int_a^b f(x)\,dx.$$

Divide the range 'a' to 'b' into n-equal parts and let the points of division be

$$x_0=a,\ x_1=x_0+h,\ldots\ldots\ldots x_n=x_0+nh=b.$$

Let the values of y at these points of division be $y_0, y_1, \ldots y_n$.

The method is to represent the integrand $f(x)$ by an interpolation formula and then to integrate this formula between the desired limits. Thus representing $f(x)$ by Newton's formula of forward differences.

$$f(x)=y_0+U\Delta y_0+\frac{U(U-1)}{2\,!}\Delta^2 y_0+\frac{U(U-1)(U-2)}{3\,!}\Delta^3 y_0+\ldots$$

where $U=\dfrac{x-x_0}{h}$

$$\therefore\qquad I=\int_a^b f(x)dx=\int_{x_0}^{x_0+nh} f(x)\,dx$$

$$=h\int_0^n\left\{ y_0+U\Delta y_0+\frac{U(U-1)}{2\,!}\Delta^2 y_0+\frac{U(U-1)(U-2)}{3\,!}\Delta^3 y_0\right.$$

$$+\frac{U(U-1)(U-2)(U-3)}{4\,!}\Delta^4 y_0$$

$$\left.+\frac{U(U-1)(U-2)(U-3)(U-4)}{5\,!}\Delta^5 y_0\right\}$$

$$\left.+\frac{U(U-1)(U-2)(U-3)(U-4)(U-5)}{6\,!}\Delta^6 y_0+\ldots\ldots\right\}dU$$

$$=h\left[ny_0+\frac{n^2}{2}n\Delta y_0+\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2\,!}+\left(\frac{n^4}{4}-n^3+n^2\right)\frac{\Delta^3 y_0}{3\,!}\right.$$

$$+\left(\frac{n^5}{5}-\frac{3}{2}n^4+\frac{11}{3}n^3-3n^2\right)\frac{\Delta^4 y_0}{4\,!}+\left(\frac{n^6}{6}-2n^5+\frac{35}{4}n^4\right.$$

$$\left.-\frac{50}{3}n^3+12n^2\right)\frac{\Delta^5 y_0}{5\,!}+\left(\frac{n^7}{7}-\frac{15\,n^6}{6}+17\,n^5-\frac{225\,n^4}{4}\right.$$

$$\left.\left.+\frac{274}{3}n^3-60n^2\right)\frac{\Delta^6 y_0}{6\,!}+\ldots\ldots\right] \qquad \ldots(A)$$

This formula is general quadrature formula.

(1) Trapezoidal rule.

Putting $n=1$ in (A)

$$I_1=\int_{x_0}^{x_1} f(x)dx=h\left\{y_0+\frac{1}{2}\Delta y_0\right\}$$

Second and higher differences have been neglected as, since the interval of integration extends from x_0 to $x_1=x_0+h$, there are only two values and with these there can be no differences higher than the one.

$$\therefore \qquad I_1=h\left\{y_0+\frac{y_1-y_0}{2}\right\}=\frac{h}{2}(y_0+y_1)$$

Similarly $\quad I_2=\displaystyle\int_{x_1}^{x_2} f(x)\,dx=\frac{h}{2}(y_1+y_2)$

....

$$I_n=\int_{x_{n-1}}^{x_n} f(x)\,dx=\frac{h}{2}(y_{n-1}+y_n)$$

$$\therefore \quad I=\int_{x_0}^{x_0+nh} f(x)\,dx=\int_{x_0}^{x_1} f(x)\,dx+\int_{x_1}^{x_2} f(x)\,dx+\ldots\ldots+\int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$=I_1+I_2+\ldots\ldots+I_n$$

$$=\frac{h}{2}\left\{ y_0+2(y_1+y_2+\ldots\ldots+y_{n-1})+y_n\right\}$$

$$=h\left\{\frac{(y_0+y_n)}{2}+(y_1+y_2+\ldots\ldots+y_{n-1})\right\}$$

(2) Simpson's one-third rule.

Putting $n=2$ in (A) and neglecting third and higher differences.

$$I_1=\int_{x_0}^{x_2} f(x)\,dx=h\left\{ 2y_0+2\Delta y_0+\frac{1}{3}\ \Delta^2 y_0 \right\}$$

$$=h\left\{ 2y_0+2(y_1-y_0)+\frac{1}{3}\ (E-1)^2 y_0 \right\}$$

$$=h\left\{ 2y_0+2(y_1-y_0)+\frac{1}{3}\ (y_2-2y_1+y_0) \right\}$$

$$=\frac{h}{3}\left\{ y_0+4y_1+y_2\right\}$$

Similarly

$$I_2=\int_{x_2}^{x_4} f(x)\,dx=\frac{h}{3}\ \{y_2+4y_3+y_4\}$$

....

$$I_{n/2}=\int_{x_{n-2}}^{x_n} f(x)\,dx=\frac{h}{3}\ \{y_{n-2}+4y_{n-1}+y_n\} \quad \text{(assuming } n \text{ to be even)}$$

$$\therefore \quad I=\int_{x_0}^{x_n} f(x)dx=I_1+I_2+\ldots\ldots+I_{n/2}$$

$$=\frac{h}{3}\ [(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+\ldots\ldots+y_{n-2}+4y_{n-1}+y_n)$$

$$=\frac{h}{3}\ [(y_0+y_n)+4(y_1+y_3+\ldots\ldots+y_{n-1})+2(y_2+y_4+\ldots+y_{n-2})]$$

(3) Simpson's three-eighth rule.

Putting $n=3$ in (A) and neglecting all differences above the third.

$$I_1=\int_{x_0}^{x_3} f(x)dx=h\left\{3y_0+\frac{9}{2}\,\Delta y_0+\frac{9}{4}\,\Delta^2 y_0+\frac{3}{8}\,\Delta^3 y_0\right\}$$

$$=h\left\{3y_0+\frac{9}{2}\,(y_1-y_0)+\frac{9}{4}\,(E-1)^2 y_0+\frac{3}{8}\,(E-1)^3 y_0\right\}$$

$$=h\left\{3y_0+\frac{9}{2}\,(y_1-y_0)+\frac{9}{4}\,(E^2-2E+1)y_0\right.$$
$$\left.+\frac{3}{8}\,(E^3-3E^2+3E-1)y_0\right\}$$

$$=\frac{3}{8}\,h\left\{y_0+3y_1+3y_2+y_3\right\}$$

Similarly,

$$I_2=\int_{x_3}^{x_6} f(x)\,dx=\frac{3}{8}\,h\left\{y_3+3y_4+3y_5+y_6\right\}$$

....

$$I_{n/3}=\int_{x_{n-3}}^{x_n} f(x)\,dx=\frac{3}{8}\,h\left\{y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right\}$$

(assuming n to be a multiple of 3)

$$\therefore\quad I=\int_{x_0}^{x_n} f(x)\,dx=I_1+I_2+\ldots\ldots I_{n/3}$$

$$=\frac{3}{8}\,h\,[(y_0+y_n)+3\{(y_1+y_2)+(y_4+y_5)+\ldots\ldots\ldots$$
$$+(y_{n-2}+y_{n-1})\}+2(y_3+y_6+\ldots\ldots+y_{n-3})]$$

(4) Weddle's rule.

Putting $n=6$ in (A) and neglecting differences above sixth.

$$I_1=\int_{x_0}^{x_6} f(x)\,dx=h\left[6y_0+18\,\Delta y_0+27\,\Delta^2 y_0+24\,\Delta^3 y_0+\frac{123}{10}\,\Delta^4 y_0\right.$$
$$\left.+\frac{33}{10}\,\Delta^5 y_0+\frac{41}{140}\,\Delta^6 y_0\right]$$

$$=h\Big[\,6y_0+18(E-1)y_0+27(E-1)^2y_0+24(E-1)^3y_0$$

$$+\frac{123}{10}(E-1)^4y_0+\frac{33}{10}(E-1)^5y_0+\frac{3}{10}(E-1)^6y_0\Big]-\frac{1}{140}h\,\Delta^6y_0$$

$$=\frac{3h}{10}\Big\{y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\Big\}-\frac{1}{140}h\,\Delta^6y_0$$

Choosing 'h' *s.t.* the sixth differences are small, the last term can be neglected.

Then $I_1=\dfrac{3h}{10}\Big\{y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\Big\}$

Similarly,

$$I_2=\int_{x_6}^{x_{12}} f(x)dx=\frac{3h}{10}(y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}+y_{12})$$

....

$$I_n=\int_{x_{n-6}}^{x_n} f(x)dx=\frac{3h}{10}\{y_{n-6}+5y_{n-5}+y_{n-4}+6y_{n-3}+y_{n-2}+5y_{n-1}+y_n\}$$

(assuming n to be a multiple of 6)

$$\therefore\quad I=\int_{x_0}^{x_n} f(x)dx=I_1+I_2+\ldots\ldots+I_{n/6}$$

$$=\frac{3h}{10}[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6)+(y_6+5y_7+y_8$$

$$+6y_9+y_{10}+5y_{11}+y_{12})+\ldots\ldots+(y_{n-6}+5y_{n-5}+y_{n-4}$$
$$+6y_{n-3}+y_{n-2}+5y_{n-1}+y_n)]$$

Note. Since in 'Trapezoidal Rule' second and higher difference are neglected, y is assumed to be linear. Similarly in 'Simpson one-third rule', 'Simpson's three-eighth rule' and 'Weddle's rule' y is assumed to be polynomials of degree second, third and sixth respectively.

Ex. 5·12. *Evaluate* $I=\displaystyle\int_{1\cdot0}^{1\cdot3}\sqrt{x}\,dx$

by (1) *Simpson's rule.*

(2) *Trapezoidal rule.*

Sol. Take $h=0{\cdot}1$

x :	1·0	1·1	1·2	1·3
$y=\sqrt{x}$:	1·000000	1·048809	1·095445	1·140175

(1) By Simpson's three-eighth rule

$$I=\frac{3}{8}h[(y_0+y_3)+3\{y_1+y_2\}]$$

$$=\frac{3}{8}(0{\cdot}1)[2{\cdot}140175+6{\cdot}432762]$$

$$=0{\cdot}321485$$

(2) By Trapezoidal rule.

$$I=h\left[\frac{y_0+y_3}{2}+(y_1+y_2)\right]$$

$$=(0{\cdot}1)[1{\cdot}0700875+2{\cdot}144254]$$

$$=0{\cdot}32143415\simeq 0{\cdot}321434$$

Ex-5-13 *Evalute* $I=\int_0^{\pi/2}\cos x\,dx$

by (1) *Trapezoidal rule.*

(2) *Simpson's rule.*

Sol. Take $h=\frac{\pi}{2}$

x	$y=\cos x$	x	$y=\cos x$
0	1·000000	$\frac{6\pi}{20}$	0·587785
$\frac{\pi}{20}$	0·987688	$\frac{7\pi}{20}$	0·453990
$\frac{2\pi}{20}$	0·951057	$\frac{8\pi}{20}$	0·309017
$\frac{3\pi}{20}$	0·891007	$\frac{9\pi}{20}$	0·156434
$\frac{4\pi}{20}$	0·809017	$\frac{\pi}{20}$	0·000000
$\frac{5\pi}{20}$	0·707107		

(1) By Trapezoidal rule,

$$I=h\left[\frac{y_0+y_{10}}{2}+(y_1+y_2+\ldots\ldots+y_9)\right]$$

$$=\frac{\pi}{20}[0{\cdot}500000+5{\cdot}853102]$$

$$=0{\cdot}9983446\simeq0{\cdot}998345$$

(2) By Simpson's one-third rule.

$$I=\frac{h}{3}\;[(y_0+y_{10})+(4y_1+y_3+\ldots+y_9)+2(y_2+y_4+\ldots+y_8)]$$

$$=\frac{\pi}{20}[1{\cdot}000000+12{\cdot}784904+5{\cdot}313752]$$

$$=1{\cdot}000406$$

Ex. 5·14. *Evaluate* $I=\int_0^6\frac{dx}{1+x^2}$

by (1) *Weddle's rule.*
(2) *Simpson's one-third rule*
(3) *Simpson's three-eighth rule.*

Sol. Divide the range of integration into twelve equal parts by taking $h=0{\cdot}5$

x	$f(x)=\frac{1}{1+x^2}$	x	$f(x)=\frac{1}{1+x^2}$
0·0	1·000000	3·5	0·0754717
0·5	0·800000	4·0	0·0588235
1·0	0·500000	4·5	0·0470588
1·5	0·307692	5·0	0·0384615
2·0	0·200000	5·5	0·0320000
2·5	0·137931	6·0	0·0270270
3·0	0·100000		

(1) By Wedcle's rule,

$$I=\frac{3}{10}(0{\cdot}5)[(y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6)$$
$$+(y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}+y_{12})]$$

$$=(0{\cdot}15)[8{\cdot}335807+1{\cdot}0440233]$$

$$=1{\cdot}406974545\simeq1{\cdot}407$$

(2) By Simpson's one-third rule.

$$I=\frac{0\cdot 5}{3}[(y_0+y_{12})+4(y_1+y_3+\ldots\ldots+y_{11})$$
$$+2(y_2+y_4+\ldots+y_{10})]$$

$$=\frac{0\cdot 5}{3}[1\cdot 0270270+5\cdot 6006140+1\cdot 7945700]$$

$$=1\cdot 4037018\simeq 1\cdot 404$$

(3) By Simpson's three-eighth rule

$$I=\frac{3}{8}(0\cdot 5)[(y_0+y_{12})+3\{(y_1+y_2)+(y_4+y_5)+(y_7+y_8)$$
$$+(y_{10}+y_{11})\}+2(y_3+y_6+y_9)]$$

$$=\frac{3}{8}(0\cdot 5)[1\cdot 0270270+5\cdot 5280631+0\cdot 9095016]$$
$$=1\cdot 39961\simeq 1\cdot 400$$

Ex. 5-15. *Compute by Simpson's rule the value of the integral*

$$I=\int_{200}^{1000}\frac{dx}{\log_{10} x}$$

taking eight subintervlas.

Sol. Here $h=100$

x	$\log_{10} x$	$y=(\log_{10} x)^{-1}$	x	$\log_{10} x$	$y=(\log_{10} x)^{-1}$
200	2·3010	0·434594	700	2·8451	0·3514815
300	2·4771	0·403698	800	2·9031	0·344459
400	2·6021	0·384305	900	2·9542	0·338501
500	2·6990	0·370508	1000	3·0000	0·333333
600	2·7782	0·359945			

By Simpson's one-third rule,

$$I=\frac{100}{3}[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$

$$=\frac{100}{3}[0\cdot 767927+5\cdot 856754+2\cdot 177418]$$

$$=293\cdot 4033\simeq 293\cdot 4$$

Ex. 5-16. *Compute $\log_e 2$ using a suitable quadrature formula with 7 ordinates to evaluate*

$$\int_0^3\frac{dx}{1+x}$$

Sol. Divide the range of integration in to six parts by taking $h=0{\cdot}5$

x	$f(x)=\dfrac{1}{1+x}$	x	$f(x)=\dfrac{1}{1+x}$
0·0	1·000000	2·0	0·333333
0·5	0·666667	2·5	0·285714
1·0	0·500000	3·0	0·250000
1·5	0·400000		

By Weddle's rule,

$$I=\int_0^3 \frac{dx}{1+x}=\frac{3}{10}\,(0{\cdot}5)\{1{\cdot}000000+5(0{\cdot}666667)+0{\cdot}500000$$
$$+6(0{\cdot}400000)+(0{\cdot}333333)+5(0{\cdot}285714)+(0{\cdot}250000)\}$$

$=1{\cdot}3867857$

Also $\quad I=\displaystyle\int_0^3 \frac{dx}{1+x}=\Big|\log(1+x)\Big|_0^3=\log_e 4=2\log_e 2$

$\therefore \qquad 2\log_e 2=1{\cdot}3867857$

$\therefore \qquad \log_e 2=0{\cdot}69339285\simeq 0{\cdot}693$

Ex. 5-17. *Applying 'Simpson's one-third rule' evaluate*

$$\int_0^1 \frac{dx}{1+x}$$

correct to three decimal places.

Sol. Divide the range of integration into 10 equal parts by taking $h=0{\cdot}1$

x	$f(x)=\dfrac{1}{1+x}$	x	$f(x)=\dfrac{1}{1+x}$
0·0	1·000000	0·6	0·625000
0·1	0·909091	0·7	0·588235
0·2	0 833333	0·8	0·555556
0·3	0·769231	0·9	0·526316
0·4	0·714286	1·0	0·500000
0·5	0·666667		

$$\therefore \quad I=\int_0^1 \frac{1}{1+x}=\frac{h}{3}\,\{(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)$$
$$+2(y_2+y_4+y_6+y_8)\}$$

$$=\frac{0\cdot 1}{3}\{1\cdot 500000+4(3\cdot 459540)+2(2\cdot 728175)\}$$

$$=0\cdot 69315$$

$$\simeq 0\cdot 693$$

Ex. 5·18. *Evalute* $I=\int_{0\cdot 2}^{1\cdot 4}(\sin x-\log x+e^x)dx$

by (1) ***Simpson's rule***

(2) ***Weddle's rule***

taking $h=0\cdot 1$ *in each case,*

Sol. Let $y=\sin x-\log x+e^x$.

x	$\sin x$	$\log x$	e^x	y
0·2	0·198669	−1·609438	1·221403	3 029510
0·3	0·295520	−1·203973	1·349859	2·849352
0·4	0·389418	−0·916291	1·491825	2·797534
0·5	0·479426	−0·693147	1·648721	2·821294
0·6	0·564642	−0·510826	1·822119	2·897587
0·7	0·644218	−0·356675	2·013753	3 014646
0·8	0·717356	−0·223143	2·225541	3·166040
0·9	0·783327	−0·105360	2·459603	3·348290
1·0	0·841471	0·000000	2·718282	3·559753
1·1	0·891207	0·095310	3·004166	3·800063
1·2	0·932039	0·182322	3·320117	4 069834
1·3	0·963558	0·262364	3·669297	4·370491
1·4	0·985450	0·336472	4·055200	4·704178

(1) By Simpson's one-third rule,

$$I=\frac{h}{3}\left[(y_0+_{12})+4(y_1+y_3+y_5+y_7+y_9+y_{11})\right.$$
$$\left.+2(y_2+y_4+y_6+y_8+y_{10})\right.$$

$$=\frac{0\cdot 1}{3}[7\cdot 733688+80\cdot 816544+32\cdot 981496)]$$

$$=4\cdot 0510576\simeq 4\cdot 05106$$

(2) **Weddle's rule.**

$$I=\frac{3h}{10}[y_0+y_2+y_4+y_8+y_{10}+y_{12})+2y_6$$
$$+5(y_1+y_5+y_7+y_{11})+6(y_3+y_9)]$$

$$=\frac{3(0\cdot1)}{10}[21\cdot058396+6\cdot332080+67\cdot913895+39\cdot728142]$$

$$=4\cdot05097539\simeq4\cdot05098$$

Ex. 5-19. *Show that* $\int_0^1 y_x\,dx=\frac{1}{12}(5y_1+8y_0-y_{-1})$ *(approximately)*

Sol. By Lagrange's formula

$$y_x\simeq\frac{(x-0)(x-1)}{(-1-0)(-1-1)}y_{-1}+\frac{(x+1)(x-1)}{(0+1)(0-1)}y_0+\frac{(x+1)(x-0)}{(1+1)(1-0)}y_1$$

$$\simeq\frac{1}{2}(x^2-x)y_{-1}-(x^2-1)y_0+\frac{1}{2}(x^2+x)y_1$$

$$\therefore \int_0^1 y_x dx\simeq\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2}\right)y_{-1}-\left(\frac{1}{3}-1\right)y_0+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}\right)y_1$$

$$\simeq\frac{1}{12}(5y_1+8y_0-y_{-1})$$

Ex. 5-20. *Show that*

$$\int_{-1/2}^{1/2} y_x dx=\frac{1}{2}\left\{y_{-1/2}+y_{1/2}\right\}+\frac{1}{24}\left\{\Delta y_{-3/2}-\Delta y_{1\cdot2}\right\}$$ *(approximately)*

Sol. By Lagrange's formula

$$y_x=\frac{\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\left(x-\frac{3}{2}\right)}{\left(-\frac{3}{2}+\frac{1}{2}\right)\left(-\frac{3}{2}-\frac{1}{2}\right)\left(-\frac{3}{2}-\frac{3}{2}\right)}y_{-3/2}$$

$$+\frac{\left(x+\frac{3}{2}\right)\left(x-\frac{1}{2}\right)\left(x-\frac{3}{2}\right)}{\left(-\frac{1}{2}+\frac{3}{2}\right)\left(-\frac{1}{2}-\frac{1}{2}\right)\left(-\frac{1}{2}-\frac{3}{2}\right)}y_{-1/2}$$

$$+\frac{\left(x+\frac{3}{2}\right)\left(x+\frac{1}{2}\right)\left(x-\frac{3}{2}\right)}{\left(\frac{1}{2}+\frac{3}{2}\right)\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{3}{2}\right)}y_{1/2}$$

$$+\frac{\left(x+\frac{3}{2}\right)+\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}{\left(\frac{3}{2}+\frac{3}{2}\right)\left(\frac{3}{2}+\frac{1}{2}\right)\left(\frac{3}{2}-\frac{1}{2}\right)}y_{3/2}$$

$$=-\frac{1}{6}\left(x^3-\frac{3}{2}x^2-\frac{1}{4}x+\frac{3}{8}\right)y_{-3/2}$$

$$+\frac{1}{2}\left(x^3-\frac{1}{2}x^2-\frac{9}{4}x+\frac{9}{8}\right)y_{-1/2}-\frac{1}{2}\left(x^3+\frac{1}{2}x^2-\frac{9}{4}x-\frac{9}{8}\right)y_{1/2}$$

$$+\frac{1}{6}\left(x^3+\frac{3}{2}x^2-\frac{1}{4}x-\frac{3}{8}\right)y_{3/2}$$

$$\therefore \int_{-1/2}^{1/2} y_x dx = -\frac{1}{24}y_{-3/2}+\frac{13}{24}y_{-1/2}+\frac{13}{24}y_{1/2}-\frac{1}{24}y_{3/2}$$

$$=\frac{1}{2}\left\{y_{-1/2}+y_{1/2}\right\}+\frac{1}{24}\left\{\left(y_{-1/2}-y_{-3/2}\right)-\left(y_{3/2}-y_{1/2}\right)\right\}$$

$$=\frac{1}{2}\left\{y_{-1/2}+y_{1/2}\right\}+\frac{1}{24}\left\{\Delta y_{-3/2}-\Delta y_{1/2}\right\}$$

EXERCISE

1. Find the first derivative of the function tabulated below at the point $x=1{\cdot}002$.

x :	1·00	1·01	1·02	1·3	1·04	1·05
$f(x)$;	0·841471	0·846832	0·852108	0·857299	0·862404	0·867423

[Ans. 0·538673]

2. Find the values of $f'(10)$, $f'(15)$ and $f'(12)$ from the following table :

x :	10	11	12	13	14	15
$f(x)$:	3·162278	3·316625	3·464102	3·605551	3·741657	3·872983

[Ans. 0 158109 ; 0·1291 0·144337]

3. Find the value of $\cos 1{\cdot}76=\left[\frac{d}{dx}\{\sin x\}\right]_{x=1{\cdot}76}$ using the following table :

x	$\sin x$	x	$\sin x$
1·70	0·99166481	1·82	0 96910913
1·72	0·98888977	1·84	0·96398300
1·74	0·98571918	1·86	0·95847128
1·76	0·98215432	1·88	0·95257619
1·78	0·97819661	1·90	0·94630009
1·80	0·97384763		

[Ans. 0·18807675]

4. Find the values of $f'(0{\cdot}6)$ and $f''(0{\cdot}6)$ from the following table :

x :	0·4	0·5	0·6	0·7	0·8
$f(x)$:	1·5836494	1·7974426	2·0442376	2·3275054	2·6510818

[Ans. 2·644225 : 3·64424]

5. Find the values of $f'(1{\cdot}0)$, $f''(1{\cdot}0)$, $f'''(1{\cdot}0)$, $f'(1{\cdot}10)$, $f''(1{\cdot}10)$ and $f'''(1{\cdot}10)$ from the following table :

x	$f(x)$	x	$f(x)$
1·00	1·00000	1·20	1·09544
1·05	1·02470	1·25	1·11803
1·10	1·04881	1·30	1·14017
1·15	1·07238		

[Ans. 0·50024 : −0·256 ; 0·4 ; 0·4767 ; −0·216 ; 0·320]

6. Find the values of $f'(1{\cdot}72)$, $f'(1{\cdot}7)$ $f'(1{\cdot}5)$, $f'(2{\cdot}0)$, $f''(1{\cdot}7)$, $f''(1{\cdot}5)$ and $f''(2{\cdot}0)$ from the following table (For central values use Stirling's formula) .

x :	1·5	1·6	1·7	1·8	1·9	2·0
$f(x)$:	0·405465	0·470004	0·530628	0·587787	0·641854	0·693147

[Ans. 0·581393 ; 0·588230 ; 0·666702 ; 0 500027 ; −0·345858 : −0·445391 ; −0·248809]

7. Find the values $f'(0{\cdot}425)$, $f'(0{\cdot}65)$, $f''(0{\cdot}425)$ and $f''(0{\cdot}65)$ from the following table :

x :	0·4	0·5	0·6	0·7	0·8
$f(x)$:	0·389418	0·479426	0·564642	0·644218	0·717356

[Ans. 0·911056 ; 0·796092 ; −0·412735 ; −0·604942].

8. Prove (1) of Ex. 5-3 by (*a*) the method of Ex. 5-4 by using operators.

9. Prove (*i*) of Ex. 5-5 by (*a*) the method of Ex. 5-4 (*b*) by using operators.

10. Find the expressions of $f''(x_0)$ and $f'''(x_0)$ in terms of backward differences.

11. Evaluate $I=\displaystyle\int_2^{10}\frac{dx}{1+x}$ by dividing the range into eight equal parts. [Ans. 1·299]

12. Evaluate $I=\displaystyle\int_1^{5}\frac{dx}{x}$ by Simpson's rule. [Ans. 1·62]

13. Evaluate $I=\int_{4}^{5\cdot 2} \frac{1}{x}\, dx$ by (1) Simpson's rule (2) Weddle's rule. (Taking $h=0\cdot 2$). [Ans. 0·262364 ; 0·262364]

14. Evaluate $I=\int_{30^\circ}^{90^\circ} \log_{10} \sin x\, dx$ by Simpson's rule (taking ten subintervals). [Ans. $-0\cdot 095$]

15. Using Simpson's rule and the table :

x :	0·5	0·6	0·7	0·8	0·9	1·0	1·1
y :	0 4804	0·5669	0·6490	0·7262	0·7985	0·8658	0·9281

Evaluate the integrals.

(i) $\int_{0\cdot 5}^{1\cdot 1} xy\, dx$ (ii) $\int_{0\cdot 5}^{1\cdot 1} y^2 dx$ (iii) $\int_{0\cdot 5}^{1\cdot 1} x^2 y dx$

(iv) $\int_{0\cdot 5}^{1\cdot 1} y^3\, dx$.

[Ans. 0·3585, 0·3201, 0 3104, 0·2444]

16. Compute $I=\int_{4}^{5\cdot 2} \log_e x\, dx$ by (i) Simpson's rule (ii) Weddle's rule. (Taking $h=0\cdot 2$). [Ans. 1·827847]

17. Evaluate $\int_{0}^{4} e^x\, dx$ using Simpson's rule and the following data :

x :	0	1	2	3	4
e^x :	1	2·72	7·39	20·09	54·60

18. Evaluate $\int_{0}^{0\cdot 3} (1-8x^3)^{1/2}\, dx$ using Simpson's three-eighth rule.

[Ans. 0·29159]

19. Compute $\int_{0}^{\pi/2} \sin x\, dx$ by (1) Trapezoidal rule (2) Simpson's rule. (using 11 ordinates). [Ans. 0·9981, 1·0006]

6

Curve Fitting and Method of Least Squares

6.1. Introduction

Fitting of curves to a set of numerical data is of considerable importance—Theoretical as well as practical. It is based on the following principle known as principle of least squares.

Principle of Least Squares

It says that the best or most probable value of the measured quantity is that value for which the sum of the squares of the errors is least.

6.2. Solving System of Linear Equations

Consider the system of equations

$$a_{11}x_1+a_{12}x_2+\dots\dots+a_{1n}x_n=b_1$$
$$a_{21}x_1+a_{22}x_2+\dots\dots+a_{2n}x_n=b_2$$
$$\dots\dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$a_{m1}x_1+a_{m2}x_2+\dots\dots+a_{mn}x_n=b_m$$

In the case when $m > n$, the equations are solved by writing

$$S=\sum_{i=1}^{m} (b_i-a_{i1}x_1-a_{i2}x_2\dots\dots-a_{in}x_n)^2$$

and minimizing S. From calculus the equations determining x_1, x_2, ...x_n so that S is minimum are

$$\frac{\partial S}{\partial x_1}=0,\ \frac{6S}{\partial x_2}=0,\ \dots\dots\frac{\partial S}{\partial x_n}=0$$

These equations are called **Normal Equations** *and the values of $x_1, x_2, \dots, x_n$ obtained from these are called* **best or most plausible values.**

Ex. 6-1. *Form the normal equations and hence find the most plausible values of x and y from the following.*

$$x+y=3{\cdot}01,\ 2x-y=0{\cdot}03,\ x+3y=7{\cdot}03,\ 3x+y=4{\cdot}97$$

Sol.

Let $S=(x+y-3{\cdot}01)^2+(2x-y-0{\cdot}03)^2+(x+3y-7{\cdot}03)^2$
$+(3x+y-4{\cdot}97)^2$

Normal equations are

$$\frac{\partial S}{\partial x}=0=\frac{\partial S}{\partial y}$$

Now $\frac{\partial S}{\partial x}=0$ imply

$$2(x+y-3{\cdot}01)+4(2x-y-0{\cdot}03)+2(x+3y-7{\cdot}03)+6(3x+y-4{\cdot}97)=0.$$

i.e., $15x+5y-25{\cdot}01=0$ ...(1)

and $\frac{\partial S}{\partial y}=0$ imply

$$2(x+y-3{\cdot}01)-2(2x-y-0{\cdot}03)+6(x+3y-7{\cdot}03)+2(3x+y-4{\cdot}97)=0$$

$5x+12y-29{\cdot}04=0$ ...(2)

Solving (1) and (2)

$$x=0{\cdot}9995,\ y=2{\cdot}0035.$$

Ex. 6-2. *Find the most plausible values of x, y and z from the equations given below :*

$$x+2y+z=1,\quad 2x+y+z=4,\quad -x+y+2z=3 \text{ and } 4x+2y-5z=-7.$$

Sol.

Let $S=(x+2y+z-1)^2+(2x+y+z-4)^2+(-x+y+2z-3)^2$
$+(4x+2y-5z+7)^2$

Normal equations are

$$0=\frac{\partial S}{\partial x}=2(x+2y+z-1)+4(2x+y+z-4)-2(-x+y+2z-3)$$
$$+8(4x+2y-5z+7)$$

i.e., $22x+11y-19z+22=0$ ...(1)

$$0=\frac{\partial S}{\partial y}=4(x+2y+z-1)+2(2x+y+z-4)+2(-x+y+2z-3)$$
$$+4(4x+2y-5z+7).$$

i.e., $11x+10y-5z+5=0$ ...(2)

and $$0=\frac{\partial S}{\partial z}=2(x+2y+z-1)+2(2x+y+z-4)+4(-x+y+2z-3)$$
$$-10(4x+2y-5z+7)$$

i.e., $-19x-5y+31z-46=0$ ...(3)

From (1), (2) and (3)

$x=1{\cdot}17.,\quad y=-0{\cdot}75,\quad z=2{\cdot}08.$

Ex. 6-3. *Find the most plausible values of x, y and z from the following equations :*

$$x-y+2z=3,\ 3x+2y-5z=5,\ 4x+y+4z=21,\ -x+3y+3z=14.$$

Sol.

Let $S=(x-y+2z-3)^2+(3x+2y-5z-5)^2+(4x+y+4z-21)^2$
$\qquad +(-x+3y+3z-14)^2$

Normal equations are

$$0=\frac{\partial S}{\partial x}=2(x-y+2z-3)+6(3x+2y-5z-5)$$
$$+8(4x+y+4z-21)-2(-x+3y+3z-14)$$

i.e., $\qquad 27x+6y-88=0 \qquad$ ...(1)

$$0=\frac{\partial S}{\partial y}=-2(x-y+2z-3)+4(3x+2y-5z-5)$$
$$+2(4x+y+4z-21)+6(-x+3y+3z-14)$$

i.e., $\qquad 6x+15y+z-70=0 \qquad$ ...(2)

$$0=\frac{\partial S}{\partial z}=0=4(x-y+2z-3)-10(3x+2y-5z-5)$$
$$+8(4x+y+4z-21)+6(-x+3y+3z-14)$$

i.e., $\qquad y+54z-107=0 \qquad$ ...(3)

From (1), (2) and (3)

$x=2{\cdot}47,\quad y=3{\cdot}55,\quad z=1{\cdot}92.$

Ex. 6-4. *A man is three times as old as his son. Ten years hence his age will be twice the age of his son. Five years before the age of the man was five times that of his son. Find their present ages.*

Sol. Let x and y be the ages of the man and his son.

Then $\qquad x=3y,$

$\qquad x+10=2(y+10) \qquad$ *i.e.,* $x-2y-10=0$

and $\qquad x-5=5(y-5) \qquad$ *i.e.,* $x-5y+20=0$

Let $S=(x-3y)^2+(x-2y-10)^2+(x-5y+20)^2$

Normal equations are

$$0=\frac{\partial S}{\partial x}=2(x-3y)+2(x-2y-10)+2(x-5y+20)$$

i.e., $\qquad 3x-10y+10=0 \qquad$ ...(1)

and $\quad 0=\frac{\partial S}{\partial y}=-6(x-3y)-4(x-2y-10)-10(x-5y+20)$

i.e., $\qquad -10x+38y-80=0 \qquad$ ...(2)

From (1) and (2),

$$x=30, \quad y=10.$$

6.3. The Method of Least Squares.

Let $y=f(x)$ be the formula (containing m unknown parameters $a_1, a_2 \ldots\ldots a_m$), whose form is to be inferred from the results of experiment or observation and in which the unknown parameters are to be determined from experimental or observational data $(x_1, y_1), (x_2, y_2) \ldots\ldots (x_n, y_n)$ $(n>m)$. These sets of simultaneous values of x and y would, when substituted in the formula, give n equations in m unknowns $a_1, a_2, \ldots\ldots a_m$ and from these equations the best values of $a_1, a_2 \ldots\ldots a_m$ are to be obtained. To solve this problem method of Least Squares is used.

The method of Least Squares says that the best representative formula is that for which the sum of the squares of the residuals (i.e., most probable value—measured value) is minimum. Since the squares of the residuals are positive, the requirement that their sum shall be as small as possible ensures that the numerical values of the residuals will be small.

Let $\quad \gamma_i=f(x_i)$

$\therefore$ *Residual for* $x=x_i \quad =\gamma_i-y_i$

$\therefore$ *By the method of Least squares* $a_1, a_2, \ldots\ldots a_m$ *are to be obtained so that*

$$S=\sum_{i=1}^{n} \{\gamma_i-y_i\}^2=\sum_{i=1}^{n} \{f(x_i)-y_i\}^2$$

is minimum. From calculus the equations determining $a_1, a_2 \ldots\ldots a_m$ *are*

$$\frac{\partial S}{\partial a_1}=0=\frac{\partial S}{\partial a_2}=\ldots\ldots=\frac{\partial S}{\partial a_m}$$

These equations are called **Normal Equations.**

6.4. Curve Fitting

(i) Fitting of Parabolic Curves.

To derive the least square equations for fitting a curve of the type.

$$y=a_0+a_1x+a_2x^2+\ldots\ldots+a_mx^m \qquad (a_m \neq 0)$$

to a set of n points.

Let (x_i, y_i) $i=1, 2, \ldots n$ be the given data and

$$\gamma_i=a_0+a_1x_i+a_2x_i^2+\ldots\ldots+a_mx_i^m$$

Let $\quad S=\sum_{i=1}^{n} (\gamma_i-y_i)^2=\sum_{i=1}^{n} \{a_0+a_1x_i+\ldots\ldots+a_mx_i^m-y_i\}^2$

By the method of least squares S is to be minimized. Normal equations are

$$\frac{\partial S}{\partial a_0}=0=\frac{\partial S}{\partial a_1}=\ldots\ldots=\frac{\partial S}{\partial a_n}$$

$$\text{Now } \frac{\partial S}{\partial a_0}=\sum_{i=1}^{n} 2\{a_0+a_1x_i+a_2x_i^2+\ldots\ldots+a_mx_i^m-y_i\}$$

$\therefore$ First normal equation is

$$\sum_{i=1}^{n} y_i=na_0+a_1\sum_{i=1}^{n} x_i+a_2\sum_{i=1}^{n} x_i^2+\ldots\ldots+a_m\sum_{i=1}^{n} x_i^m \quad \ldots(1)$$

Similarly $\frac{\partial S}{\partial a_1}=0, \ldots\ldots \frac{\partial S}{\partial a_m}=0$ imply

$$\sum_{i=1}^{n} x_iy_i=a_0\sum_{i=1}^{n} x_i+a_1\sum_{i=1}^{n} x_i^2+\ldots\ldots+a_m\sum_{i=1}^{n} x_i^{m+1} \quad \ldots(2)$$

$$\sum_{i=1}^{n} x_i^2y_i=a_0\sum_{i=1}^{n} x_i^2+a_1\sum_{i=1}^{n} x_i^3+\ldots\ldots a_m\sum_{i=1}^{n} x_i^{m+2} \quad \ldots(3)$$

....

$$\sum_{i=1}^{n} x_i^my_i=a_0\sum_{i=1}^{n} x_i^m+a_1\sum_{i=1}^{n} x_i^{m+1}+\ldots\ldots a_m\sum_{i=1}^{n} x_i^{2m} \ldots(m+1)$$

Eqs. (1) to $(m+1)$ are required equations.

6.4.1. Corollary. *Fitting of a straight line.*

The equation of a straight line is

$$y=a_0+a_1x$$

$\therefore$ The normal equations are

$$\Sigma y=na_0+a_1\Sigma x$$

and
$$\Sigma xy=a_0\Sigma x+a_1\Sigma x^2$$

The calculations are simplified by taking the variables v and u (to be chosen suitably) instead of y and x. Generally, u is chosen s.t. $\Sigma u=0$.

Ex. 6-5. *Fit a straight line trend by the method of least squares to the following data :*

6.6

Year	*Milk consumption (Million Gallons)*
1940	102·3
1941	101·9
1942	105·8
1943	112·0
1944	114·8
1945	118·7
1946	124·5
1947	102·9

Sol. Let x and y be the variables for years and Milk consumption.

x	u	y	v	u^2	uv	
1940	−4	102·3	−9·7	16	38·8	
1941	−3	101·9	−10·1	9	30·3	
1942	−2	105·8	−6·2	4	12·4	
1943	−1	112·0	0	1	0	$u=x-1944$
1944	0	114·8	2·8	0	0	$v=y-112$
1945	1	118·7	6·7	1	6·7	
1946	2	124·5	12·5	4	25·0	
1947	3	102·9	−9·1	9	−27·3	
	−4		−13·1	44	84·9	

Let the straight line to be fitted be

$$v=a+bu$$

where the co-efficients 'a' and 'b' are to be determined from the normal equations

$$\Sigma v=na+b\Sigma u$$

$$\Sigma uv=a\Sigma u+b\Sigma u^2$$

Substituting the values of Σv, Σu etc.

$$-13{\cdot}1=8a-4b \qquad \ldots(i)$$

$$85{\cdot}9=(-4a)+44b \qquad \ldots(ii)$$

$\therefore \qquad b=1{\cdot}89, \quad a=-0{\cdot}69$

$\therefore$ Eq. of the straight line is

$$y=-0{\cdot}69+1{\cdot}89u$$

i.e., $\qquad y-112=-0{\cdot}69+1{\cdot}89\,(x-1944).$

Ex. 6-6. *Fit a straight line trend by the method of least squares to the following series.*

year	*Price Index*
1951	107
1952	110
1953	114
1954	112
1955	115
1956	113

Sol. Let x and y be the variables for years and Price Index.

Years x	u	*Price Index* y	v	u^2	uv	
1951	-3	107	-5	9	15	
1952	-2	110	-2	4	4	
1953	-1	114	2	1	-2	$u=(x-1954)$
1954	0	112	0	0	0	$v=(y-112)$
1955	1	115	3	1	3	
1956	2	113	1	4	2	
	-3		-1	19	22	

Let the equation of the straight line to be fitted be

$$v=a+bu$$

where the co-efficients 'a' and 'b' are to be determined from normal equations

$$\Sigma v=na+b\Sigma u$$

$$\Sigma uv=a\Sigma u+b\Sigma u^2$$

Substituting the values of Σu, Σv etc.

$$-1=6a-3b \qquad \ldots(1)$$

and $$22=(-3)a+19b \qquad \ldots(2)$$

Multiplying (2) by '2' and adding to (1) we get

$$43=35(b)$$

or $$b=1{\cdot}23$$

$\therefore$ From (1)

$$6a=2{\cdot}69$$

or $$a=0{\cdot}45$$

$\therefore$ The equation is

$$v=0{\cdot}45+1{\cdot}23u$$

or $$y-112=0{\cdot}45+1{\cdot}23\,(x-1954).$$

Ex. 6-7. *Compute the straight line trend equation for the data below by the method of least squares; determine the annual trend estimates of each year.*

Year	*Sales in thousands of Rs.*
1949	25
1950	30
1951	40
1952	50
1953	45

Sol. Let x and y be the variables for years and sales.

Years x	*Sales* y	u	v	u^2	uv	*Total Values*	
1949	25	−2	−15	4	30	26	
1950	30	−1	−10	1	10	32	$u=(x-1951)$
1951	40	0	0	0	0	38	
1952	50	1	10	1	10	44	$v=(y-40)$
1953	45	2	5	4	10	50	
		0	−10	10	60		

Let the equation of the straight line be

$$v=a+bu$$

where the co-efficient 'a' and 'b' are to be determined from the normal equations

$$\Sigma v=na+b\Sigma u$$

$$\Sigma vu=a\Sigma u+b\,\Sigma u^2$$

Substituting the values of Σu etc.

$$-10=5a \quad \text{or} \quad a=-2$$

and $$60=10b \quad \text{or} \quad b=6$$

$\therefore$ The equation is

$$v=-2+6u$$

or $$(y-40)=-2+6(x-1951)$$

Annual Trend estimates for each year are shown in the table.

Ex. 6-8. *Show that the line of fit to the following data is given by* $y=0\cdot7x+11\cdot28$:

x :	0	5	10	15	20	25
y :	12	15	17	22	24	30

Sol.

x	y	u	v	u^2	uv	
0	12	−3	−5	9	15	$u=\dfrac{(x-15)}{5}$
5	15	−2	−2	4	4	$v=(y-17)$
10	17	−1	0	1	0	
15	22	0	5	0	0	
20	24	1	7	1	7	
25	30	2	13	4	26	
		−3	18	19	52	

Let the equation of the line be

$$v=a+bu$$

The normal equations are

$$\Sigma v=na+b\ \Sigma u$$

and $$\Sigma vu=a\ \Sigma u+b\ \Sigma u^2$$

Substituting the values of Σu etc.,

$$18=6a-3b \quad \ldots(1)$$

$$52=-3a+19b \quad \ldots(2)$$

$\therefore$ $$b=3\cdot486$$

$$a=4\cdot743$$

$\therefore$ The equation to the straight line is

$$v=4\cdot743+3\cdot486u$$

or $$y-17=4\cdot743+3\cdot486\left(\frac{x-15}{5}\right)$$

or $$y=0\cdot7x+11\cdot28.$$

6.4-2. Corollary. *Fitting of a second degree parabola.*

The equation of a second degree parabola is

$$y=a_0+a_1x+a_2x^2$$

$\therefore$ The normal equations are

$$\Sigma y = na_0 + a_1\Sigma x + a_2\Sigma x^2$$
$$\Sigma xy = a_0\Sigma x + a_1\Sigma x^2 + a_2\Sigma x^3$$
and
$$\Sigma x^2 y = a_0\Sigma x^2 + a_1\Sigma x^3 + a_2\Sigma x^4.$$

Ex. 6·9 *Fit a parabolic curve of second degree to the following data taking x as the independent variable :*

x :	0	1	2	3	4
y :	1	1·8	1·3	2·5	6·3

Find out the difference between the actual value of y and the value of y obtained from the fitted curve when $x=2$.

Sol.

x	y	u	v	u^2	u^3	u^4	uv	vu^2	
0	1	−2	−1·5	4	−8	16	3·0	−6·0	
1	1·8	−1	−0·7	1	−1	1	0·7	−0·7	$u=x-2$
2	1·3	0	−1·2	0	0	0	0	0	
3	2·5	1	0	1	1	1	0	0	$v=y-2{\cdot}5$
4	6·3	2	3·8	4	8	16	7·6	15·2	
		0	0·4	10	0	34	11·3	8·5	

Let the equation of the parabola be

$$v = a + bu + cu^2$$

The normal equations are

$$\Sigma v = na + b\,\Sigma u + c\,\Sigma u^2$$
$$\Sigma vu = a\,\Sigma u + b\,\Sigma u^2 + c\Sigma u^3$$
$$\Sigma vu^2 = a\,\Sigma u^2 + b\Sigma u^3 + c\Sigma u^4$$

Substituting the values of Σu, Σv etc.,

$$0{\cdot}4 = 5a + 10c \qquad \cdots(1)$$
$$11{\cdot}3 = 10b$$
or
$$b = 1{\cdot}13$$
$$8{\cdot}5 = 10a + 34c \qquad \cdots(2)$$
$\therefore$
$$c = 0{\cdot}55,\ a = -1{\cdot}02$$

$\therefore$ The equation of the second degree parabola fitted to the given data is

$$v = -1{\cdot}02 + 1{\cdot}13u + 0{\cdot}55u^2$$
$$y = 1{\cdot}48 + 1{\cdot}13(x-2) + 0{\cdot}55(x-2)^2$$

$\therefore$ Value of y for $x=2$ obtained from the fitted curve

$$=1{\cdot}48$$

Also, actual value of y for $x=2=1{\cdot}3$.

$\therefore$ Difference$=1{\cdot}3-1{\cdot}48=-0{\cdot}18$

Ex. 6-10. *Fit a parabolic curve of Regression of y on x to pairs of values :*

x :	1·0	1·5	2·0	2·5	3·0	3·5	4·0
y :	1·1	1·3	1·6	2·0	2·7	3·4	4·1

Sol.

x	u	y	v	u^2	u^3	u^4	uv	u^2v	
1·0	−3	1·1	−16	9	−27	81	48	−144	
1·5	−2	1·3	−14	4	−8	16	28	−56	$u=\frac{x-2{\cdot}5}{0{\cdot}5}$
2·0	−1	1·6	−11	1	−1	1	11	−11	
2·5	0	2·0	−7	0	0	0	0	0	
3·0	1	2·7	0	1	1	1	0	0	$v=\frac{y-2{\cdot}7}{0{\cdot}1}$
3·5	2	3·4	7	4	8	16	14	28	
4·0	3	4·1	14	9	27	81	42	126	
	0		−27	28	0	196	143	−57	

Let the curve to be fitted be

$$v=a+bu+cu^2$$

The normal equations are :

$$\Sigma v=na+b\,\Sigma u+c\,\Sigma u^2$$
$$\Sigma uv=a\,\Sigma u+b\,\Sigma u^2+c\,\Sigma u^3$$
$$\Sigma u^2v=a\,\Sigma u^2+b\,\Sigma u^3+c\,\Sigma u^4$$

Substituting the values of Σv, Σu etc.

$$-27=7a+28c \qquad \ldots(1)$$
$$143=28\,b$$

or $$b=\frac{143}{28}=5{\cdot}107$$

$$-57=28a+196c \qquad \ldots(2)$$

Multiplying (1) by '4' and subtracting from (2)

$$84c=51$$

or $$c=0{\cdot}607$$

Multiplying (1) by '7' and subtracting from (2)

$$21a=-132$$

or $$a=-6{\cdot}286.$$

$\therefore$ The equation of the second degree parabola fitted to the given data is

$$v=(-6{\cdot}286)+(5{\cdot}107)u+(0{\cdot}607)u^2$$

or $$\left(\frac{y-2{\cdot}7}{0.1}\right)=(-6{\cdot}286)+(5{\cdot}107)\left(\frac{x-2{\cdot}5}{0{\cdot}5}\right)+(0{\cdot}607)\left(\frac{x-2{\cdot}5}{0{\cdot}5}\right)^2$$

$$y=1{\cdot}0354-0{\cdot}1926x+(0{\cdot}2428)x^2$$

Ex. 6-11. *Fit a second degree parabola to the following data, taking x as the independent variable :*

x :	1	2	3	4	5	6	7	8	9
y :	2	6	7	8	10	11	11	10	9

Sol.

x	u	y	v	u^2	u^3	u^4	uv	u^2v	
1	−4	2	−6	16	−64	256	24	−96	
2	−3	6	−2	9	−27	81	6	−18	
3	−2	7	−1	4	−8	16	2	−4	
4	−1	8	0	1	−1	1	0	0	$u=x-5$
5	0	10	2	0	0	0	0	0	
6	1	11	3	1	1	1	3	3	
7	2	11	3	4	8	16	6	12	$v=y-8$
8	3	10	2	9	27	81	6	18	
9	4	9	1	16	64	256	4	16	
	0		2	60	0	708	51	−69	

Let the parabola of second degree to be fitted be

$$v=a+bu+cu^2$$

The normal equations for this are :

$$\Sigma v=na+b\,\Sigma u+c\,\Sigma u^2$$
$$\Sigma vu=a\,\Sigma u+b\,\Sigma u^2+c\,\Sigma u^3$$
$$\Sigma vu^2=a\,\Sigma u^2+b\,\Sigma u^3+c\,\Sigma u^4$$

Substituting the values of Σu, Σv etc., in these equations :

$$2=9a+60c \qquad \ldots(1)$$
$$51=60b$$

or $$b=0{\cdot}85$$

$$-69=60a+708c$$

i.e. $$-23=20a+236c \quad \ldots(2)$$

Multiplying (1) by '(20)' and (2) by '9' and subtracting

$$924c=-247$$

or $$c=-0{\cdot}267$$

$\therefore$ From (1),

$$a=\left(\frac{18{\cdot}02}{9}\right)=2{\cdot}002$$

Thus the equation of the parabola of second degree is

$$v=2{\cdot}002+(0{\cdot}85)u-(0{\cdot}267)\,u^2$$

Substituting the values of u and v

$$(y-8)=2{\cdot}002+(0{\cdot}85)(x-5)-(0{\cdot}267)(x-5)^2.$$

$$y=(-0{\cdot}923)+(3{\cdot}52)x-(0{\cdot}267)\,x^2.$$

(2) **Fitting of Curves of the type $y=ax^b$.**

Taking logarithm

$$\log_{10}y=\log_{10}a+b\log_{10}x$$

or $$\gamma=A+bX$$

(*where* $\gamma=\log_{10}y,\quad A=\log_{10}a$ *and* $X=\log_{10}x$)

The constants A and b can be obtained as in (6.4.1) by using γ *and X instead of y and x. From A, a is obained on taking antilog.*

(3) **Fitting of Curves of the type $y=ab^x$**

Taking logarithm

$$\log_{10}y=\log_{10}a+x\log_{10}b$$

or $$\gamma=A+xB$$

(*where* $\gamma=\log_{10}y,\quad A=\log_{10}a,\ B=\log_{10}b$)

The constants A, B and hence a, b are obtained as in (2).

Ex. 6-12. *The population of a state at ten yearly intervals is given below :*

Year	1911	1921	1931	1941	1951	1951	1971	1981
Population in Millions	3·9	5·3	7·3	9·6	12·9	17·1	23·2	30·5

By fitting a curve of the form $y=ab^x$ to this data estimate the population for 1991.

Year x	Population y	$\gamma=\log_{10} y$	u	u^2	$u\gamma$	
1911	3·9	0·5911	−4	16	−2·3644	
1921	5·3	0·7243	−3	9	−2·1729	
1931	7·3	0·8633	−2	4	−1·7266	
1941	9·6	0·9823	−1	1	−0·9823	$u=\dfrac{x-1951}{10}$
1951	12·9	1·1106	0	0	0	
1961	17·1	1·2330	1	1	1·2330	
1971	23·2	1·3655	2	4	2·7310	
1981	30·5	1·4843	3	9	4·4529	
		8·3544	−4	44	1·1707	

The equation to the curve to be fitted is

$$y=ab^x$$

Taking $\log_{10}$

$$\log_{10} y=\log_{10} a+x \log_{10} b$$

or

$$\gamma=A+xB$$

where $\gamma=\log_{10} y$, $A=\log_{10} a$, $B=\log_{10} b$

Replacing x by u

$$\gamma=A+Bu$$

The normal equations are

$$\Sigma\gamma=nA+B\Sigma u$$

$$\Sigma u\gamma=A\Sigma u+B\Sigma u^2$$

Substituting the values of $\Sigma\gamma$ etc.,

$$8·3544=8A-4B \qquad \ldots(i)$$

and

$$1·1707=(-4)\,A+44B \qquad \ldots(ii)$$

Multiplying (ii) by '2' and adding to (i)

$$10·6958=84B$$

or

$$B=0·12733$$

∴ From (i)

$$8A=8·86372$$

$$A=1·107965$$

∴ The equation is

$$\gamma=1·107965+0·12733u$$

$$=1·107965+0·12733\left(\frac{x-1951}{10}\right)$$

$\therefore$ For $x=1991$,

$$\begin{aligned} Y &= 1{\cdot}107965+0{\cdot}50932 \\ &= 1{\cdot}617285 \\ &\simeq 1{\cdot}6173 \end{aligned}$$

$\therefore$ $y=41{\cdot}43$

$\therefore$ Population for the year $1991=41{\cdot}43$ millions.

Ex. 6-13. (*a*) *Derive the least-square equations for fitting a curve of the type* $y=ax^2+\dfrac{b}{x}$ *to a set of n points.*

(*b*) *Fit the curve* $y=ax^2+\dfrac{b}{x}$ *to the data given below :*

x :	1	2	3	4
y :	$-1{\cdot}51$	$0{\cdot}99$	$3{\cdot}88$	$7{\cdot}66$

Sol. (*a*) Let (x_i, y_i) $i=1, 2, \ldots\ldots n$ be the given data and

$$Y_i=ax_i^2+\frac{b}{x_i}$$

Let $$S=\sum_{i=1}^{n}\{Y_i-y_i\}^2=\sum_{i=1}^{n}\left\{ax_i^2+\frac{b}{x_i}-y_i\right\}^2$$

Normal equations are

$$0=\frac{\partial S}{\partial a}=\sum_{i=1}^{n} 2\left\{ax_i^2+\frac{b}{x_i}-y_i\right\}x_i^2$$

i.e., $$\sum_{i=1}^{n} y_i x_i^2=a\sum_{i=1}^{n} x_i^4+b\sum_{i=1}^{n} x_i \qquad \ldots(1)$$

and $$0=\frac{\partial S}{\partial b}=\sum_{i=1}^{n} 2\left\{ax_i^2+\frac{b}{x_i}-y_i\right\}\left(\frac{1}{x_i}\right)$$

i.e., $$\sum_{i=1}^{n}\frac{y_i}{x_i}=a\sum_{i=1}^{n} x_i+b\sum_{i=1}^{n}\left(\frac{1}{x_i^2}\right) \qquad \ldots(2)$$

(1) and (2) are required equations.

(b)

x	y	x^2	x^4	$\frac{1}{x}$	$\frac{1}{x^2}$	yx^2	$\frac{y}{x}$
1	−1·51	1	1	1·0000	1·0000	−1·5100	−1·5100
2	0·99	4	16	0·5000	0·2500	3·9600	0·4950
3	3·88	9	81	0·3333	0·1111	34·9200	1·2933
4	7·66	16	256	0·2500	0·0625	122·5600	1·9150
10			354		1·4236	159·9300	2·1933

Substituting values in eqs. (1) and (2)

$$159{\cdot}9300=354a+10b \quad \text{...(3)}$$

and

$$2{\cdot}1933=10a+1{\cdot}4236b \quad \text{...(4)}$$

From (3) and (4)

$$a=0{\cdot}509, \qquad b=-2{\cdot}04$$

∴ The equation of the curve best fitted to the given data is

$$y=0{\cdot}509x^2-\frac{2{\cdot}04}{x}$$

Ex. 6-14. *Derive the least-square equations for fitting a curve of the type $y=ax+\frac{b}{x}$ to a set of n points.*

Sol. Let $(x_i, y_i) i=1, 2\ldots\ldots n$ be the given data and

$$Y_i=ax_i+\frac{b}{x_i}$$

Let
$$S=\sum_{i=1}^{n}(Y_i-y_i)^2=\sum_{i=1}^{n}\left(ax_i+\frac{b}{x_i}-y_i\right)^2$$

Normal equations are

$$\frac{\partial S}{\partial a}=0=\sum_{i=1}^{n}2\left(ax_i+\frac{b}{x_i}-y_i\right)(x_i)$$

i.e.,
$$\sum_{i=1}^{n}(ax_i^2+b-x_iy_i)=0$$

i.e. $$\sum_{i=1}^{n} x_i y_i = nb + a \sum_{i=1}^{n} x_i^2 \quad ...(1)$$

and $$\frac{\partial S}{\partial b} = 0 = \sum_{i=1}^{n} 2\left(ax_i + \frac{b}{x_i} - y_i \right)\left(\frac{1}{x_i} \right)$$

i.e., $$na + b \sum_{i=1}^{n} \frac{1}{x_i^2} = \sum_{i=1}^{n} \frac{y_i}{x_i} \quad ...(2)$$

(1) and (2) are required equations.

Ex. 6-15. *Three independent measurements on each of three angles A, B, C of a triangle are as follows :*

A : 39·5	39·3	39·6
B : 60·3	62·2	60·1
C : 80·1	80·3	80·4

Obtain the best estimates of the three angles, taking into account the relation that sum of the angles is equal to 180.

Sol. Let the measurements for angles A, B and C be denoted by x, y and z respectively.

Then $\Sigma x = 39{\cdot}5 + 39{\cdot}3 + 39{\cdot}6 = 118{\cdot}4$

$\Sigma y = 60{\cdot}3 + 62{\cdot}2 + 60{\cdot}1 = 182{\cdot}6$

and $\Sigma z = 80{\cdot}1 + 80{\cdot}3 + 80{\cdot}4 = 240{\cdot}8$

Let α, β and γ be the true values of angles A, B and C respectively.

Then $\gamma = 180 - \alpha - \beta$.

Let $S = \Sigma\{(\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2\}$

$= \Sigma\{(\alpha - x)^2 + (\beta - y)^2 + (180 - \alpha - \beta - z)^2\}$

By the principle of least squares, S is to be minimized. Normal equations are

$$0 = \frac{\partial S}{\partial \alpha} = \Sigma\{(2\alpha - x) - 2(180 - \alpha - \beta - z)\}$$

i.e., $2\Sigma\alpha + \Sigma\beta = \Sigma x - \Sigma z + \Sigma 180.$

i.e., $6\alpha + 3\beta = 118{\cdot}4 - 240{\cdot}8 + 540 = 417{\cdot}6.$

i.e., $2\alpha + \beta = 139{\cdot}2 \quad ...(1)$

and $$0 = \frac{\partial S}{\partial \beta} = \Sigma\{2(\beta - y) - 2(180 - \alpha - \beta - z)\}$$

which implies.

$$\alpha+2\beta=160\cdot6$$

From (1) and (2)

$$\alpha=39\cdot27,\ \beta=60\cdot67 \qquad \ldots(2)$$

$$\gamma=180-\alpha-\beta=80\cdot06.$$

EXERCISE

1. Find the most plausible values of x and y from the following equations :

$$x+y=3,\ x-y=2,\ x+2y-4=0,\ x=2y+1.$$

[**Ans.** $x=2\cdot5,\ y=0\cdot7$]

2. A and B are two brothers. A is ten years older than B. Five years before A's age was twice that of B's. Five years hence twice the age of A will be same as three times that of B. Find their present ages. [**Ans.** 25, 15]

3. Fit a straight line to the data given below :

x :	6	7	7	8	8	8	9	9	10
y :	5	5	4	5	4	3	4	3	3

[**Ans.** $y=-0\cdot5x+8$]

4. Fit a straight line to the following data treating 'y' as the dependent variable.

x :	1	2	3	4	5
y :	5	7	9	10	11

[**Ans.** $y=3\cdot9+1\cdot5x$]

5. Fit a straight line to the data given below ; showing the production of a commodity in different years :

Year x :	1981	1982	1983	1984	1985
Production y : (1000 tons)	10	12	8	10	14

[**Ans.** $y=0\cdot6\,x+10\cdot8$]

6. Fit a straight line to the following data regarding x as the independent variable :

x :	1	1	2	3	4
y :	1	1·8	3.3	4·5	6·3

[**Ans.** $y=0\cdot72+1\cdot33x$]

7. The weights of a calf taken at weekly intervals are given below. Fit a straight line using the method of least squares and calculate the average rate of growth per week.

Age :	1	2	3	4	5	6	7	8	9	10
Weight :	52·5	58·7	65·0	70·2	75·4	81·1	87·2	95·5	102·2	108·4

[**Ans.** $y=79\cdot62+6\cdot16(x-5\cdot5)$; 6·16]

8. Find the equation of the straight line which comes nearest to passing through the following points :

x :	0·5	1·0	1·5	2·0	2·5	3·0
y :	0·31	0·82	1·29	1·85	2·51	3·02

[**Ans.** $y=-0{\cdot}285+1{\cdot}10x$]

9. Below are given figures of production of a sugar factory.

year :	1971	1972	1973	1974	1975	1976	1977	1978
Production : (in thousand mounds)	80	90	92	83	94	99	92	110

Fit a straight line trend by the method of least squares to the above data.

[**Ans.** $y=94+3(x-1975)$]

10. The profits Rs. y of a certain company in the xth year of its existence are given by :

x :	1	2	3	4	5
y :	1250	1400	1650	1950	2300

Show that the parabolic regression of y on x is

$$y=1140+72x+32{\cdot}15x^2$$

11. Fit a second degree parabola to the data given below :

x :	0	0·1	0·2	0·3	0·4	0·5
y :	3·1950	3·2299	3·2532	3·2611	3·2516	3·2282
x :	0·6	0·7	0·8	0·9		
y :	3·1807	3·1266	3·0594	2·9759		

[**Ans.** $y=3{\cdot}1951+0{\cdot}4425x-0{\cdot}7653x^2$]

12. Fit a second degree parabola to the following data :

x :	0	1	2	3	4
y :	1	5	0	22	38

[**Ans.** $y=5{\cdot}914+9{\cdot}1(x-2)+3{\cdot}643(x-2)^2$]

13. Fit the curve $y=ae^{bx}$ to the data given below :

x :	0	2	4
y :	5·012	10	31·62

($e=2{\cdot}71828$)

[**Ans.** $y=4{\cdot}642\ e^{0{\cdot}46x}$]

14. Fit the curve $y=ab^x$ to the data given below :

x :	2	3	4	5	6
y :	144	172·8	207·4	248·8	298·5

[**Ans.** $y=100(1{\cdot}2)^x$]

15. Fit the curve $y=ae^{bx}$ to the following data :

x :	1	2	3	4	5	6	7	8
y :	15·3	20·5	27·4	36·6	49·1	65·6	87·8	117·6

$\left[\text{Ans.}\quad y=11{\cdot}58e^{-0{\cdot}2898x}\right]$

16. Fit $y=ae^{bx}$ to the following data :

x :	2·5	5·0	7·5	10·0	12·5	15·0
y :	76	52	35	25	16	11

$\left[\text{Ans.}\quad y=(113{\cdot}4)e^{-0{\cdot}1549x}\right]$

17. Fit the curve of the type $xy^a=b$ to the following data :

x :	0·5	1·0	1·5	2·0	2·5	3·0
y :	1·62	1·00	0·75	0·62	0·52	0·46

18. Use the method of least squares to determine 'a' and 'b' in the formula $y=ax+bx^2$ for the following data :

x :	1	2	3	4	5
y :	1·8	5·1	8·9	14·1	19·8

Calculate the value of y for $x=2$.

[Ans. $a=1{\cdot}521$, $b=0{\cdot}49$, $5{\cdot}006$]

19. Fit $y=a+bx^2$ to the following data :

x :	5	7	9	11	12
y :	290	560	1044	1810	2300

[Ans. $y=130{\cdot}71+1{\cdot}2572x^2$]

7

Probability

7.1. Introduction

A fundamental principle involved in the experiments of science and engineering is that if these experiments are repeatedly performed under very nearly identical conditions the results are essentially the same. But, there are experiments in which results will not be essentially the same even though conditions may be nearly identical. Such experiments will be called **random experiments** or simply **experiments**.

Below are certain terms which will be used subsequently.

Trial. *Performing of an experiment is called trial.*

Cases. *Various possible outcomes of a trial are termed as cases.*

Event. *It is used to represent the aim with which the experiment is performed.*

Sample space. *It is the set of all possible outcomes of an experiment.*

Event is a subset of sample space and cases are its members i.e., subsets consisting of single members.

(E.L.) Equally Likely Cases (Events). *Cases (Events) are called equally likely when none of them can be preferred rather than the other.*

(M.E.) Mutually Exclusive Cases (Events). *Cases (Events) are called mutually exclusive when no two of them can occur simultaneously.*

Exhaustive Cases (Events). *A set of cases (events) is said to be exhaustive if it includes all possible outcomes of a trial.*

Favourable Cases. *The cases which entail the happening of an event are said to be favourable to an event.*

In a trial, there is always uncertainty as to whether a particular event will occur or not. To measure this uncertainty the idea of probability (or chance) was introduced. It is the number between 0 and 1. If the event is sure to occur its probability is taken to be 1 and if the event is sure not to occur its probability is taken to be zero. If the probability is $\frac{1}{4}$, it means that there are 25% chances for the event to occur and 75% chances for the event not to occur.

Below are two definitions of probability of an event.

7.1·1. Mathematical Definition

Let 'n' be the number of cases which are equally likely, mutually exclusive and exhaustive and 'm' of these are favourable to the happening of an event 'A'. Then the probability of the happening of A is defined to be $\frac{m}{n}$'.

7.1.2. Statistical Definition

If a trial is repeated a number of times under essentially the same conditions, then limiting value of the ratio of the number of times the event happens to the number of trials, as the number of trials increases indefinitely, is called the probability of the happening of that event. (It is assumed that the ratio approaches a finite and unique limit).

Both these definitions have serious difficulties, the first because the word "equally likely" is vague and second because of the vagueness of infinite number of trials. Because of these difficulties, the following axiomatic approach to probability was introduced.

7.1.3. Axioms of Probability

Let S be a sample space. Let C be the class of events in S. To each A in C is associated a real numbe P (A) st.

(1) $P(A) \geqslant 0, \forall A \in C$

(2) $P(S)=1$

(3) If $A_1, A_2, \ldots$ are any number of mutually exclusive events in C then

$$P(A_1+A_2+\ldots)=P(A_1)+P(A_2)+\ldots$$

Obviously, P is a real valued function defined on C. P is called a probability function and $P(A)$ the probability of the event A.

Remark. Starting with definition (7.1.1) axion (3) of (7.1.3) can be proved. In view of this $P(A)$ is taken to be $\frac{O(A)}{O(S)}$, where $O(A)$ denotes the order of A.

Odds in favour of and against an event.

Odds in favour of an event $= \dfrac{\textit{Prob. of happening}}{\textit{Prob. of non-happening}}$

Odds against an event $= \dfrac{\textit{Prob of non-happening}}{\textit{Prob. of happening}}$.

To find the no. of ways of getting a certain sum in rolling dice.

No. of ways of getting a sum 'r' in rolling 'n' f-faced dice

$=$ *co-efficient of* x^r *in* $(x+x^2+\ldots x^f)^n$

Ex. 7-1. *The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. Find the chance of each.*

Sol. Let p and p' be the chances of happening of two events.

Then $$p=p'^2$$

Odds against the first event $= \dfrac{1-p}{p}$

and odds against the second event $= \dfrac{1-p'}{p'}$

$$\therefore \quad \frac{1-p}{p}=\left(\frac{1-p'}{p'}\right)^3$$

$$\therefore \quad \frac{1-p'^2}{p'^2}=\frac{(1-p')^3}{p'^3}$$

which implies $p'=\dfrac{1}{3}$

$$\therefore \quad p=\frac{1}{9}.$$

Ex. 7-2. *The sum of two positive quantities is equal to 2n. Find the chance that the product of two quantities is not less than 3/4 times their greatest product.*

Sol. Let x be one quantity.

Then $(2p-x)$ is the other quantity

Let $$y=x(2n-x)$$

$$\frac{dy}{dx}=2n-2x$$

y is maximum when $\dfrac{dy}{dx}=0$

i.e., $$2n-2x=0$$

i.e., $x=n$

$\therefore$ Maximum value of $y=n^2$

$\therefore$ x is to be such that

$$x(2n-x)\nless\frac{3}{4}n^2$$

i.e., $3n^2-8nx+4x^2 \ngtr 0$

i.e., $(3n-2x)(n-2x)\ngtr 0$

$$\therefore \qquad \frac{n}{2}<x<\frac{3n}{2}$$

$\therefore$ No. of favourable cases $=\dfrac{3n}{2}-\dfrac{n}{2}=n$

and total no. of cases $=2n$

Reqd. prob. $=\dfrac{n}{2n}=\dfrac{1}{2}$.

Ex. 7·3. *What is the chance that (a) a leap year selected at random will contain 53 Sundays, (b) a non-leap year selected at random will contain 53 Sundays ?*

Sol. (*a*) In a leap year there are 366 days *i.e.*, 52 weeks and 2 days. Remaining two days can be any two days of the week. Different possibilities are :

Sunday	and Monday
Monday	and Tuesday
Tuesday	and Wednesday
Wednesday	and Thursday
Thursday	and Friday
Friday	and Saturday
Saturday	and Sunday

In order to have 53 Sundays, out of remaining two days one must be Sunday.

No. of cases favourable to the event of having one Sunday out of 2 days $=2$

Total number of cases $=7$

$\therefore$ Reqd. prob. $=\dfrac{2}{7}$.

(*b*) In a non-leap year there are 365 days *i.e.*, 52 weeks and 1 days. Remaining 1 day can be any day of the week.

$\therefore$ Total no. of cases $=7$.

There will be 53 Sundays if the remaining one day is Sunday.

$\therefore$ No. of favourable cases $=1$

$\therefore$ Reqd. prob. $=\frac{1}{7}$.

Ex. 7-4. *Two cards are drawn at random from a well-suffled pack of 52. Show that the chance of drawing two aces is* $\frac{1}{221}$.

Sol. Total number of ways of drawing two cards out of $52={}^{52}c_2$. In a pack of 52, there are 4 aces out of which 2 aces can be drawn in ${}^4c_2$ ways.

$$\therefore \text{ Reqd. prob.} =\frac{{}^4c_2}{{}^{52}c_2}$$

$$=\frac{4\cdot3}{52\cdot51}=\frac{1}{221}.$$

Ex. 7-5. *From a pack of 52 cards, two are drawn at random. Find the chance that one is a king and the other a queen.*

Sol. Total number of cases $={}^{52}c_2$

Since in a pack there are 4 kings and 4 queens, number of favourable cases $={}^4c_1.{}^4c_1$.

$$\therefore \text{ Reqd. prob.} =\frac{{}^4c_1.{}^4c_1}{{}^{52}c_2}$$

$$=\frac{4\cdot4}{52\cdot51}2\cdot1$$

$$=\frac{8}{663}.$$

Ex. 7-6. *Four cards are drawn from a well shuffled pack of cards. What is the probability that they are from four different suits ?*

Sol. Total number of cases $={}^{52}c_4$

Since there are 13 cards of each suit, no. of ways of drawing 4 cards belonging to different suits

$$=({}^{13}c_1)({}^{13}c_1)({}^{13}c_1)({}^{13}c_1)=13^4$$

$$\therefore \text{ Reqd. Prob.} =\frac{(13)^4}{{}^{52}c_4}=\frac{2197}{20825}$$

Ex. 7-7. *From a set of 17 cards numbered 1, 2, ... 17, one is drawn at random. What is the chance that*

(*i*) *Its number is a multiple of 3 or of 7 ?*

(*ii*) *Its number is a multiple of 3 or 5 or both ?*

Sol. (*i*) Total number of cases$={}^{17}c_1=17$.

The number on the card drawn will be a multiple of 3 or of 7 if it is 3, 6, 7, 9, 12, 14 or 15.

$\therefore$ Number of favourable cases$=7$

$$\therefore \text{ Reqd. prob. } = \frac{7}{17}$$

(*ii*) The number on the card drawn will be a multiple of 3 or 5 or both if it is 3, 5, 6, 9, 10, 12 or 15.

$\therefore$ Number of favourable cases.

$=7$

$$\therefore \text{ Reqd. prob. } = \frac{7}{17}.$$

Ex. 7·8. (*a*) *If n biscuits are distributed at random among N beggars, what is the chance that a particular beggar receives $r(<n)$ biscuits ?*

Sol. Since one biscuit can be given in N ways, number of ways of distributing n biscuits among N beggars$=N^n$.

If one particular beggar receives r biscuits, remaining $(n-r)$ biscuits are to be distributed among $(N-1)$ beggars and this can be done in $(N-1)^{n-r}$ ways.

r biscuits to be given to one particular beggar can be chosen in ${}^nc_r$ ways.

$\therefore$ Number of favourable cases$={}^nc_r.(N-1)^{n-r}$

$$\therefore \text{ Reqd. prob. } = \frac{{}^nc_r(N-1)^{n-r}}{N^n}$$

(*b*) *What is the most probable number of biscuits distributed to a particular beggar ?*

Sol. Let $$P(r)=\frac{{}^nc_r(N-1)^{n-r}}{N^n}$$

Most probable number of biscuits distributed to a particular beggar is that value of r which is *s.t.*

$$P(r-1)<P(r)>P(r+1)$$

Consider $$P(r-1)<P(r)$$

i.e., $$\frac{{}^nc_{r-1}(N-1)^{n-r+1}}{N^n}<\frac{{}^nc_r(N-1)^{n-r}}{N^n}$$

i.e., $$\frac{n\,!}{(r-1)\,!\,(n-r+1)\,!}(N-1)<\frac{n\,!}{r\,!\,(n-r)\,!}$$

i.e., $$\frac{N-1}{n-r+1}<\frac{1}{r}$$

i.e., $$r(N-1) \leqslant (n+1)-r$$

i.e., $$r \leqslant \frac{n+1}{N}$$

Consider $$P(r) \geqslant P(r+1)$$

i.e., $$\frac{{}^nc_r(N-1)^{n-r}}{N^n} \geqslant \frac{{}^nc_{r+1}(N-1)^{n-r-1}}{N^n}$$

i.e., $$\frac{n!}{r!(n-r)!}(N-1) \geqslant \frac{n!}{(r+1)!(n-r-1)!}$$

i.e., $$\frac{(N-1)}{n-r} \geqslant \frac{1}{r+1}$$

i.e., $$r(N-1)+(N-1) \geqslant n-r$$

i.e., $$r \geqslant \frac{n+1}{N}-1$$

$\therefore$ Most probable value of r is such that

$$\frac{n+1}{N}-1 \leqslant r \leqslant \frac{n+1}{N}$$

Since r is the integer, it is the greatest integer less than $\frac{n+1}{N}$ if $\frac{n+1}{N}$ is not an integer. In case $\frac{n+1}{N}$ is an integer, r can take both values $\frac{n+1}{N}$ and $\frac{n+1}{N}-1$.

Ex. 7-9. *Two different digits are chosen at random from the set 1, 2, 3......8. Show that the probability, that the sum of the digits will be equal to 5 is the same as the probability that their sum will exceed 13 each being $\frac{1}{14}$. Also show that the chance of both digits exceeding 5 is $\frac{3}{28}$.*

Sol. Total number of ways of choosing 2 digits $= {}^8c_2 = 28$.

(*i*) Different possibilities of getting 2 digits with sum 5 are :

one-digit	2nd-digit
1	4
2	3

$\therefore$ Number of favourable cases $= 2$

$\therefore$ Prob. of choosing 2 digits with sum $5 = \frac{1}{14}$.

(*ii*) Different possibilities of getting 2 digits with sum exceeding 13 are :

1st-digit	2nd-digit
6	8
7	8

$\therefore$ Number of favourable cases $=2$

$\therefore$ Probability of choosing 2 digits with sum exceeding 13

$$=\frac{1}{14}.$$

(*iii*) There are only three digits exceeding 5 and out of these three digits two can be chosen in ${}^3c_2$ ways.

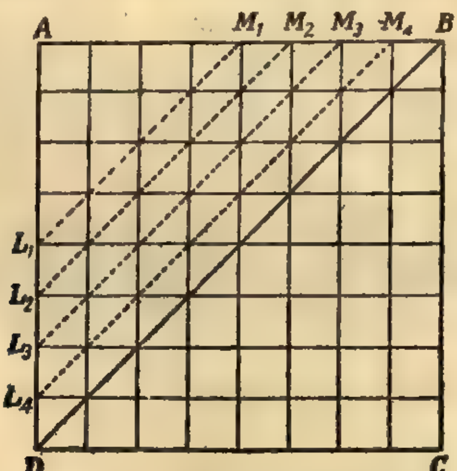
$\therefore$ Number of favourable cases $={}^3c_2=3$

$\therefore$ Reqd. prob. $=\frac{3}{28}$.

Ex. 7-10. *If four squares are chosen at random on a chess-board, find the chance that they should be in a diagonal line.*

Sol. A chess board is a square divided into 64 equal squares parallel to the sides.

Diagonal BD divides the board in two equal $\triangle$s ABD and CBD. In $\triangle ABD$, four squares along a diagonal line can be chosen along L_1M_1, L_2M_2, L_3M_3, L_4M_4 or DB which contain respectively 4, 5, 6, 7, 8 squares.



$\therefore$ Number of ways of selecting 4 squares in

$$\triangle ABD={}^4c_4+{}^5c_4+{}^6c_4+{}^7c_4+{}^8c_4$$

This is also the number of ways of selecting 4 squares in each of $\triangle$sBCD, ACD and ABC.

$\therefore$ Total number of ways of selecting 4 squares along a diagonal line in the sqaure $ABCD = 4\{{}^4c_4 + {}^5c_4 + {}^6c_4 + {}^7c_4\} + 2.{}^8c_4$

$$= 364$$

(This is because the diagonals BD and AC are common to $\triangle$sABD, BCD and ACD, ABC).

Also total number of ways of selectin 4 squares out of 64

$$= {}^{64}c_4$$

$\therefore$ Reqd. prob$= \dfrac{364}{{}^{64}c_4} = \dfrac{13}{22692}$.

Ex. 7-11. *A five-figure number is formed by the digits 0, 1, 2, 3, 4 (without repetition). Find the prob that the number formed is divisible by 4.*

Sol. Total number of ways of arranging digits 0, 1, 2, 3, 4

$$= 5\,! = 120\,!$$

If a number start with '0' the remaining 4 digits can be arranged in $4\,! = 24\,!$ ways and hence total number of five-figure numbers$= 120 - 24 = 96$.

Now the number ending with 04, 20, 40, 12, 24 and 32 are divisible by 4.

No. of numbers ending with 04 = Total number of ways of arranging digits 1, 2, 3 $= 3\,! = 6$.

Evidently this is also the number of numbers ending with 20 and 40 respectively.

For the numbers ending with 12, out of total number of ways of arranging remaining three digits 0, 3, 4 the number of ways in which '0' occurs first are to be discarded.

($\because$ these will give four figure numbers).

$\therefore$ No. of numbers ending with $12 = 3\,! - 2\,! = 4$.

Evidently this is also the number of numbers ending with 24 and 32 respectively.

$\therefore$ Total number of five-figure numbers, formed by the digits 0, 1, 2, 3, 4, which are divisible by 4

$$= 3(6+4) = 30$$

$\therefore$ Reqd. prob.$= \dfrac{30}{96} = \dfrac{5}{16}$.

Ex. 7-12. *Out of $(2n+1)$ tickets consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in A.P.*

Sol. Total number of cases $= {}^{2n+1}C_3$

Different possibilities drawing tickets with their numbers in A.P. are :

$$1, 2, 3\ ;\ 1, 3, 5\ ;\ldots\ldots\ldots 1, (n+1), (2n+1)$$
$$2, 3, 4\ ;\ 2, 4, 6\ ;\ldots\ldots\ldots 2, (n+1), 2n$$
$$3, 4, 5\ ;\ 3, 5, 7\ ;\ldots\ldots\ldots 3, (n+2), (2n+1)$$
$$4, 5, 6\ ;\ 4, 6, 8\ ;\ldots\ldots\ldots 4, (n+2), 2n$$
$$5, 6, 7\ ;\ 5, 7, 9\ ;\ldots\ldots\ldots 5, (n+3), (2n+1)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and so on.

Number of terms in first sequence $= n$

" " " " 2nd " $= (n-1)$

" " " " 3rd " $= (n-1)$

" " " " 4th " $= (n-2)$

" " " " 5th " $= (n-2)$

and so on.

$\therefore$ No. of favourable cases $= n+2\{(n-1)+(n-2)+\ldots\ldots 1\}$

$$= n+2\frac{(n-1)n}{2} = n^2$$

$$\therefore \qquad \text{Reqd. prob.} = \frac{n^2}{{}^{2n+1}C_3} = \frac{3n}{4n^2-1}.$$

Ex. 7-13. *Four cards are drawn out at random from a full deck of 52. Find the probabilities of the following contingencies.*

(a) The cards are of the four different suits and of different denominations (1, 2, king etc.)

(b) There is at least one ace-card.

(c) Only two of the four suits are represented.

Sol. (*a*) Total number of ways $= {}^{52}C_4$.

First card (say A_1) can be any out 52. Then the second card (say A_2) must be from the 36 cards obtained on discarding the cards belonging to the suit and denomination of A_1, the third card (say A_3) must be from the 22 cards obtained on discarding the cards belonging to the suits and denominations of A_1 and A_2 and the fourth card must be from the 10 cards obtained on discarding the cards belonging to the suits and denominations of A_1, A_2 and A_3.

$\therefore$ No. of favourable cases $= {}^{52}C_1 . {}^{36}C_1 . {}^{22}C_1 . {}^{10}C_1$

$$\therefore \quad \text{Reqd. prob} = \frac{{}^{52}C_1 . {}^{36}C_1 {}^{22}C_1 {}^{10}C_1}{{}^{52}C_4}$$

(*b*) and (*c*) are left as exercises.

Ex. 7-14. *Out of 3n consecutive numbers 3 are selected at random. Find the chance that their sum is divisible by 3.*

Sol. Let $3n$ consecutive numbers be $p+1, \ldots\ldots\ldots\ldots p+3n$ where p is any integer. These $3n$ numbers can be arranged as follows :

$$\left.\begin{array}{lll} p+1 & p+2 & p+3 \\ p+4 & p+5 & p+6 \\ p+7 & p+8 & p+9 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ p+3n-2 & p+3n-1 & p+3n \end{array}\right\} \quad \ldots(\mathbf{A})$$

Numbers in three columns have the property that the sum of any three numbers in any particular column is divisible by 3. Now each column consists of n numbers and hence number of ways of selecting 3 numbers from any one particular column$={}^nc_3$.

Also the numbers in (A) are s.t., if three numbers are chosen one from each column, their sum is divisible by 3.

Now number of ways of selecting three numbers from (A) one from each column$=n^3$.

$\therefore$ No. of favourable cases

$$=n^3+3.{}^nc_3$$

$$=\frac{n}{2}\,[3n^2-3n+2].$$

Also total number of cases

$$={}^{3n}c_3=\frac{n}{2}\,(9n^2-9n+2)$$

$\therefore$ Reqd. prob.$=\dfrac{3n^2-3n+2}{9n^2-9n+2}$.

Ex. 7-15. *If 6n tickets numbered 0, 1, 2, ... 6n−1 are placed in a bag and three are drawn out, show that the chance that the sum of the numbers on them is equal to 6n is* $\dfrac{3n}{(6n-1)(6n-2)}$.

Sol. Total number of cases$={}^{6n}c_3$.

Different possibilities of drawing tickets with the sum of their numbers equal to $6n$ are :

$0, 1, 6n-1$; $0, 2, 6n-2$;......; $0, 3n-1, 3n+1$
$1, 2, 6n-3$; $1, 3, 6n-4$; .....; $1, 3n-1, 3n$
$2, 3, 6n-5$; $2, 4, 6n-6$;......; $2, 3n-2, 3n$
$3, 4, 6n-7$; $3, 5, 6n-8$;......; $3, 3n-2, (3n-1)$
$4, 5, 6n-9$; $4, 6, 6n-10$; .....; $4, 3n-3, 3n-1$
....

$2n-2, 2n-1, 2n+3$; $2n-2, 2n, 2n+2, 2n-1, 2n, 2n+1$

No. of terms in first sequence $=3n-1$

" " " " 2nd " $=3n-2$

" " " " 3rd " $=3n-4$

" " " " 4th " $=3n-5$

" " " " 5th " $=3n-7$

and so on.

$\therefore$ Total number of ways of drawing tickets with the sum of their numbers equal to $6n$

$$=\{(3n-1)+(3n-2)\}+\{(3n-4)+(3n-5)\}+\{(3n-7)+(3n-8)\} +\ldots\ldots+2+1$$

$$=\{(3n-1)+(3n-4)+(3n-7)+\ldots\ldots+2\} +\{(3n-2)+(3n-5)+(3n-8)+\ldots\ldots+1\}$$

$$=\frac{n}{2}\{4+3(n-1)\}+\frac{n}{2}\{2+3(n-1)\}=3n^2$$

$$\therefore \text{ Reqd. prob.}=\frac{3n^2}{{}^{6n}c_3}=\frac{3n}{(6n-1)(6n-2)}.$$

Ex. 7-16 *Four different objects 1, 2, 3, 4 are distributed at random on four places marked 1, 2, 3, 4. What is the probability that none of the objects occupies the place corresponding to its number?*

Sol. Total number of ways of distributing 4 objects on 4 places $=4\,!=24$.

Number of ways in which all the four objects can occupy their places $=1$.

If three objects occupy their places, 4rth will also do so. So this is contained in possibility discussed above.

If two objects occupy their places, remaining two can go wrong by occupying each other's position *i.e*, in only one way.

Since out of 4, two objects can be chosen in $4c_2$ ways, number of ways in which only two objects can occupy their places.

$$=4c_2\times 1=6.$$

If one object occupies its position, any one of the ramaining three can go wrong in 2 ways by occupying the positions of other two.

Since out of 4, one object can be chosen in $4c_1$ ways, number of ways in which only one object can occupy its place

$$=4c_1\times 2=8.$$

$\therefore$ Total number of ways in which at least one object can occupy its place

$$=1+6+8=15$$

$\therefore$ Total number of ways in which none of the objects occupies the place corresponding to its number

$$=24-15=9$$

$$\therefore \quad \text{Reqd. prob.}=\frac{9}{24}=\frac{3}{8}.$$

Ex. 7·17. *A and B stand in a ring with 10 other persons. If the arrangement of 12 persons is at random, find the chance that there are exactly three persons between A and B.*

Sol. There are in all 12 persons who are to stand in a ring. Fixing the position of one person remaining 11 persons can stand in 11 ! ways.

$\therefore$ Total number of ways in which 12 persons can stand in a ring$=11\,!$.

Out of 10 persons three are to stand between A and B. These three can be chosen in ${}^{10}c_3$ ways and can be arranged in 3 ! ways.

Also number of ways of arranging remaining 7 persons$=7\,!$

Since A and B can interchange their position, number of ways of having exactly 3 persons between A and $B=2.{}^{10}c_3.3\,!\,7\,!$

$$=2.\frac{10\,!}{3\,!\,7\,!}.3\,!.7\,!$$

$$=2.10\,!$$

$$\therefore \quad \text{Reqd. prob.}=\frac{2.10\,!}{11\,!}=\frac{2}{11}.$$

Ex. 7·18 *The first 12 letters of the alphabet are written at random. Find the chance that there are exactly 4 letters between A and B.*

Sol. Total number of ways$=12\,!$

Different possibilities are :

1	2	3	4	5	6	7	8	9	10	11	12
A	.	.	.	.	*B*	.	.	.	.	.	.
.	*A*	.	.	.	.	*B*	.	.	.	.	.
.	.	*A*	.	.	.	.	*B*	.	.	.	.
.	.	.	*A*	.	.	.	.	*B*	.	.	.
.	.	.	.	*A*	.	.	.	.	*B*	.	.
.	.	.	.	.	*A*	.	.	.	.	*B*	.
.	.	.	.	.	.	*A*	.	.	.	.	*B*

Out of remaining 10 letters, 4 letters are to lie between A and B. These four can be chosen in ${}^{10}c_4$ ways and can be arranged in 4 ! ways.

Also number of ways of arranging remaining 6 letters $=6\,!$ and A and B can interchange their positions.

$\therefore$ Total number of ways of having exactly 4 letters between A and B.

$$=7.2.{}^{10}c_4.4\,!\,.\,6\,!$$

$$=7.2.\frac{10\,!}{6\,!\,4\,!}.4\,!\,.\,6\,!$$

$$=7.2.10\,!$$

$$\therefore \qquad \text{Reqd. prob.}=\frac{7.2.10\,!}{12\,!}=\frac{7}{66}.$$

Ex. 7-19. *If the letters of the word 'REGULATIONS' be arranged at random, what is the chance that there will be exactly 4 letters between the 'R' and the 'E'?*

Sol. There are in all 11 letters to be arranged.

Total number of ways of arranging 11 letters

$$=11\,!$$

Different possibilities are :

1	2	3	4	5	6	7	8	9	10	11
R	.	.	.	.	*E*	.	.	.	.	.
.	*R*	.	.	.	.	*E*	.	.	.	.
.	.	*R*	.	.	.	.	*E*	.	.	.
.	.	.	*R*	.	.	.	.	*E*	.	.
.	.	.	.	*R*	.	.	.	.	*E*	.
.	.	.	.	.	*R*	.	.	.	.	*E*

Therefore, as in last example, total number of ways of having exactly 4 letters between the '*R*' and the '*E*'

$$=6\cdot 2.{}^{9}c_4.\ 4\,!\ 5\,!$$
$$=6\cdot 2.\ 9\,!$$

$$\text{Therefore, reqd. prob.}=\frac{6\cdot 2.\ 9\,!}{11\,!}$$

$$=\frac{6\cdot 2}{11\cdot 10}=\frac{6}{55}.$$

Ex. 7-20. *Show that the chance of throwing an odd number with a die is* $\frac{1}{2}$.

Sol. There are six faces of a die marked with numbers from 1 to 6.

$\therefore$ Total number of cases $=6$.

Out of six faces, three are marked with odd numbers *viz* 1, 3 and 5.

$\therefore$ Number of favourable cases=3.

$\therefore$ Reqd. prob$=\frac{3}{6}$

$=\frac{1}{2}$.

Ex. 7·21. *In a single throw with two dice, find the chances of throwing (i) eight (ii) eleyen.*

Sol. Total number of cases$=6\times6=36$

(*i*) The sum 8 can be obtained in either of the following ways :

First die	Second die
6	2
5	3
4	4
3	6
2	6

$\therefore$ The number of favourable cases=5

$\therefore$ Reqd. prob$=\frac{5}{36}$.

(*ii*) The sum 11 can be obtained in either of the following ways :

First die	Second die
6	5
5	6

$\therefore$ Number of favourable cases=2.

$\therefore$ Reqd. probability$=\frac{1}{18}$.

Ex. 7·22. *Find the chance of throwing a total of 3 or 5 or 11 with two dice.*

Sol. Consider the expression

$$(x+x^2+\ldots\ldots+x^6)^2$$

$$=\left\{\frac{x(1-x^6)}{1-x}\right\}^2$$

$$=x^2(1-x^6)^2(1-x)^{-2}$$

$$=x^2(1-2x^6+x^{12})(1+2x+3x^2+4x^3+\ldots)$$

$\therefore$ Number of ways of getting a total of 3

$$=\text{co-efficient of } x^3=2$$

Number of ways of getting a total of 5

$$=\text{co-efficient of } x^5=4$$

and number of ways of getting a total of 11

$$=\text{co-efficient of } x^{11}=10-8=2$$

$\therefore$ Number of ways of getting a total of 3 or 5 or 11

$$=2+4+2=8$$

Total no. of cases $=6^2=36$

$\therefore$ Reqd. prob. $=\frac{8}{36}=\frac{2}{9}$.

Ex. 7-23. *Show that the chance of throwing 15 with 3 dice is* $\frac{5}{108}$

Sol. Consider the expression

$$(x+x^2+\ldots+x^6)^3$$

$$=x^3\frac{(1-x^6)^3}{(1-x)^3}$$

$$=x^3(1-3x^6+3x^{12}-x^{18})(1-x)^{-3}$$

$$=x^3(1-3x^6+3x^{12}-x^{18})\left(1+3x+\frac{3.4}{2!}x^2+\frac{3.4.5}{3!}x^3+\right.$$

$$\left.\ldots\ldots\ldots+\frac{3.4.5\ldots(r+2)}{r!}x^r+\ldots\ldots\right)$$

Therefore, number of ways of getting a total of 15

$$=\text{co-efficient of } x^{15}$$

$$=\frac{3.4.5.6.7.8.9.10.11.12.13.14}{12!}-3.\frac{3.4.5.6.7.8}{6!}+3$$

$$=10.$$

Total number of cases $=6^3=216$

$\therefore$ Reqd. prob. $=\frac{10}{216}=\frac{5}{108}$.

Ex. 7-24. *Show that, in a single throw with two dice, the chance of throwing more than 7 is equal to that of throwing less than 7 each being* $\frac{5}{12}$.

Sol. Consider the expression

$$(x+x^2+\ldots\ldots+x^6)^2$$
$$=x^2(1-x^6)^2(1-x)^{-2}$$
$$=x^2(1-2x^6+x^{12})(1+2x+3x^2+4x^3+\ldots\ldots+(r+1)x^r+\ldots\ldots)$$
$$=x^2(1+2x+3x^2+4x^3+5x^4+6x^5+5x^6+4x^7+3x^8+2x^9+x^{10}+\ldots)$$

Therefore, number of ways of getting a number less than 7

$=$sum of co-efficients of x^2, x^3, x^4, x^5 and x^6

$=1+2+3+4+5=15$

and number of ways of getting a number greater than 7

$=5+4+3+2+1=15$

Total number of cases$=6^2=36$

Therefore, probability of getting a number greater than 7

$=$probability of getting a number less than 7

$$=\frac{15}{36}=\frac{5}{12}.$$

Ex. 7-25. *Each co-efficient in the equation $ax^2+bx+c=0$ is determined by throwing an ordinary die. Find the prob that the equation will have real roots.*

Sol. The roots of the given equation will be real if $b^2 \geqslant 4ac$.

Now various possible values of co-efficients in the equation are 1, 2, 3, 4, 5, 6.

Since there are 3 co-efficients, total number of cases

$=6^3=216.$

Now various possibilities are :

ac	a	c	$4ac$	b	No. of cases
1	1	1	4	2, 3, 4, 5, 6	$5\times1=5$
2	2 1	1 2	8	3, 4, 5, 6	$4\times2=8$
3	1 3	3 1	12	4, 5, 6	$3\times2=6$
4	1 2 4	4 2 1	16	4, 5, 6	$3\times3=9$
5	1 5	5 1	20	5, 6	$2\times2=4$
6	1 2 3 6	6 3 2 1	24	5, 6	$2\times4=8$

7	$\begin{cases}1 & 7\\ 7 & 1\end{cases}$	This is not possible as on a die number greater than '6' can't occur.			
8	$\begin{cases}2 & 4\\ 4 & 2\end{cases}$	32	6		$1\times2=2$
9	3 3	36	6		$1\times1=1$

(Values 10, 11 etc of '*ac*' are not possible as $b^2 \ngtr 36$).

$\therefore$ Total number of favourable cases

$=43$

$\therefore$ Reqd. prob.$=\dfrac{43}{216}$.

Ex. 7-26. *Four tickets marked 00, 01, 10, 11 respectively are placed in a bag. A ticket is drawn at random five times, being replaced each time. Find the prob that the sum of the numbers on tickets thus drawn is 23.*

Sol. Number of favourable cases = co-efficient of x^{23} in

$(x^0+x^1+x^{10}+x^{11})^5$ *i.e.*, $(1+x+x^{10}+x^{11})^5$

i.e., $(1+x)^5(1+x^{10})^5$

i.e., $(1+5x+10x^2+10x^3+5x^4+x^5)(1+5x^{10}+10x^{20}+10x^{30}+5x^{40}+x^{50})$

$\therefore$ Number of favourable cases$=100$

Total number of ways of drawing cards$=4^5$

$\therefore$ Reqd. prob.$=\dfrac{100}{4^5}=\dfrac{25}{256}$.

Ex. 7-27. *An urn contains a white balls and b black balls. If $\alpha+\beta$ balls are drawn from this urn, find the probability that among them there will be exactly α white and β black balls.*

Sol. Total number of cases$={}^{a+b}c_{\alpha+\beta}$

Number of ways of drawing α white balls

$={}^{a}c_{\alpha}$

and number of ways of drawing β black balls

$={}^{b}c_{\beta}$

$\therefore$ Number of ways of having α white and β black balls among $(\alpha+\beta)$ balls drawn

$={}^{a}c_{\alpha}\cdot{}^{b}c_{\beta}$

$\therefore$ Reqd. prob.$=\dfrac{{}^{a}c_{\alpha}\cdot{}^{b}c_{\beta}}{{}^{a+b}c_{\alpha+\beta}}$.

7.2. Notations

(1) Capital letters A, B, C etc., denote events.

(2) (A), (B) etc., denote the happening of events A, B etc.

(3) $(\overline{A})$, $(\overline{B})$ etc., denote the non-happenings of A, B etc.

(4) (AB) denotes the simultaneous happening of A and B.

(5) (A+B) denotes the happening of at least one of the events A and B.

(6) P(A), P(B) etc., denote the probabilities of the happening of the events A, B etc.

(7) P (A/B) denotes the conditional probability of the happening of the event A when it is known that the event B has already happened.

7-3. Theorem of Total Probability

It states that the probability of the happening of any one of the several mutually exclusive events is the sum of the probabilities of the happening of separate events i.e.,

$$P(A_1+A_2+\ldots\ldots+A_n)=P(A_1)+P(A_2)+\ldots\ldots+P(A_n)$$

where A_1, A_2, ... A_n are M.E. events.

Sol. Let A_1, A_2 ... A_n be n mutually exclusive events. Then ws are to show that

$$P(A_1+A_2+\ldots+A_n)=P(A_1)+P(A_2)+\ldots\ldots+P(A_n)$$

Let N be the number of cases which are equally likely, mutually exclusive and exhaustive. Out of these let

no. of cases favourable to $A_1=m_1$

no. of cases favourable to $A_2=m_2$

....

....

no. of cases favourable to $A_n=m_n$

Since A_1, A_2, ... A_n are mutually exclusive, the cases m_1, m_2, ...m_n are quite distinct and non-overlapping.

∴ No. of cases which are favourable to $(A_1+A_2+\ldots+A_n)$ (*i.e.,* occurrence of any of the events A_1, A_2 ... A_n)

$$=m_1+m_2+\ldots+m_n$$

$$\therefore \quad P(A_1+A_2+\ldots+A_n)=\frac{m_1+m_2+\ldots+m_n)}{N}$$

$$=\frac{m_1}{N}+\frac{m_2}{N}+\ldots+\frac{m_n}{N}$$

Now by def.,

$$P(A_1)=\frac{m_1}{N} \text{ etc.}$$

$$\therefore \quad P(A_1+A_2+\ldots+A_n)=P(A_1)+P(A_2)+\ldots\ldots+P(A_n)$$

Ex. 7·28. *Generalize the theorem of total probability for non-mutually exclusive events.*

Sol. Let $A_1, A_2, \ldots\ldots, A_n$ be the events which are not mutually exclusive.

Consider A_1 and A_2 to events. Two mutually exclusive and exhaustive forms in which A_1 can happen are

(*i*) A_1 happens and A_2 does not happens *i.e.*, $(A_1\bar{A}_2)$

(*ii*) A_1 happens and A_2 also happens *i.e.*, (A_1A_2).

Let m_1 and m_2 be the number of cases favourable to (A_1A_2) and (A_1A_2) respectively.

Then number of cases favourable to $(A_1)=m_1+m_2$

$$\therefore \qquad P(A_1)=\frac{m_1+m_2}{n}=\frac{m_1}{n}+\frac{m_2}{n}$$

Where n is the number of equally likely, mutually exclusive and exhaustive cases.

$$\therefore \qquad P(A_1)=P(A_1\bar{A}_2)+P(A_1A_2) \qquad \ldots(1)$$

Interchanging A_1 and A_2,

$$P(A_2)=P(\bar{A}_1A_2)+P(A_1A_2) \qquad \ldots(2)$$

Now the three mutually exclusive and exhaustive forms in which (A_1+A_2) can happen are

(*i*) A_1 happens and A_2 does not happen *i.e.*, $(A_1\bar{A}_2)$.

(*ii*) A_1 happens and A_2 also happens *i.e.*, (A_1A_2).

(*iii*) A_2 happens and A_1 does not not happen *i.e.*, $(\bar{A}_1A_2)$.

$\therefore$ As before, $P(A_1+A_2)=P(A_1\bar{A}_2)+P(A_1A_2)=P(\bar{A}_1A_2) \qquad \ldots(3)$

Subtracting (1) and (2) from (3)

$$P(A_1+A_2)-P(A_1)-P(A_2)=-P(A_1A_2)$$

$$\therefore \qquad P(A_1+A_2)=P(A_1)+P(A_2)-P(A_1A_2) \qquad \ldots(4)$$

Now $P(A_1+A_2+A_3)=P(A_1+\overline{A_2+A_3})$

$$=P(A_1)+P(A_2+A_3)-P(A_1\overline{A_2+A_3})$$

$$=P(A_1)+P(A_2+A_3)-P(A_1A_2+A_1A_3)$$

$$=P(A_1)+\{P(A_2)+P(A_3)-P(A_2A_3)\}$$

$$-\{P(A_1A_2)+P(A_1A_3)-P(A_1A_2A_3)\}$$

$$\{\because \quad (A_1A_2A_1A_3)\equiv(A_1A_2A_3)\}$$

$$=\sum_{i=1}^{3} P(A_i)-\sum_{\substack{i,j=1\\ i<j}}^{3} P(A_iA_j)+(-1)^{3-1}P(A_1A_2A_3)$$

$\therefore$ In general,

$$P(A_1+A_2+\ldots+A_n)=\sum_{i=1}^{n} P(A_i)-\sum_{\substack{i,j=1\\ i<j}}^{n} P(A_iA_j)$$

$$+\sum_{\substack{i,j,k=1\\ i<j<k}}^{n} P(A_iA_jA_k)\ldots\ldots+(-1)^{n-1} P(A_1A_2\ldots A_n)$$

Note. If $A_1, A_2,\ldots A_n$ are mutually exclusive, $P(A_iA_j)=0$, $P(A_iA_jA_k)=0$ etc.

$$\therefore \quad P(A_1+A_2+\ldots+A_n)=P(A_1)+P(A_2)+\ldots+P(A_n)$$

7.4. Independent Events. Def *Two events are said to be independent (in probability sense) if the probability of happening of one does not depend on the happening or non-happening of the other.*

Theorem of Compound Probability. *It states that the probability of the simultaneous occurrence of two non-mutually exclusive events is equal to the probability of happening of one multiplied by the conditional probability of the other when it is known that first has already happened.*

i.e.,
$$P(AB)=\begin{cases} P(A)\,P(B/A) \\ \quad\text{or} \\ P(B)P(A/B)\end{cases}$$

Proof. Let A and B be two non-mutually exclusive events. Then we have to show that

$$P(AB)=\begin{cases} P(A)P(B/A) \\ \quad\text{or} \\ P(B)P(A/B)\end{cases}$$

Two mutually exclusive and exhaustive forms in which A can happen are :

(1) A happens and B does not happen *i.e.*, $(A\,\bar{B})$

(2) A happens and B also happens *i.e.*, (AB)

Let m_1 and m_2 be the number of cases favourable to $(A\bar{B})$ and (AB) respectively and n the number of cases which are equally likely, mutually exclusive and exhaustive.

Then number of cases favourable to $(A)=m_1+m_2$

$$\therefore \qquad P(A)=\frac{m_1+m_2}{n}$$

Now by def., $P(AB=)\frac{m_2}{n}$

$$=\frac{m_1+m_2}{n}\cdot\frac{m_2}{m_1+m_2}=P(A).\frac{m_2}{m_1+m_2}$$

Assuming the occurrence of A, out of n only (m_1+m_2) cases are left, out of which m_2 are also favourable to B.

$\therefore$ $\frac{m_2}{m_1+m_2}$ gives the conditional probability of B when it is given that A has occurred *i.e.*,

$$\frac{m_2}{m_1+m_2}=P(B/A)$$

$$\therefore \quad P(AB)=P(A)P\ (B/A)$$

Interchanging A and B

$$P(AB)=P(B)P(A/B)$$

Note. (1) If A and B are independent, prob of happening of B (or A) is not effected by the happening or non-happening of A (or B).

$$\therefore \quad P\ (B/A)=P(B) \text{ and } P(A/B)=P(A)$$

$$\therefore \quad P(AB)=P(A)P(B)$$

Ex. 7-29. *It is given that*

$$P(A_1+A_2)=\frac{5}{6},\ P(A_1A_2)=\frac{1}{3},\ P(\bar{A}_2)=\frac{1}{2}$$

where $P\ (\bar{A}_2)$ *stand for the probability that* A_2 *does not happen. Determine* $P(A_1)$ *and* $P(A_2)$: *Hence show that the events* A_1 *and* A_2 *are independent.*

Sol. Since total probability is always unity,

$$P(A_2)+P(\bar{A}_2)=1$$

$$\therefore \quad P(A_2)=1-P(\bar{A}_2)$$

$$=1-\frac{1}{2}=\frac{1}{2}.$$

By additive law for non-mutually exclusive events,

$$P(A_1+A_2)=P(A_1)+P(A_2)-P(A_1A_2)$$

$$\therefore \quad \frac{5}{6}=P(A_1)+\frac{1}{2}-\frac{1}{3}$$

$$\therefore \quad P(A_1)=\frac{2}{3}$$

$$\therefore \quad P(A_1)P(A_2)=\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{3}=P(A_1\,A_2)$$

$\therefore$ Events A_1 and A_2 are independent.

Ex. 7·30. *Discuss and criticise the following :*

$$P(A)=\frac{2}{3},\ P(B)=\frac{1}{4},\ P(C)=\frac{1}{6}$$

where A, B and C are mutually exclusive events.

Sol. Since three events A, B and C are mutually exclusive, by total probability theorem

$$P(A+B+C)=P(A)+P(B)+P(C)$$
$$=\frac{2}{3}+\frac{1}{4}+\frac{1}{6}$$
$$=\frac{13}{12}>1$$

Since the probability is always less than unity, the statement is wrong.

Ex. 7·31. *Two packs of cards are made up in such a way that the first pack consists of 39 red cards and 13 black cards ; second pack consists of 39 black cards and 13 red cards. A sampling experiment is carried out in the following way : A card is drawn from the first pack, if it is red, a second card is drawn from the same pack after replacing the first red card. The colour of the second card drawn from the first pack is noted. If the first card drawn from the first pack is black, then the second card is drawn from the second pack and the colour of the second card is noted. Both the cards are then replaced in their respective packs. What is the probability that the second card is red.*

Sol. Two different possibilities are :

(1) First card drawn is red.

(2) First card drawn is black.

(1) Now the probability of drawing a red card from the first pack

$$=\frac{39}{39+13}$$
$$=\frac{39}{52}=\frac{3}{4}$$

Since the first card drawn is red, the second card is also to be drawn from the first pack but after replacing the first red card.

$\therefore$ Probability of drawing a red card in second draw $=\frac{3}{4}$.

This is the conditional probability of drawing a red card in second draw when it is known that card drawn in first draw was red.

$\therefore$ By the theorem of compound probability, the probability of drawing red cards in first and second draws

$$=\frac{3}{4}\cdot\frac{3}{4}=\frac{9}{16}.$$

(2) The probability of drawing a black card from the first pack

$$=\frac{13}{39+13}=\frac{13}{52}=\frac{1}{4}$$

Since the first card drawn is black, the second card is to be drawn from the second pack.

$\therefore$ Conditional probability of drawing a red card when it is known that a black card was drawn in first draw.

$$=\frac{13}{39+13}=\frac{1}{4}$$

$\therefore$ Probability of drawing a black card in first draw and a red card in second draw

$$=\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)=\frac{1}{16}$$

$\therefore$ Since two possibilities are mutually exclusive, by the theorem of total probability.

Reqd. prob. $=\frac{9}{16}+\frac{1}{16}=\frac{10}{16}=\frac{5}{8}$

Ev. 7·32. *From each of three married couples one of the partners is selected at random. What is the probability of their being all of one sex ?*

Sol. Probability of selecting a partner (male or female) from either couple

$$=\frac{1}{2}$$

There are only two mutually exclusive possibilities :

(1) All the partners are of male sex,

(2) All the partners are of female sex.

By compound probability theorem, probability of either possibility

$$=\left(\frac{1}{2}\right)^3=\frac{1}{8}$$

$\therefore$ By theorem of total probability,

reqd. prob. $\qquad = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

Ex. 7-33. *In above question, show that the probability of choosing two men and one woman is* $\frac{3}{8}$.

Sol. Three possibilities are :

1st couple	*2nd couple*	*3rd couple*
M	*M*	*W*
M	*W*	*M*
W	*M*	*M*

Probability of choosing a partner (male or female) from either couple

$$=\frac{1}{2}$$

By compound probability theorem, probability of either possibility

$$=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$=\frac{1}{8}$$

By theorem of total probability, probability of choosing two men and one woman

$$=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$$

$$=\frac{3}{8}$$

Ex. 7-34. *A number x is chosen at random from the integers 1, 2, 3, ... n and A and B denote the event that x is a multiple of 2 and 3 respectively. Show that A and B are independent events when $n=96$ but not when $n=100$.*

Sol. The no. of integers in 1, 2, ...n, which are divisible by the integer m is the greatest integer less than $\frac{n}{m}$

(*i*) When $\qquad n=96$

No. of integers which are divisible by 2

$$=\frac{96}{2}=48$$

No. of integers which are divisible by 3

$$=\frac{96}{3}=32$$

No. of integers which are divisible by 3 and 2 both *i.e.*, by 6

$$=\frac{96}{6}=16$$

$\therefore$ $P(A)$=Probability that x is a multiple of 2

$$=\frac{48}{96}=\frac{1}{2}$$

$P(B)$=Probability that x is a multiple of 3

$$=\frac{32}{96}=\frac{1}{3}$$

and $P(AB)$=Probability that x is a multiple of 2 and 3 both *i.e.*, 6

$$=\frac{16}{96}=\frac{1}{6}$$

$\therefore$ $P(AB)=P(A)P(B)$

$\therefore$ Events A and B are independent.

(*ii*) When $n=100$

No. of integers which are divisible by 2

$$=\frac{100}{2}=50$$

No. of integers which are divisible by 3

$$=\text{greatest integer less than } \frac{100}{3}=33$$

No. of integers which are divisible by 3 and 2 both *i.e.*, 6

$$=\text{greatest integer less than } \frac{100}{6}=16$$

$\therefore$ $$P(A)=\frac{50}{100}=\frac{1}{2}$$

$$P(B)=\frac{33}{100}$$

and $$P(AB)=\frac{16}{100}$$

$\therefore$ $P(AB)\neq P(A)P(B)$

$\therefore$ Events A and B are not independent.

Ex. 7-35. *A bag contains 5 white and 4 black balls. A ball is drawn from this bag and replaced and then a second draw of a ball is made. What is the probability that the two balls drawn were of different colours.*

Sol. The rwo different possibilities are :

(1) The first draw gives white ball and the second draw gives black ball.

(2) The first draw gives black ball and the second draw gives white ball.

Since the ball drawn in the first draw is replaced, the probabilities of drawing either ball (white or black) in two draws are same.

Now the probability of drawing a white ball

$$=\frac{5}{9}$$

and the probability of drawing a black ball

$$=\frac{4}{9}$$

$\therefore$ By the theorem of compound probability, probability of possibility (1)

$$=\frac{5}{9}\times\frac{4}{9}=\frac{20}{81}$$

and probability of possibility (2)

$$=\frac{4}{9}\times\frac{5}{9}=\frac{20}{81}$$

$\therefore$ By the theorem of total probability, the probability of getting two balls of different colours in two draws

$$=\frac{20}{81}+\frac{20}{81}=\frac{40}{81}$$

Ex. 7-36. *An urn contains 4 white and 5 black balls, a second urn contains 5 white and 4 black balls. One ball is transferred from the first to second urn, then a ball is drawn from the second urn. What is the probability that it is white ?*

Sol. There are two different possibilities.

(1) The ball transferred from the first to second urn is white.

(2) The ball transferred from the first to second urn is black.

The probability of drawing a white ball from second urn in these two possibilities will be different. So we consider these two possibilities separately.

(1) Now probability of drawing a white ball from the first urn

$$=\frac{^4c_1}{^9c_1}=\frac{4}{9}$$

Since the ball transferred from first to second urn is white, total number of white balls in second run

$$=5+1=6.$$

The number of balls in second urn

$$=6+4=10$$

$\therefore$ Probability of drawing a white ball from second urn

$$=\frac{6}{10}=\frac{3}{5}$$

$\therefore$ By the theorem of compound probability, the probability of transferring a white ball and then drawing a white ball from the second urn

$$=\frac{4}{9}\cdot\frac{3}{5}=\frac{4}{15}$$

(2) Probability of drawing a black ball from the first urn

$$=\frac{^5c_1}{^9c_1}=\frac{5}{9}$$

Total number of white balls in second urn$=5$
and number of balls in second urn$=10$

$\therefore$ Probability of drawing a white ball from the second urn

$$=\frac{5}{10}=\frac{1}{2}$$

$\therefore$ The probability of transferring a black ball and then drawing a white ball from the second urn

$$=\frac{5}{9}\cdot\frac{1}{2}=\frac{5}{18}$$

The two possibilities (1) and (2) are mutually exclusive Hence by the theorem of total probability, the probability of transferring a ball from first urn to second and then drawing a white ball from the second.

$$=\frac{4}{15}+\frac{5}{18}=\frac{49}{90}$$

Ex. 7-37. *Three urns contain respectively 1 white, 2 black balls : 2 W and 1 B balls, 2 white and 2 black balls. One ball is transferred from the first urn into the second; then one from the latter is transferred into the third. Finally one ball is drawn from the third urn. What is the probability of its being white.*

Sol. There are in all four different possibilities :

(1) The white ball is transferred from the first to the second urn and then a white ball is transferred from the second to the third urn.

(2) The white ball is transferred from the first to the second urn and then the black ball is transferred from the second to the third urn.

(3) A black ball is transferred from the first to the second urn and then a white ball is transferred from the second to the third urn.

(4) A black ball is transferred from the first to the second urn and then a black ball is transferred from the second to the third urn.

(1). Probability of drawing the white ball from the first urn

$$=\frac{1}{3}$$

After transferring the white ball from the first to the second urn,

Number of white balls in the second urn $=3$

and total number of balls in the second urn $=4$

$\therefore$ Probability of drawing a white ball from the second urn after transferring the white ball from the first to the second urn

$$=\frac{3}{4}$$

After transferring a white ball from the second to the third urn,

Number of white balls in the third urn $=3$

and total number of balls in the third urn $=5$

$\therefore$ The probability of drawing a white ball from the third urn after transferring a white ball from the second to the third urn

$$=\frac{3}{5}$$

$\therefore$ By the theorem of compound probability, the probability of transferring the white ball from the first to the second urn ; then a white ball from the second to the third and then drawing a white ball from the third urn

$$=\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{3}{5}=\frac{9}{60}$$

2. After transferring the white ball from the first to the second urn, the number of black balls in the second urn $=1$.

$\therefore$ The probability of drawing the black ball from the second urn after transferring the white ball from the first to the second urn.

$$=\frac{1}{4}$$

After transferring the black ball from the second to the third urn,

the number of white balls in the third urn $=2$

$\therefore$ Probability of drawing a white ball from the third urn after transferring the black ball from the second to the third urn

$$=\frac{2}{5}.$$

$\therefore$ The probability of transferring the white ball from the first to the second ; then the black ball from the second to the third and then drawing a white ball from the third urn

$$=\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{2}{5}=\frac{2}{60}$$

(3) Proceeding as in above two cases, the probability of transferring a black ball from the first to the second urn : then a white from the second to the third urn and then drawing a white ball from the third urn

$$=\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}=\frac{12}{60}.$$

(4) The probability of transferring a black ball from the first to the second ; a black ball from the second to the third and then drawing a while ball from the third urn

$$=\frac{2}{3}\cdot\frac{2}{4}\cdot\frac{2}{5}=\frac{8}{60}.$$

Since the four possibilities are mutually exclusive, by the theorem of total probability, the probability of transferring a ball from the first to the second ; then a ball from the second to the third and then drawing a white ball from the third urn

$$=\frac{9}{60}+\frac{2}{60}+\frac{12}{60}+\frac{8}{60}=\frac{31}{60}$$

Ex. 7-38. *In a bag there are six balls of which 3 are white and 3 are black. They are drawn successively without replacement. What is the chance that the colours are alternate ?*

Sol. Let (W) be the event that in a draw white ball appears and (B) be the event that blackball appears. The possible sequences are

$$(W)(B)(W)(B)(W)(B)$$

and

$$(B)(W)(B)(W)(B)(W)$$

Probability of drawing a white (or black) ball in first draw

$$=\frac{3}{6}=\frac{1}{2}$$

Probability of drawing a black (or white) ball in second draw when the ball drawn in first draw is white (or black)

$$=\frac{3}{5}$$

Probability of drawing a white (or black) ball in third draw when in first two draws white (or black) and black (or white) balls have been drawn

$$=\frac{2}{4}=\frac{1}{2}$$

Probability of drawing a black (or white) ball in fourth draw when in third draw the ball drawn is white (or black)

$$=\frac{2}{3}$$

Probability of drawing a white (or black) ball in fifth draw

$$=\frac{1}{2}$$

and probability of drawing a black (or white) ball in sixth draw=1.

$\therefore$ By compound probability theorem, probability of either sequence

$$=\left(\frac{1}{2}\right)\left(\frac{3}{5}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)(1)$$

$$=\frac{1}{20}$$

$\therefore$ By total probability theorem, probability of getting balls of alternate colours$=2\times\frac{1}{20}=\frac{1}{10}$.

Ex. 7-39. *An urn contains 3 white and 5 black balls. One ball is drawn and its colour unnoted laid aside. Then another ball is drawn. Find the probability that it is white.*

Sol. There are two mutually exclusive possibilities :

(1) In the first draw a white ball appears.

Prob of drawing a white ball in first draw

$$=\frac{3}{8}$$

Number of white balls in the urn before second draw

$=2.$

Therefore, conditional probability of drawing a white ball in second draw, when in first draw a white ball has been drawn

$$=\frac{2}{7}$$

Therefore, by compound probability theorem, prob of drawing a white ball in second draw

$$=\frac{3}{8}\cdot\frac{2}{7}=\frac{6}{56}$$

(2) In the first draw a black ball appears.

Prob of drawing a black ball in first draw

$$=\frac{5}{8}$$

Number of white balls in the urn before second draw

$$=3$$

Therefore, conditional prob of drawing a while ball in second draw, when in first draw a black ball has been drawn

$$=\frac{3}{7}$$

Therefore, prob of drawing a white ball in second draw

$$=\frac{5}{8}\ \frac{3}{7}=\frac{15}{56}$$

$\therefore$ Reqd. prob. $=\frac{6}{56}+\frac{15}{56}=\frac{21}{56}$

Ex. 7·40. *A lady declares that by taking a cup of tea with milk she can discriminate whether the milk or tea-infusion was first added to the cup. It is proposed to test this assertion by means of an experiment with 10 cups of tea, five made in one way and five in the other and presenting them to the lady for judgement in random order.*

Calculate the probability on the null hypothesis (i.e., the lady has no discrimination power) that the lady would judge correctly all the ten cups, being known to her that 5 are of each kind.

Suppose that the tea cups were presented to the lady in five pairs, each pair to consist of cups of each kind in a random order. How would the probability of correctly judging with every cup on the null hypothesis be altered in this case ?

Sol. (*a*) When tea cups are presented in random order :

No. of ways of presenting 10 cups, 5 of each kind

$$=\frac{10\,!}{5\,!\,5\,!}$$

$$=252$$

Out of these 252 ways, the cups are presented in any one manner. The lady has to find which one is that method.

$\therefore$ No. of favourable cases$=1$.

$\therefore$ Reqd. prob.$=\frac{1}{252}$.

(*b*) When cups are presented in 5 pairs :

Two two ways of presenting a pair are :

$$MI; \quad IM$$

where '*M*' stands for the cup prepared by taking milk first and '*I*' for the cup prepared by taking infusion first.

When a pair is presented to the lady, she has to find, out of these two which one is the method used.

$\therefore$ Probability of correct Judging for each pair

$$=\frac{1}{2}.$$

As the presentation of various pairs is independent of each other, by the theorem of compound probability, the joint probability of correctly judging the 5 pairs

$$=\left(\frac{1}{2}\right)^5=\frac{1}{32}.$$

Ex. 7·41. *In a group of equal number of men and women 10% men and 45% women are unemployed. What is the probability that a person selected at random is employed?*

Sol. Probability for a man to be unemployed

$$=\frac{10}{100}=\frac{1}{10}$$

and probability for a woman to be unemployed

$$=\frac{45}{100}=\frac{9}{20}$$

$\therefore$ Probability for a man to be employed

$$=1-\frac{1}{10}=\frac{9}{10}$$

and probability for a woman to be employed

$$=1-\frac{9}{20}=\frac{11}{20}.$$

Since the group contains equal number of men and women, probability of selecting a man$=\frac{1}{2}$ and same is probability of selected a woman.

$\therefore$ Probability of selecting an employed man

$$=\left(\frac{1}{2}\right)\left(\frac{9}{10}\right)$$

and probability of selecting an employed woman

$$=\left(\frac{1}{2}\right)\left(\frac{11}{20}\right)$$

The selected person may be either man or woman. Hence by the theorem of total probability, probability of selecting an employed person

$$=\left(\frac{1}{2}\right)\left(\frac{9}{10}\right)+\left(\frac{1}{2}\right)\left(\frac{11}{20}\right)$$

$$=\frac{1}{2}\left\{\frac{9}{10}+\frac{11}{20}\right\}=\frac{29}{40}.$$

Ex. 7-42. *In a random sample of 1000 residents of a large city, 700 were found to be reading newspaper A and 400 were found to be reading newspaper B. On the hypothesis that the habits of reading newspapers A and B are independent of each other, (i) what is the chance that a person selected at random would be reading both the newspapers (ii) How many persons out of 1000 should be expected to be reading both newspapers ?*

Sol. Probability for a person to be reading newspaper A

$$=\frac{700}{1000}=\frac{7}{10}$$

and probability for a person to be reading newspaper B

$$=\frac{400}{1000}=\frac{2}{5}$$

Since the habits of reading newspapers A and B are independent of each other, by the theorem of compound probability, the probability that a person selected at random would be reading both the newspapers

$$=\left(\frac{7}{10}\right)\left(\frac{2}{5}\right)=\frac{7}{25}$$

$\therefore$ No. of persons out of 1000 to be reading both the newspapers

$$=\frac{7}{25}\times 1000=280.$$

Ex. 7-43. *The probabilities of n independent events are $p_1, p_2, \ldots p_n$. Find an expression for probability that at least one of the events will happen.*

Sol. Since total prob is unity,

Prob. of non-happening of 1st event $=1-p_1$

" " " " " 2nd " $=1-p_2$

....

" " " " " nth " $=1-p_n$

Since the events are independent, by compound prob. theorem,

Prob. of non-happening of all the events.

$$=(1-p_1)(1-p_2)\ldots\ldots(1-p_n)$$

$\therefore$ Prob. of the happening of at least one of the events

$$=1-(1-p_1)(1-p_2)\ldots\ldots(1-p_n)$$

Ex. 7-44. *A problem in statistics is given to students whose chance of solving it are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved ?*

Sol. The problem will be solved if at least one student solves it.

Here $p_1=\frac{1}{2}, p_2=\frac{1}{3}, p_3=\frac{1}{4}$

$$\therefore \text{ Reqd. prob.} \quad =1-\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)$$

$$=\frac{3}{4}$$

Ex. 7-45. *Three groups of children contain 3 girls and 1 boy. 2 girls and 2 boys and 3 boys and 1 girl. One child is selected at random from each group. Show that the chance that the three selected consist of 1 girl and 2 boys is $\frac{13}{32}$.*

Sol. Probabilities of selecting a boy from three groups are $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ respectively and probabilities of selecting a girl from three groups are $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$ respectively.

Different mutually exclusive possibilities of required selection are :

1st group	*2nd group*	*3rd group*
1 girl	1 boy	1 boy
1 boy	1 girl	1 boy
1 boy	1 boy	1 girl

$$=\frac{7}{20}$$

$$\left\{\begin{array}{l}\because \quad \text{Prob. of } A \text{ speaking truth}=\frac{75}{100}=\frac{3}{4} \\ \quad\ \ \text{Prob. of } B \text{ speaking truth}=\frac{80}{100}=\frac{4}{5}\end{array}\right\}$$

Tharefore, A and B are likely to contradict each other in $\frac{7}{20}\times 100=35\%$ of the cases.

Ex. 7-49. *The probability that a teacher will give an unannounced test during any class meeting is $\frac{1}{5}$. If a student is absent twice, what is the probability that he will miss at least one test ?*

Sol. The student will not miss any test, if on the two days he is absent the teacher does not give any test.

Probability for a teacher not giving any test on the two days (when the student is absent)

$$=\left(1-\frac{1}{5}\right)\left(1-\frac{1}{5}\right)=\frac{16}{25}$$

$\therefore$ Probability that the student will not miss any test

$$=\frac{16}{25}$$

Since total probability is unity,
probability that the student will miss at least one test

$$=1-\frac{16}{25}=\frac{9}{25}$$

Ex. 7-50. *A can solve 75% of the problems and B can solve 70%. What is the probability that either A or B can solve a problem chosen at random ?*

Sol. Let (A) and (B) be the events that A and B solve the problem respectively.

Then $\quad P(A)=\frac{3}{4}$

and $\quad P(B)=\frac{7}{10}$

$$\begin{aligned}\text{Now } P(A+B)&=P(A)+P(B)-P(AB)\\&=P(A)+P(B)-P(A)P(B)\\&=\frac{3}{4}+\frac{7}{10}-\frac{21}{40}\\&=\frac{37}{40}\end{aligned}$$

Therefore, probability that either A or B can solve the problem

$$=\frac{37}{40}.$$

Ex. 7·51. *If 4 whole numbers taken at random are multiplied together, show that the chance that the last digit in the product is 1, 3, 7 or 9 is* $\frac{16}{625}$

Sol. In all there are ten digits *viz.*, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Digits 1, 3, 7 and 9 have the property that when any two of them are multiplied, the last digit in the product is one of these four digits.

Therefore, if the last digit in the product of four whole numbers is to be 1, 3, 7 or 9, each whole number must have its last digit 1, 3, 7, or 9.

Probability for a whole number to have its last digit 1, 3, 7 or 9

$$=\frac{4}{10}=\frac{2}{5}$$

Therefore, by compound probability theorem, probability that all the four whole numbers have their last digit 1, 3, 7 or 9

$$=\left(\frac{2}{5}\right)^4$$

$$=\frac{16}{625}$$

Therefore, reqd. prob. $=\frac{16}{625}$.

Ex. 7·52. *A six faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. What is the probability that the sum of the two numbers thrown is even ?*

Sol. Let p be the probability for an odd number.

Then $2p$ is the probability for an even number.

When the die is thrown, either even number turns up or odd number turns up.

$\therefore\quad 2p+p=\text{Total prob.}=1$

$\therefore\quad p=\frac{1}{3}$

$\therefore\quad$ Probability for an odd number$=\frac{1}{3}$

Probability for an even number$=\frac{2}{3}$

The sum of the two numbers thrown will be even if in both throws either we get even numbers or odd numbers. Now probability that in two throws even numbers turns up

$$=\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)=\frac{4}{9}$$

and probability that in two throws odd numbers turn up

$$=\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)=\frac{1}{9}$$

$\therefore$ By the theorem of total probability, probability that the sum of the two numbers thrown is even

$$=\frac{4}{9}+\frac{1}{9}=\frac{5}{9}$$

Ex. 7-53. *A and B throw with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 beofre A throws 6. If A begins, show that his chance of winning is* $\frac{30}{61}$.

Sol. Let (A) and (B) be the events that A gets 6 and B gets 7 with a pair of dice respectively.

Then $\qquad P(A)=\frac{5}{36}$ and $P(B)=\frac{6}{36}=\frac{1}{6}$

Therefore $\qquad P(\bar{A})=1-\frac{5}{36}=\frac{31}{36}$ and $P(\bar{B})=1-\frac{1}{6}=\frac{5}{6}$

Since A begins, he can win in following mutually exclusive ways :

$$(A),\ (\bar{A}\ \bar{B}\ A),\ (\bar{A}\ \bar{B}\ \bar{A}\ \bar{B}\ A),\ldots\ldots\ldots$$

Therefore, by theorem of total probability, probability that A wins

$$=P(A)+P(\bar{A}\ \bar{B}\ A)+P(\bar{A}\ \bar{B}\ \bar{A}\ \bar{B}\ A)+\ldots\ldots\ldots\ldots$$

Since throws are independent, by compound probability theorem,

probability that A wins

$$=P(A)+P(\bar{A})P(\bar{B})P(A)+P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A)+\ldots\ldots$$

$$=\frac{5}{36}\left\{1+\left(\frac{31}{36}\cdot\frac{5}{6}\right)+\left(\frac{31}{36}\cdot\frac{5}{6}\right)^2+\ldots\ldots\right\}$$

$$=\frac{5}{36}\cdot\frac{1}{1-\frac{31}{36}\cdot\frac{5}{6}}$$

$$=\frac{30}{61}.$$

Ex. 7·54. *A and B take turns in throwing two dice, the first to throw 9 being awarded the prize. Show that their chance of winning are in the ratio 9 : 8.*

Sol. Let (A) and (B) be two events that A and B get 9 in a throw respectively.

Then $\quad P(A)=P(B)=\dfrac{4}{36}=\dfrac{1}{9}$

Therefore $P(\bar{A})=P(\bar{B})=1-\dfrac{1}{9}=\dfrac{8}{9}$

Since A begins, he can win in following mutually exclusive ways :

$$(A),\ (\bar{A}\ \bar{B}\ A),\ \bar{A}\ \bar{B}\ \bar{A}\ \bar{B}\ A),\ldots\ldots\ldots$$

Therefore, by theorem of total probability, probability that A wins

$$=P(A)+P(\bar{A}\ \bar{B}\ \bar{A})+P(\bar{A}\ \bar{B}\ \bar{A}\ \bar{B}\ A)+\ldots\ldots$$

$$=P(A)+P(\bar{A})P(\bar{B})P(A)+P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A)+\ldots.$$

$$=\frac{1}{9}\left\{1+\left(\frac{8}{9}\right)^2+\left(\frac{8}{9}\right)^4+\ldots\ldots\right\}$$

$$=\frac{1}{9}\cdot\frac{1}{1-\frac{64}{81}}=\frac{9}{17}.$$

Since total probability is unity and one of two players is to win, probability that B wins

$$=1-\frac{9}{17}=\frac{8}{17}$$

Therefore, chances of winning are in the ratio 9 : 8.

Ex. 7·55. *A, B and C in order toss a coin. The first one to throw a head wins. What are their respective chances of winning assuming that the game may continue indefinitely ?*

Sol. Let (A), (B) and (C) be the events that A, B and C get head in a toss respectively.

Then $\qquad P(A)=P(B)=P(C)=\dfrac{1}{2}$

$\therefore \qquad P(\bar{A})=P(\bar{B})=P(\bar{C})=1-\dfrac{1}{2}=\dfrac{1}{2}$

Since A begins, he can win in following mutually exclusive ways :

$$(A),\ (\bar{A}\ \bar{B}\ \bar{C}\ A),\ (\bar{A}\ \bar{B}\ \bar{C}\ \bar{A}\ \bar{B}\ \bar{C}\ A),\ \ldots\ldots\ldots$$

Therefore, by theorem of total probability, probability that A wins

$$=P(A)+P(\bar{A}\,\bar{B}\,\bar{C}\,A)+P\,(\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,\bar{B}\,\bar{C}\,A(+\ldots\ldots$$

$$=P(A)+P(\bar{A})P(\bar{B})P(\bar{C})P(A)+\{P(\bar{A})P(\bar{B})P(\bar{C})\}^2P(A)+\ldots\ldots\ldots$$

$$=\frac{1}{2}+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^7+\ldots\ldots$$

$$=\frac{1}{2}\left\{1+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^6+\ldots\ldots\right\}=\frac{1}{2}\cdot\frac{1}{1-\frac{1}{8}}=\frac{4}{7}$$

B can win in following mutually exclusive ways :

$(\bar{A}B)$, $(\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,\bar{B})$, $(\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,B)$,.........

$\therefore$ Probability that B wins

$$=P(\bar{A}\,B)+P(\bar{A}\bar{B}\,\bar{C}\,\bar{A}\,B)+P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,\bar{B}\,\bar{C}\,\bar{A}\,B)+\ldots\ldots$$

$$=P(\bar{A})P(B)+P(\bar{A})P(\bar{B})P(\bar{C})P(\bar{A})P(B)$$
$$+\{P(\bar{A})P(\bar{B})P(\bar{C})\}^2P(\bar{A})P(B)+\ldots\ldots$$

$$=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^8+\ldots\ldots$$

$$=\frac{\frac{1}{4}}{1-\frac{1}{8}}=\frac{2}{7}$$

$\therefore$ Probability that C wins

$$=1-\frac{4}{7}-\frac{2}{7}=\frac{1}{7}$$

Ex. 7·56. *A and B toss a coin alternately on the understanding that the first to obtain head wins the toss. Show that their respective chances of winning are* $\frac{2}{3}$ *and* $\frac{1}{3}$.

Sol. Let (A) and (B) be the events that A and B get head in a toss respectively.

Then $$P(A)=P(B)=\frac{1}{2}$$

$\therefore$ $$P(\bar{A})=P(\bar{B})=1-\frac{1}{2}=\frac{1}{2}$$

Since A begins, he can win in following mutually exclusive ways :

(A), $(\bar{A}\,\bar{B}\;A)$, $(\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A)$,.........

Therefore by theorem of total probability, probability that A wins

$$=P(A)+P(\bar{A}\,\bar{B}\,A)+P(\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A)+\dots\dots\dots$$

$$=P(A)+P(\bar{A})P(\bar{B})P(A)+\{P(\bar{A})P(\bar{B})\}^2\,P(A)+\dots\dots\dots$$

$$=\frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\dots\dots$$

$$=\frac{\frac{1}{2}}{1-\frac{1}{4}}=\frac{2}{3}$$

$\therefore$ Probability that B wins

$$=1-\frac{2}{3}=\frac{1}{3}$$

Ex. 7-57. *A coin is tossed three times. Find the chance that head and tail will show alternately.*

Sol. Let (H) and (T) denote the occurrence of head and tail in a toss respectively. There are two possibilities :

$$(H)(T)(H)$$

and $$(T)(H)(T)$$

Probability of either possibility

$$=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$$

Therefore, by total probability theorem, probability of having head and tail alternately

$$=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$$

Ex. 7-58. *A and B are two independent witnesses (i.e., there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Show that the probability that this statement is true is*

$$\frac{xy}{1-x-y+2xy}$$

Sol. Let (A_1) and (A_2) be the events that A and B agree in a statement and their statement is correct respectively.

Then $$P(A_1)=x.y+(1-x)(1-y)$$

$$=1-x-y+2xy$$

and $$P(A_1A_2)=xy$$

By compound probability theorem,

$$P(A_1A_2)=P(A_1)\,P(A_2/A_1)$$

$$\therefore \qquad P(A_2/A_1)=\frac{P(A_1A_2)}{P(A_1)}$$

$$=\frac{xy}{1-x-y+2xy}$$

7-59. *A coin is tossed $(m+n)$ times $(m>n)$. Show that the probability of getting m consecutive heads is*

$$\frac{n+2}{2^{m+1}}$$

Sol. Let p_n denote the probability of getting m consecutive heads in $(m+n)$ tossing. Two mutually exclusive ways of having m consecutive heads in $(m+n)$ tossing are :

(*i*) m consecutive heads occur in $(m+n-1)$ tossings.

(*ii*) Only $(m-1)$ consecutive heads occur in $(m+n-1)$ tossings. In order to have m consecutive heads, head must appear in $(m+n)$th toss and there must be a tail in nth toss.

Probability of possibility $(i)=p_{n-1}$

and probability of possibility $(ii)=$(Probability of tail in nth trial)$\times$(Probability of heads in last m trials)

$$=\frac{1}{2}\cdot\left(\frac{1}{2}\cdot\frac{1}{2}\cdots\cdots\cdots\; m \text{ times}\right)$$

$$=\frac{1}{2^{m+1}}$$

By theorem of total probability,

$$p_n=p_{n-1}+\frac{1}{2^{m+1}}$$

or $$\frac{1}{2^{m+1}}=p_n-p_{n-1}=p_{n-1}-p_{n-2}=\cdots\cdots\cdots=p_1-p_0$$

$$\therefore \qquad p_n=p_0+\frac{n}{2^{m+1}}$$

$p_0=$probability of getting m consecutive heads in $(m+0)$ tossings

$$=\frac{1}{2^m}$$

$$\therefore \qquad p_n=\frac{1}{2^m}+\frac{n}{2^{m+1}}$$

$$=\frac{n+2}{2^{m+1}}.$$

Ex. 7·60. (*a*) *What is the probability that two numbers chosen at random will be prime to each other?*

(*b*) *Four positive integers are chosen at random. Find the probability of their having a common factor.*

Sol. (*a*) Let 'a' and 'b' be any two numbers and 'r' a prime number. When 'a' is divided by 'r' the possible remainders are

$$0, 1, 2, \ldots\ldots\ldots (r-1)$$

Therefore, probability that 'a' is divisible by 'r'

i.e., probability of getting zero remainder $= \frac{1}{r}$.

Similarly probability that 'b' is divisible by 'r' $= \frac{1}{r}$

By compound probability theorem,

probability that 'r' divides both 'a' and 'b' $= \frac{1}{r^2}$

Therefore, probability that 'a' and 'b' don't have a common factor 'r'

$$= \left(1 - \frac{1}{r^2}\right)$$

Therefore, probability that 'a' and 'b' are prime to each other

$$= \prod_r \left(1 - \frac{1}{r^2}\right) \qquad r = 2, 3, 5, 7, \ldots\ldots$$

$$= \frac{6}{\pi^2}$$

(*b*) Let a, b, c and d be four integers and r a prime number. Then prob. that all the four numbers a, b, c, d are divisible by r

$$= \frac{1}{r^4}$$

Therefore prob. that a, b, c, d do not have r as a common factor

$$= \left(1 - \frac{1}{r^4}\right)$$

Therefore, prob. that four integers do not have any common factor

$$= \prod_r \left(1 - \frac{1}{r^4}\right) \qquad r = 2, 3, 5, 7 \ldots$$

$$= \frac{90}{\pi^4}$$

Therefore, prob. that four integers have a common factor

$$=1-\frac{90}{\pi^4}$$

Ex. 7-61. *Cards are dealt one by one from a well shuffled pack until an ace appears. Show that the probability that exactly n cards are dealt before the first ace, is*

$$\frac{(51-n)(50-n)(49-n)}{13.51.50.49}$$

If cards continue to be dealt until a second ace appears, prove that the probability that exactly r cards are dealt in all before the second ace, is

$$\frac{r(51-r)(50-r)}{13.17.50.49}$$

Sol. Since n cards are to be dealt before first ace appears, these n cards are to be dealt from the remaining 48 cards leaving aside 4 aces.

$\therefore$ Probability for drawing n cards (not containing any ace) from a pack

$$=\frac{{}^{48}C_n}{{}^{52}C_n}$$

Evidently $(n+1)$th card must be an ace to be drawn out of 4 aces present in remaining $(52-n)$ cards.

Probability for having $(n+1)$th card an ace

$$=\frac{{}^{4}C_1}{{}^{52-n}C_1}$$

By the theorem of compound probability, probability of dealing exactly n cards before first ace

$$=\frac{{}^{48}C_n}{{}^{52}C_n}\cdot\frac{{}^{4}C_1}{{}^{52-n}C_1}$$

$$=\frac{(51-n)(50-n)(49-n)}{13\cdot51\cdot50.49}$$

Out of r cards dealt before a second ace appears, one card is on ace which was drawn in $(n+1)$th draw,

Number of ways of dealing r cards containing one ace

$$={}^{4}C_1\,{}^{48}C_{r-1}$$

$\therefore$ Probability of drawing r cards containing one ace

$$=\frac{{}^{4}C_1.{}^{48}C_{r-1}}{{}^{52}C_r}$$

Evidently $(r+1)$th card must be an ace to be drawn out of 3 aces present in remaining $(52-r)$ cards.

Probability of having $(r+1)$th card a second ace

$$=\frac{{}^{3}C_{1}}{{}^{52-r}C_{1}}$$

Therefore, by compound probability theorem probability of dealing exactly r cards before second ace

$$=\frac{{}^{4}C_{1}\,{}^{48}C_{r-1}}{{}^{52}C_{r}}\cdot\frac{{}^{3}C_{1}}{{}^{52-r}C_{1}}$$

$$=\frac{r(51-r)(50-r)}{13.17.50.49}$$

Ex. 7·62. *Cards are dealt one by one from an ordinary pack (without replacements) until two aces have appeared. Find the most probable number of cards to be turned up.*

Sol. Let x be the total number of cards dealt until the second ace appears and $P(x)$ be its probability.

Evidently xth card must be an ace and among remaining $(x-1)$ cards there must be one ace.

Number of ways of drawing $(x-1)$ cards including one ace

$$={}^{4}C_{1}.{}^{48}C_{x-2}$$

$\therefore$ Prob. of drawing $(x-1)$ cards including one ace

$$=\frac{{}^{4}C_{1}.{}^{48}C_{x-2}}{{}^{52}C_{x-1}}$$

Since xth card, which is to be an ace, is to be drawn from $52-(x-1)=53-x$ cards containing 3 aces, prob. that xth card is an ace

$$=\frac{3}{53-x}$$

Therefore by compound probability theorem,

$$P(x)=\frac{{}^{4}C_{1}.{}^{48}C_{x-2}}{{}^{52}C_{x-1}}\cdot\frac{3}{53-x}$$

$$=\frac{(x-1)(52-x)(51-x)}{13.17.50.49}$$

Most probable number of cards is that value of x for which

$$P(x-1)<P(x)>P(x+1)$$

Consider $\qquad P(x-1)<P(x)$

i.e., $\dfrac{(x-2)(53-x)(52-x)}{13.17.50.49}<\dfrac{(x-1)(52-x)(51-x)}{13.17.50.49}$

i.e., $$x<\frac{55}{3}$$

Consider $$P(x)>P(x+1)$$

i.e., $$\frac{(x-1)(52-x)(51-x)}{13.17.50.49}>\frac{x(51-x)(50-x)}{13.17.50.49}$$

i.e., $$x>\frac{52}{3}$$

Therefore, most probable number x of cards is such that

$$\frac{52}{3}<x<\frac{55}{3}$$

Since x is to be an integer,

$$x=18$$

$\therefore$ Most probable number of cards dealt

$$=18.$$

Ex. 7·63. *Of three independent events the prob. that the first only should happen is $\frac{1}{4}$; the prob. that the second only should happen is $\frac{1}{8}$ and the prob. that the third only should happen is $\frac{1}{12}$. Obtain the unconditional probabilities of the three events.*

Sol. Let A_1, A_2 and A_3 be the events and p_1, p_2, p_3 be their unconditional probabilities.

Now $$P(A_1\bar{A}_2\bar{A}_3)=\frac{1}{4},$$

i.e., $$P(A_1)P(\bar{A}_2)P(\bar{A}_3)=\frac{1}{4}$$

($\because$ A, B, C are independent)

i.e., $$p_1(1-p_2)(1-p_3)=\frac{1}{4} \quad ...(1)$$

$$P(\bar{A}_1A_2\bar{A}_3)=\frac{1}{8}$$

i.e., $$(1-p_1)(p_2)(1-p_3)=\frac{1}{8} \quad ...(2)$$

and $$P(\bar{A}_1\bar{A}_2A_3)=\frac{1}{12}$$

i.e., $$(1-p_1)(1-p_2)p_3=\frac{1}{12} \quad \ldots(3)$$

From (1), (2) and (3)

$$p_1=\frac{1}{2}, \qquad p_2=\frac{1}{3} \text{ and } p_3=\frac{1}{4}.$$

Ex. 7·64. *An integer is chosen at random from the first two hundred digits, what is the prob. that the integer chosen is divisible by 6 or 8 ?*

Sol. Let A and B be the events that the number chosen is divisible by 6 and 8 respectively.

Since $6\times33=198$ and $8\times25=200$, there are 33 and 25 numbers upto 200 which are divisible by 6 and 8 respectively.

$$\therefore \quad P(A)=\frac{33}{200}, \quad P(B)=\frac{25}{200}.$$

Now L.C.M. of 6 and 8=24

The number which is divisible by 24 is divisible by 6 and 8 both.

$\because$ The number of numbers which are divisible by 6 and 8 both

$$=\text{greatest integer less than } \left\{\frac{200}{24}\right\}$$

$$=8$$

$$\therefore \quad P(AB)=\frac{8}{200}$$

Now prob. that the integer chosen is divisible by 6 or 8

$$=P(A+B)$$
$$=P(A)+P(B)-P(AB)$$
$$=\frac{33}{200}+\frac{25}{200}-\frac{8}{200}$$
$$=\frac{50}{200}=\frac{1}{4}.$$

Ex. 7·65. *If n integers taken at random are multiplied together, show that (a) the chance that the last digit of the product is 1, 3, 7 or 9 is $\left(\frac{2}{5}\right)^n$; (b) the chance of its being 2, 4, 6 or 8 is $\frac{4^n-2^n}{5^n}$; (c) of its being 5 is $\frac{5^n-4^n}{10^n}$; and (d) of its being '0' is $\frac{10^n-8^n-5^n+4^n}{10^n}$.*

Sol. (*a*) The last digit in the product will be 1, 3, 7 or 9 if each of the n integers end with either of these four digits. Now since there are in all 10 digits, prob. of selecting these four digits

$$=\frac{4}{10}$$

This is also the prob for a integer to end with either 1, 3, 7 or 9.

$\therefore$ By compound prob theorem, prob that the last digit in the product of n integers is 1, 3, 7 or 9 = prob that all n integers end with 1, 3, 7 or 9

$$=\left(\frac{4}{10}\right)^n=\left(\frac{2}{5}\right)^n.$$

(*b*) The last digit in the product will be 1, 2, 3, 4, 6, 7, 8 or 9 iff each of the n integers end with either of these eight digits.

$\therefore$ Prob for a integer to end with either of these 8 digits

$$=\frac{8}{10}=\frac{4}{5}$$

$\therefore$ Prob for the last digit in the product of n integers to be 1, 2, 3, 4, 6, 7, 8 or $9=\left(\frac{4}{5}\right)^n$.

Now the prob. for the last digit in the product of n integers to be 1, 3, 7 or $9=\left(\frac{2}{5}\right)^n$ {from (*a*)}

$\therefore$ By total prob. theorem, prob. of having the last digit in the product of n integers to be 2, 4, 6 or $8=\dfrac{4^n-2^n}{5^n}$.

(*c*) The last digit in the product will be 5 iff at least one integer end with 5 and other with 1, 3, 7 or 9.

Now prob. that n integers end with 1, 3, 5, 7 or 9.

$$=\left(\frac{5}{10}\right)^n$$

$\therefore$ Prob. that out of n integers at least one integer end with 5 and other with 1, 3, 7 or $9=\left(\frac{5}{10}\right)^n-$ prob. that all integers end with 1, 3, 7 or $9=\left(\frac{5}{10}\right)^n-\left(\frac{4}{10}\right)^n=\dfrac{5^n-4^n}{10^n}$.

(*d*) Since total prob. is unity, prob. that the last digit in the product is '0'

$=1-$(prob that last digit in the product is 1, 2, 3, 4, 5, 6, 7, 8 or 9).

$=1-$(prob. that the last digit in the product is 1, 3, 7 or 9)
$-$(prob. that the last digit in the product is 2, 4, 6 or 8)
$-$(prob. that the last digit in the product is 5)

$$=1-\left(\frac{4}{10}\right)^n-\left(\frac{4^n-2^n}{5^n}\right)-\frac{5^n-4^n}{10^n}$$

$$=\frac{1}{10^n}\{10^n-8^n-5^n+4^n\}.$$

Ex. 7·66. *n letters to each of which corresponds an envelope are placed in the envelopes at random. (i) What is the probability that no letter is placed in the right envelope? (ii) What is the probability that exactly r letters are placed in the right envelopes?*

Sol. Let U_n be the number of ways in which all the letters can go wrong.

There are two mutually exclusive possibilities :

(1) If any two letters occupy each other's position, the remaining $(n-2)$ letters can go wrong in U_{n-2} ways. Since two letters can be interchanged in $(n-1)$ ways, number of ways in which all the letters can go wrong$=(n-1)\ U_{n-2}$.

(2) If one letter occupies another's envelope and not vice-versa which can happen in $(n-1)$ ways, the remaining $(n-1)$ letters can go wrong in U_{n-1} ways.

Therefore, number of ways in which all the letters can go wrong$=(n-1)\ U_{n-1}$.

$$\therefore \qquad U_n=(n-1)\{U_{n-1}+U_{n-2}\}$$

or
$$U_n-nU_{n-1}=-\{U_{n-1}-(n-1)U_{n-2}\}$$

Change n to $n-1$, $n-2$, ...3

$$U_{n-1}-(n-1)U_{n-2}=-\{U_{n-2}-(n-2)U_{n-3}\}$$
$$U_{n-2}-(n-2)U_{n-3}=-\{U_{n-3}-(n-3)U_{n-4}\}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$U_3-3U_2=-\{U_2-2U_1\}$$

Multiplying

$$U_n-nU_{n-1}=(-1)^{n-2}\{U_2-2U_1\}$$

$U_1=$number of ways in which one letter out of one can go wrong

$=0$

and U_2=number of ways in which two letters out of two can go wrong

$=1$

Therefore, $U_n - nU_{n-1} = (-1)^{n-2} = (-1)^n$

or
$$\frac{U_n}{n!} - \frac{U_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!}$$

Change n to $n-1, n-2, \ldots\ldots 2$

$$\frac{U_{n-1}}{(n-1)!} - \frac{U_{n-2}}{(n-2)!} = \frac{(-1)^{n-1}}{(n-1)!}$$

$$\frac{U_{n-2}}{(n-2)!} - \frac{U_{n-3}}{(n-3)!} = \frac{(-1)^{n-2}}{(n-2)!}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{U_2}{2!} - \frac{U_1}{1!} = \frac{(-1)^2}{2!}$$

Adding

$$\frac{U_n}{n!} = \left\{\frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + \frac{(-1)^{n-1}}{n-1!} + \frac{(-1)^n}{n!}\right\}$$

Total number of ways of distributing n letters in n envelopes $= n!$

Therefore, reqd. prob. $= \dfrac{U_n}{n!} = \dfrac{1}{2!} - \dfrac{1}{3!} + \ldots\ldots + \dfrac{(-1)^n}{n!}$

(*ii*) r letters can be chosen out of n in ${}^nc_r$ ways and the rest can go wrong in U_{n-r} ways.

Therefore, reqd. prob. $= \dfrac{{}^nc_r U_{n-r}}{n!}$

$$= \frac{{}^nc_r}{n!}\left[\frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + \frac{(-1)^{n-r}}{(n-r)!}\right](n-r)!$$

$$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + \frac{(-1)^{n-r}}{(n-r)!}\right]$$

Ex. 7·67. *A player tosses a coin and is to score one point for every head turned up and two for every tail. He is to play on until his score reaches or passes n. If p_n is the chance for attaining exactly n, show that*

$$p_n = \frac{1}{2}\,(p_{n-1} + p_{n-2})$$

and hence find the value of p_n.

Sol. Two mutually exclusive possibilities of attaining score exactly n are :

(1) when score is $(n-1)$, player tosses head.

(2) when score is $(n-2)$, player tosses tail.

The possibilities of these two possibilities, by compound probability theorem, are $\frac{1}{2}\, p_{n-1}$ and $\frac{1}{2}\, p_{n-2}$.

Therefore, by total probability theorem,

$$p_n=\frac{1}{2}\, p_{n-1}+\frac{1}{2}\, p_{n-2}$$

$$=\frac{1}{2}\,(p_{n-1}+p_{n-2})$$

or $$2p_n=p_{n-1}+p_{n-2}$$

or $$2(p_n-p_{n-1})=-(p_{n-1}-p_{n-2})$$

or $$(p_n-p_{n-1})=\left(-\frac{1}{2}\right)(p_{n-1}-p_{n-2})$$

Changing n to $n-1$, $n-2$,......3

$$(p_{n-1}-p_{n-2})=\left(-\frac{1}{2}\right)(p_{n-2}-p_{n-3})$$

$$(p_{n-2}-p_{n-3})=\left(-\frac{1}{2}\right)(p_{n-3}-p_{n-4}) \qquad \ldots(1)$$

....

$$(p_3-p_2)=\left(-\frac{1}{2}\right)(p_2-p_1)$$

Multiplying

$$p_n-p_{n-1}=\left(-\frac{1}{2}\right)^{n-2}(p_2-p_1)$$

Score '2' can be attained in following two mutually exclusive ways :

(1) tossing tail in the first trial.

(2) tossing heads in first two trials.

Therefore, $p_2=\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}$

$$=\frac{1}{2}+\frac{1}{4}$$

$$=\frac{3}{4}$$

Also p_1=probability of tossing head in first trial.

$$=\frac{1}{2}$$

Therefore, $\quad p_n-p_{n-1}=\left(-\frac{1}{2}\right)^n$ ...(2)

Adding above equations (1)

$$p_n-p_2=\left(-\frac{1}{2}\right)(p_{n-1}-p_1)$$

or
$$p_n-\frac{3}{4}=-\frac{1}{2}\,p_{n-1}+\frac{1}{4}$$

$$2p_n+p_{n-1}=2 \qquad \text{...(3)}$$

Adding (2) and (3)

$$3p_n=2+\left(-\frac{1}{2}\right)^n$$

Therefore, $\quad p_n=\frac{1}{3}\left\{2+(-1)^n.\frac{1}{2^n}\right\}.$

Ex. 7-68 *Each of the n urns contains a white and b black balls. One ball is transferred from the first urn into the second, then one ball from the latter into the third and so on. Finally one ball is taken from the last urn, what is the probability of its being white ?*

Sol. Let p_k be the probability of drawing a white ball from the kth urn.

There are two possibilities :

(1) A white ball is transferred from $(k-1)$th urn to kth urn.

(2) A black ball is transferred from $(k-1)$th urn to kth urn.

In (1), number of white balls in kth urn$=a+1$.

Therefore, conditional probability of drawing a white ball from kth urn when a white ball is transferred from $(k-1)$th urn to kth urn$=\frac{a+1}{a+b+1}$.

Probability of drawing a white ball from $(k-1)$th urn$=p_{k-1}$.

Therefore, by compound probability theorem, probability of drawing a white ball from kth urn. (If possibility (1) happens)

$$=\frac{a+1}{a+b+1}\,p_{k-1}$$

Similarly probability of drawing a white ball from kth urn.

(If possibility (2) happens)$=\frac{a}{a+b+1}\,(1-p_{k-1})$

Therefore, by theorem of total probability

$$p_k=\frac{a+1}{a+b+1}\,p_{k-1}+\frac{a}{a+b+1}\,(1-p_{k-1})$$

$$=\frac{1}{a+b+1}\,p_{k-1}+\frac{a}{a+b+1}$$

Put $\quad k=2$

$$p_2=\frac{1}{a+b+1}\,p_1+\frac{a}{a+b+1}$$

But p_1=probability of drawing a white ball from first urn.

$$=\frac{a}{a+b}$$

Therefore, $p_2=\dfrac{1}{a+b+1}\cdot\dfrac{a}{a+b}+\dfrac{a}{a+b+1}$

$$=\frac{a}{a+b}$$

Similarly, $p_3=\dfrac{a}{a+b}$ and so on.

In general,

$$p_n=\frac{a}{a+b}.$$

Ex. 7·69. *In a lottery m-tickets are drawn at a time out of the total number of n tickets and returned before the next drawing is made. Show that the probability that in k drawings each of the numbers 1, 2, ...n will appear at least once is given by*

$$p_k=1-{}^nc_1\left(1-\frac{m}{n}\right)^k+{}^nc_2\left(1-\frac{m}{n}\right)^k\left(1-\frac{m}{n-1}\right)^k\ldots\ldots\ldots$$

Sol. Let (A_1), (A_2), ...(A_n) denote the events that the numbers 1, 2, ...n respectively appear at least once in k drawings. Then $(\bar{A}_1)$, $(\bar{A}_2)$, ...$(\bar{A}_n)$ denote the events that the numbers, 1, 2, ...n respectively do not appear in k drawings.

By additive law,

$$P(\bar{A}_1+\bar{A}_2+\ldots\ldots+\bar{A}_n)=\sum_{i=1}^{n}P(\bar{A}_i)-\sum_{\substack{i,\,j=1\\ i<j}}^{n}P(\bar{A}_i\bar{A}_j)+\ldots\ldots$$

Probability of appearance of ith number in one draw

$$=\frac{m}{n}$$

Therefore, probability of non-appearance of ith number in one draw

$$=\left(1-\frac{m}{n}\right)$$

Since tickets are replaced after each draw, draws are independent and hence by compound probability theorem

$P(\bar{A}_i)$=Probability of non-appearance of ith number in k drawings

$$=\left(1-\frac{m}{n}\right)^k$$

Therefore, $$\sum_{i=1}^{n} P(\bar{A}_i)={}^nc_1\left(1-\frac{m}{n}\right)^k.$$

Probability of non-appearance of any two specified numbers in one draw

$$=\frac{{}^{n-2}c_m}{{}^nc_m}$$

$$=\frac{(n-m)(n-m-1)}{n(n-1)}$$

$$=\left(1-\frac{m}{n}\right)\left(1-\frac{m}{n-1}\right)$$

Therefore, $P(\bar{A}_i\bar{A}_j)$=Probability of non-appearance of ith and jth numbers in k drawings

$$=\left(1-\frac{m}{n}\right)^k\left(1-\frac{m}{n-1}\right)^k$$

Therefore, $$\sum_{\substack{i,j=1\\ i<j}}^{n} P(\bar{A}_i\bar{A}_j)={}^nc_2\left(1-\frac{m}{n}\right)^k\left(1-\frac{m}{n-1}\right)^k$$

Similarly,

$$\sum_{\substack{i,j,k=1\\ i<j<k}}^{n} P(\bar{A}_i\bar{A}_j\bar{A}_k)={}^nc_3\left(1-\frac{m}{n}\right)^k\left(1-\frac{m}{n-1}\right)^k\left(1-\frac{m}{n-2}\right)^k$$

and so on.

Therefore, $P(\bar{A}_1+\bar{A}_2+\ldots\ldots+\bar{A}_n)$

$$={}^nc_1\left(1-\frac{m}{n}\right)^k-{}^nc_2\left(1-\frac{m}{n}\right)^k\left(1-\frac{m}{n-1}\right)^k$$

$$+{}^nc_3\left(1-\frac{m}{n}\right)^k\left(1-\frac{m}{n-1}\right)^k\left(1-\frac{m}{n-2}\right)^k-\ldots\ldots$$

Therefore, $P(A_1A_2\ldots A_n)=1-P(\bar{A}_1+\bar{A}_2+\ldots+\bar{A}_n)$

$$=1-{}^nc_1\left(1-\frac{m}{n}\right)^k+{}^nc_2\left(1-\frac{m}{n}\right)^k\left(1-\frac{m}{n-1}\right)^k$$

$$-{}^nc_3\left(1-\frac{m}{n}\right)^k\left(1-\frac{m}{n-1}\right)^k\left(1-\frac{m}{n-2}\right)^k+\ldots\ldots$$

which is the required probability.

Ex. 7·70. *We have k varieties of objects each variety consisting of the same number of objects. These objects are drawn one at a time and replaced before the next drawing. Show that the probability p_n that n and no less drawings will be required to produce objects of all varieties is given by*

$$k^{n-1}p_n=(k-1)^{n-1}-{}^{k-1}c_1(k-2)^{n-1}+{}^{k-1}c_2(k-3)^{n-1}\ldots\ldots\ldots$$

Sol. In the first draw there will be necessarily one variety, therefore the probability of drawing $(k-1)$ varieties in $(n-1)$ drawings is to be obtained. To proceed with firstly the probability of at least one out of $(k-1)$ varieties missing in $(n-1)$ drawings will be obtained.

Let (B_1), (B_2),......(B_{k-1}) denote the events that first, second,... ...$(k-1)$th. varieties are absent respectively. Evidently events (B_1), (B_2),......(B_{k-1}) are non-mutually exclusive.

Therefore by additive law,

$$P(B_1+B_2+\ldots\ldots+B_{k-1})=\sum_{i=1}^{k-1}P(B_i)-\sum_{\substack{i,j=1\\ i<j}}^{k-1}P(B_iB_j)+\ldots\ldots$$

If (B_1) happens, out of k varieties two varieties (first variety and variety which was drawn in first draw) are missing. Probability of not drawing any of the two varieties (mentioned above) in a draw

$$=\left(\frac{k-2}{k}\right)$$

Therefore $P(B_1)$ = Probability that two varieties (mentioned above) are absent in $(n-1)$ draws

$$=\left(\frac{k-2}{k}\right)^{n-1}$$

Similarly $P(B_2)=P(B_3)=\ldots\ldots=P(B_{k-1})=\left(\frac{k-2}{k}\right)^{n-1}$

If (B_1B_2) happens out of k varieties three varieties (first, second and one which was drawn in first draw) are missing.

Therefore as above,

$$P(B_1B_2)=P(B_1B_3)=\ldots\ldots=\left(\frac{k-3}{k}\right)^{n-1}$$

and so on.

$$\therefore\ P(B_1+B_2\ldots+B_{k-1})={}^{k-1}c_1\left(\frac{k-2}{k}\right)^{n-1}-{}^{k-1}c_2\left(\frac{k-3}{k}\right)^{n-1}+\ldots$$

Also probability that in $(n-1)$ draws the variety that has been drawn in first draw is absent$=\left(1-\frac{1}{k}\right)^{n-1}$

$$\therefore\ \text{Reqd. prob.}=\left(1-\frac{1}{k}\right)^{n-1}-P(B_1+B_2+\ldots\ldots+B_{k-1})$$

$$\therefore\ p_n=\left(\frac{k-1}{k}\right)^{n-1}-{}^{k-1}c_1\left(\frac{k-2}{k}\right)^{n-1}+{}^{k-1}c_2\left(\frac{k-3}{k}\right)^{n-1}\ldots\ldots$$

or $\quad k^{n-1}p_n=(k-1)^{n-1}-{}^{k-1}c_1(k-2)^{n-1}+{}^{k-1}c_2(k-3)^{n-1}+\ldots\ldots$

Ex. 7·77. *Two similar decks of n different cards are put into random order and matched against each other. If a card occupies the same position in both the decks we speak of match (coincidence). Find the probability of at least one match.*

Sol. Assume that cards of one deck be in natural order. Let (A_i) be the event that a match occurs at the ith place.

By additive law,

$$P(A_1+A_2+\ldots\ldots+A_n)=\sum_{i=1}^{n}P(A_i)-\sum_{i<j=1}^{n}P(A_iA_j)+\ldots\ldots$$
$$+(-1)^{n-1}P(A_1A_2\ldots A_n)$$

If event (A_i) happens, a match occurs at the ith place, *i.e.* in second deck ith numbered card is at the ith place while the remaining $(n-1)$ cards may be in any order.

Number of ways of distributing $(n-1)$ cards on $(n-1)$ places $=(n-1)\,!$

and total number of ways of distributing n cards on n places

$=n\,!$

$$\therefore \qquad P(A_i)=\frac{(n-1)\,!}{n\,!}=\frac{1}{n}$$

Similarly $P(A_iA_j)=\dfrac{(n-2)\,!}{n\,!}=\dfrac{1}{n(n-1)}$

and so on.

$$\therefore \quad P(A_1+A_2+\ldots\ldots+A_n)={}^nc_1.\frac{1}{n}-{}^nc_2.\frac{1}{n(n-1)}$$

$$+{}^nc_3\frac{1}{n(n-1)(n-2)}+\ldots\ldots+(-1)^{n-1}\frac{1}{n\,!}$$

$$=1-\frac{1}{2\,!}+\frac{1}{3\,!}+\ldots\ldots\frac{(-1)^{n-1}}{n\,!}$$

Ex. 7-72. *What is the probability that at least one of the players in a bridge game will get a complete suit of cards ?*

Sol. Let (A_1), (A_2), (A_3) and (A_4) denote the events that four players respectively get a complete suit of cards,

$P(A_1)$=Probability that player gets a complete suit of cards

$$=\frac{{}^4c_1}{{}^{52}c_{13}}$$

as there are four suits and player is to get one.

Similarly $P(A_2)=P(A_3)=P(A_4)=\dfrac{{}^4c_1}{{}^{52}c_{13}}$

$P(A_iA_j)$=Probability that ith and jth player get complete suit of cards

$$=\frac{{}^4c_1}{{}^{52}c_{13}}\,\frac{{}^3c_1}{{}^{39}c_{13}}$$

$i=1, 2, 3, 4$

$j=1, 2, 3, 4$ and $i \neq j$

Similarly $P(A_iA_jA_k)=\dfrac{{}^4c_1}{{}^{52}c_{13}}\cdot\dfrac{{}^3c_1}{{}^{39}c_{13}}\cdot\dfrac{{}^2c_1}{{}^{26}c_{13}}$ $\quad i, j, k=1, 2, 3, 4$

and $\qquad P(A_1A_2A_3A_4)=\dfrac{{}^4c_1}{{}^{52}c_{13}}\cdot\dfrac{{}^3c_1}{{}^{39}c_{13}}\cdot\dfrac{{}^2c_1}{{}^{26}c_{13}}\cdot\dfrac{1}{{}^{13}c_{13}}$

$$P(A_1+A_2+A_3+A_4)=\sum_{i=1}^{4}P(A_i)-\sum_{i<j=1}^{4}P(A_iA_j)$$

$$+\sum_{i<j<k=1}^{4}P(A_iA_jA_k)-P(A_1A_2A_3A_4)$$

$$={}^4C_1.\frac{{}^4C_1}{{}^{52}C_{13}}-{}^4C_2.\frac{{}^4C_1.{}^3C_1}{{}^{52}C_{13}.{}^{39}C_{13}}+{}^4C_3.\frac{{}^4C_1{}^3C_1.{}^2C_1}{{}^{52}C_{13}.{}^{39}C_{13}.{}^{26}C_{13}}$$

$$-\frac{{}^4C_1.{}^3C_1.{}^2C_1.1}{{}^{52}C_{13}.{}^{39}C_{13}.{}^{26}C_{13}{}^{13}C_{13}}$$

$$=16.\frac{13\,!.\,39\,!}{52}-\frac{6.4.3.(13\,!)^2}{52}26\,!+\frac{4.4.3.2}{52}(13\,!)^4-24\frac{(13\,!)^4}{52\,!}$$

$$=\frac{16\,.\,13!\,39!-72\,(13!)^2\,26\,!+72\,(13!)^4}{52}$$

which is the required probability.

Ex. 7-73. *Show that*

$$P(AB)\leqslant P(A)\leqslant P(A+B)\leqslant P(A)+P(B)$$

Sol. By compound prob. theorem,

$$P(AB)=P(A)P(B/A)$$

Since $\quad P(B/A)\leqslant 1,\ P(AB)\leqslant P(A)$ ...(1)

Since $A\bar{B}$, $\bar{A}B$, AB are mutually exclusive forms in which an event $(A+B)$ can happen, by total prob. theorem,

$$P(A+B)=P(A\bar{B})+P(\bar{A}B)+P(AB)$$

Similarly $\quad P(A)=P(A\bar{B})+P(AB)$

$\therefore \quad P(A+B)=P(A)+P(\bar{A}B)$

Since $\quad P(\bar{A}B)\geqslant 0,\ P(A)\leqslant P(A+B)$ ...(2)

Also $\quad P(A+B)=P(A)+P(B)-P(AB)$

Since $\quad P(AB)\geqslant 0,\ P(A+B)\leqslant P(A)+P(B)$ .. (3)

Result follows from (1), (2) and (3).

Ex. 7-74. *State and prove Baye's theorem.*

Sol. Statement. If an event E can only occur in combination with one of the mutually exclusive events $E_1, E_2, \ldots\ldots E_n$, then

$$P(E_k/E)=\frac{P(E_k)P(E/E_k)}{\sum_{i=1}^{n}P(E_i)P(E/E_i)} \qquad k=1, 2, \ldots n$$

Proof. Since the event E can occur only with the events $E_1, E_2, \ldots\ldots E_n$, the possible forms in which E can occur are

$$EE_1,\ EE_2, \ldots\ldots EE_n.$$

These forms are mutually exclusive as the events E_i are mutually exclusive.

$\therefore$ By total prob. theorem,

$$P(E)=P(EE_1)+P(EE_2)+\dots\dots\dots+P(EE_n)$$

$$=\sum_{i=1}^{n} P(EE_i)=\sum_{i=1}^{n} P(E_i)P(E/E_i)$$

(using compound prob. theorem)

Now by compound prob. theorem

$$P(EE_k)=P(E)P(E_k/E)=P(E_k)P(E/E_k)$$

$$\therefore \qquad P(E_k/E)=\frac{P(E_k)P(E/E_k)}{P(E)}$$

$$=\frac{P(E_k)P(E/E_k)}{\sum_{i=1}^{n} P(E_i)P(E/E_i)}$$

Note. The probabilities $P(E_k)$ and $P(E_k/E)$ are known as 'priori' and 'posteriori' probabilities.

Thus Baye's theorem can be stated as :

'If an event E can occur only in combination with the mutually exclusive events E_1, E_2,......E_n and if

(i) the priori probabilities $P(E_1)$, $P(E_2)$,......$P(E_n)$ corresponding to the total absence of knowledge regarding the occurrence of A

and (ii) the conditional probabilities

$$P(E/E_1),\ P(E/E_2)\dots\dots\ P(E/E_n)$$

are given, the posteriori probabilities

$$P(E_1/E),\ P(E_2/E),\ \dots\dots P(E_n/E)$$

are given by

$$P(E_k/E)=\frac{P(E_k)P(E/E_k)}{\sum_{i=1}^{n} P(E_i)P(E/E_i)} \qquad k=1, 2, \dots n\text{'}$$

Ex. 7·75. *In a bolt factory machines A, B, C manufacture respectively 25, 35 and 40 percent of the total. Out of their out put 5, 4 and 2 percent are defective bolts. A bolt is drawn from the produce and is found defective. What are the probabilities that it was manufactured by A, B and C ?*

Sol. Let E be the event that the bolt is defective and E_1, E_2, E_3 the events that the bolt is being produced by A, B, C respectively.

Then $P(E_1)=0{\cdot}25$, $P(E_2)=0{\cdot}35$, $P(E_3)=0{\cdot}40$, $P(E/E_1)=0{\cdot}05$, $P(E/E_2)=0{\cdot}04$ and $P(E/E_3)=0{\cdot}02$.

It is required to find $P(E_1/E)$, $P(E_2/E)$ and $P(E_3/E)$

By Baye's theorem

$$P(E_1/E)=\frac{P(E_1)\,P(E/E_1)}{P(E_1)\,P(E/E_1)+P(E_2)\,P(E/E_2)+P(E_3)\,P(E/E_3)}$$

$$=\frac{(0{\cdot}25)(0{\cdot}05)}{(0{\cdot}25)(0{\cdot}05)+(0{\cdot}35)(0{\cdot}04)+(0{\cdot}4)(0{\cdot}02)}$$

$$=\frac{125}{345}$$

Similarly $\quad P(E_2/E)=\dfrac{140}{345}$

and $\quad P(E_3/E)=\dfrac{80}{345}.$

Ex. 7-76. *Prove that*

$$P(A_1+A_2+\ldots+A_n)\leqslant P(A_1)+P(A_2)+\ldots\ldots+P(A_n)$$

Proof. We have *(Bool's inequality)*

$$P(A_1+A_2)=P(A_1)+P(A_2)-P(A_1A_2)$$

$$\therefore \quad P(A_1+A_2)\leqslant P(A_1)+P(A_2)$$

$$\therefore \quad P(A_1+A_2+A_3)=P(A_1+\overline{A_2+A_3})$$
$$\leqslant P(A_1)+P(A_2+A_3)$$
$$\leqslant P(A_1)+P(A_2)+P(A_3)$$

Let $\quad P(A_1+A_2+\ldots\ldots+A_m)\leqslant P(A_1)+P(A_2)+\ldots\ldots+P(A_m)$

Then $P(A_1+A_2+\ldots\ldots+A_{m+1})=P(A_1+\overline{A_2+\ldots\ldots+A_{m+1}})$
$$\leqslant P(A_1)+P(A_2+A_3+\ldots\ldots A_{m+1})$$
$$\leqslant P(A_1)+\{P(A_2)+P(A_3)+\ldots\ldots+P(A_{m+1})\}$$
$$=P(A_1)+P(A_2)+\ldots\ldots+P(A_{m+1})$$

$\therefore$ By induction result follows :

EXERCISES

1. If three squares are chosen at random on a chess board, show that the chance that they should be in a diagonal line is 7/744.
2. Three squares of a chess board being chosen at random, what is the chance that two are of one colour and one of another ?

$$\left[\textbf{Ans. } \frac{16}{21}\right]$$

3. A person writes 4 letters and 4 envelopes. If the letters are placed in the envelopes at random, what is the chance that not more than one letter is placed in the correct envelope ?

$$\left[\textbf{Ans. } \frac{17}{24}\right]$$

4. Four right-foot shoes are paired at random with the corresponding set of the left-foot shoes. Find the prob. that no correct pair is obtained. $\left[\textbf{Ans. } \frac{3}{8}\right]$

5. From a pack of 52 cards, three are drawn at random. Find the chance that these are a king, a queen and a knave.

$$\left[\textbf{Ans. } \frac{16}{5525}\right]$$

6. If two balls are drawn from a bag containing 2 white, 4 red and 5 black balls. What is the chance that (*i*) both the balls are red (*ii*) one is red and the other black ? $\left[\textbf{Ans. } \frac{6}{55}\,;\ \frac{4}{11}\right]$

7. Find the chance of throwing a sum of 9 in a single throw of two dice. $\left[\textbf{Ans. } \frac{1}{9}\right]$

8. Find the prob. of obtaining a total of 6 in a throw of 6 dice.

$$\left[\textbf{Ans. } \frac{1}{6^6}\right]$$

9. In a single throw of three dice, what is the chance of throwing (*i*) 'four-five-six', (*ii*) less than 11, (*iii*) more than 10 ?

$$\left[\textbf{Ans. } \frac{1}{36}\,;\ \frac{1}{2}\,;\ \frac{1}{2}\right]$$

10. A and B throw with 3 dice ; if *A* throws 8, what is *B*'s chance of throwing a higher number ? $\left[\textbf{Ans. } \frac{20}{27}\right]$

11. Find the chance of throwing 10 exactly in one throw with 3 dice. $\left[\textbf{Ans. } \frac{1}{8}\right]$

12. Find the chance of throwing (*i*) 18, (*ii*) 10 exactly in one throw of 4 dice. $\left[\textbf{Ant. } \frac{5}{81}\,;\ \frac{5}{81}\right]$

13. A person throws two dice, one the common cube and the other a regular tetrahedron, the number on the lowest face being

taken in the case of the tetrahedron ; what is the chance that the sum of the numbers thrown is not less than 5 ?

$\left[\text{Ans. } \frac{3}{4}\right]$

14. There are 10 tickets, 5 of which are blanks and the others are marked with the numbers 1, 2, 3, 4, 5. What is the prob. of drawing 10 in three trials, (*i*) when the tickets are replaced at every trial, (*ii*) if the tickets are not replaced ?

$\left[\text{Ans. } \frac{33}{1000}, \frac{1}{60}\right]$

15. Out of 20 consecutive numbers two are chosen at random, find the probability that their sum is odd.

$\left[\text{Ans. } \frac{10}{19}\right]$

16. A bag contains 50 tickets numbered 1, 2, ... 50 of which 5 are drawn at random and arranged in ascending order of their numbers. $(x_1 < x_2 < x_3 < x_4 < x_5)$ What is the probability that $x_3 = 30$?

$\left[\text{Ans. } \frac{{}^{29}C_2 \times {}^{20}C_2}{{}^{50}C_5}\right]$

17. Nine cards are drawn at random from a set of cards. Each card is marked with one of the numbers '1', '0' or '-1' and it is equally likely that any of the three numbers will be drawn. Find the chance that the sum of the numbers drawn is zero.

$\left[\text{Ans. } \frac{3139}{3^9}\right]$

18. A party of 21 persons take their seats at a round table. What are the odds in favour of two specified persons sitting together ?

[Ans. '1 : 9']

19. A number is chosen from each of two sets :

1, 2, 3, 4, 5, 6, 7, 8, 9 ; 1, 2, 3, 4, 5, 6, 7, 8, 9.

If p_1 denotes the probability that the sum of the numbers be 10 and p_2 the probability that their sum be 8, find $p_1 + p_2$.

$\left[\text{Ans. } \frac{16}{81}\right]$

20. A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning his bet ?

[Ans. '9 : 4']

21. Find the probability that in a random arrangement of the letters of the word 'UNIVERSITY' the two 'I's' don't come together.

22. A party of 'n' men of whom 'A', 'B' are two, form single rank. What is the chance that (*i*) A, B are next one another (*ii*) exactly 'm' men are between them (*iii*) not more than 'm' men are between them ?

$$\left[\textbf{Ans.}\quad \frac{2}{n}\ ;\ \frac{2(n-m-1)}{n(n-1)}\ ;\ \frac{(m+1)(2n-m-2)}{n(n-1)}\right]$$

23. If the letters of 'ATTEMPT' are written down at random, find the chance that (*i*) all the 'T's' are together (*ii*) no two 'T's' are together. $\left[\textbf{Ans.}\quad \frac{1}{7}\ ;\ \frac{2}{7}\right]$

24. Find the number of ways in which 'p' plus signs' and 'q minus signs' may be placed in a row so that no two minus signs are together.

25. A letter is chosen at random out of 'ASSININE' and one is chosen at random out of 'ASSASSIN'. Show that the chance that the same letter is chosen on both accasions is $\frac{1}{4}$.

26. Six cards are drawn at random from a pack of 52 cards. What is the probability that 3 will be red and 3 black ?

$$\left[\textbf{Ans.}\quad \frac{13000}{39151}\right]$$

27. A bag contains 6 white and 9 black balls. The drawings of 4 balls are made such that (*a*) the balls are replaced before the second draw (*b*) the balls are not replaced before the second draw. Find the probability that the first drawing will give 4 white and the second 4 black balls in each case.

$$\left[\textbf{Ans.}\quad \frac{6}{5915}\ ;\ \frac{3}{715}\right]$$

28. Obtain the probability that the birth-days of seven people will fall on seven different days of the week, assuming equal probability for the seven days. $\left[\textbf{Ans.}\quad \frac{6\,!}{7^6}\right]$

29. Two drawings each of 3 balls are made from a bag containing 5 white and 8 black balls, the balls being replaced before the second trial. Find the chance that the first drawing will give 3 white and the second 3 black balls. $\left[\textbf{Ans.}\quad \frac{140}{20449}\right]$

30. A and B draw from a bag containing 3 white and 4 black balls. Find their respective chances of first drawing a white ball (the balls when drawn not being replaced). $\left[\textbf{Ans.}\quad \frac{22}{35}\cdot\frac{13}{35}\right]$

31. A is one of 6 horses entered for a race and is to be ridden by one of two jockeys B and C. It is 2 : 1 that B rides A, in which case all the horses are equally likely to win ; if C rides A, his chance is trebled. What are the odds against his winning ? [**Ans.** '13 : 5']

32. A person draws a card from a pack, replaces it, and shuffles the pack. He continues doing so until he draws a 'club'. What is the chance that he will have to make (i) at least three trials (ii) exactly three trials ? $\left[\text{Ans. } \frac{9}{16} ; \frac{9}{64}\right]$

33. Three urns contain respectively 1 white and 2 black balls ; 3 white and 1 black balls and 2 white and 3 black balls. One ball is taken at random from each urn. Find the prob. that among the balls drawn there are 2 white and 1 black ball.

34. A bag contains 17 counters marked with the numbers 1 to 17. A counter is drawn and replaced ; a second drawing is then made, find the chance that the first number drawn is even and the second odd. $\left[\text{Ans. } \frac{72}{289}\right]$

35. Three urns respectively contain 1 white and 3 black, 2 white and 4 black and 3 white and 1 black balls. A ball is drawn from an urn selected at random, find the chance of its being white. $\left[\text{Ans. } \frac{4}{9}\right]$

36. Criticise the statement : 'The chance of throwing ace in the first trial is $\frac{1}{6}$ and the chance of ace in the second trial is $\frac{1}{6}$, therefore the chance of ace in two trials is $\frac{1}{3}$'.

37. Counters marked 1, 2, 3 are placed in a bag and one is withdrawn and replaced. The operation being repeated three time, what is the chance of obtaining a total of 6 ? $\left[\text{Ans. } \frac{7}{27}\right]$

38. A, B, C in order cut a pack of cards, replacing them after each cut, on the condition that the first who cuts a spade shall win a prize. Find their respective chances. $\left[\text{Ans. } \frac{16}{37} ; \frac{12}{37} ; \frac{9}{37}\right]$

39. Six persons throw for a stake, which is to be won by the one who first throws head with a coin. If they throw in succession, find the chance of the fourth person. $\left[\text{Ans. } \frac{4}{63}\right]$

40. A and B play for a prize ; A is to throw a die first and is to win if he throws 6. If he fails B is to throw and to win if he throws 6 or 5. If he fails, A is to throw again and to win with 6 or 5 or 4, and so on. Find the chance of each player.

$$\left[\textbf{Ans.}\quad A, \frac{169}{324}; B, \frac{155}{324}\right]$$

41. A certain stake is to be won by the first person who throws an ace with an octahedral die. If there are 4 persons, what is the chance of the last ? $\left[\textbf{Ans.}\ \frac{343}{1695}\right]$

42. A, B, C, D cut a pack of cards successively in the order mentioned. What are their respective chances of first cutting a spade ? $\left[\textbf{Ans.}\quad \frac{64}{175}; \frac{48}{175}; \frac{36}{175}; \frac{27}{175}\right]$

43. Five persons A, B, C, D, E throw a die in the order named until one of them throws an ace ; find their relative chances of winning, supposing the throws to continue till an ace appears.

$$\left[\textbf{Ans.}\quad 1 : \frac{5}{6} : \left(\frac{5}{6}\right)^2 : \left(\frac{5}{6}\right)^3 : \left(\frac{5}{6}\right)^4\right]$$

44. How many throws with a pair of dice are necessary in order to have the chance of getting 'double six' at least once greater than $\frac{1}{2}$? [**Ans.** 25]

45. How many throws with a single die are necessary in order to have the chance of getting an ace at least once greater than $\frac{1}{2}$? [**Ans.** 4]

46. If the prob of success be 0·01, how many trails are necessary in order that prob of at least one success is greater than $\frac{1}{2}$. [**Ans.** 69]

47. The odds against a certain event are 5 : 2 and the odds in favour of another event, independent of the former are 6 : 5. Find the chance that one at least of the events will happen.

$$\left[\textbf{Ans.}\quad \frac{52}{77}\right]$$

48. The odds that a book will be favourably reviewed by three independent critics are 5 : 2, 4 : 3 and 3 : 4 respectively. What is the probability that of the three reviews a majority will be favourable ? $\left[\textbf{Ans.}\ \frac{209}{343}\right]$

49. A throws two coins and B throws three coins. Find the chance that B will throw a greater number of heads than A.

$$\left[\textbf{Ans.}\quad \frac{1}{2}\right]$$

50. There are three works, one consisting of '3' volumes, one of '4' and the other of '1' volume. They are placed on a self at random ; prove that the chance that volumes of the same works are all together is $\frac{3}{140}$.

51. There are two bags, one of which contains 5 red and 7 white balls and the other 3 red and 12 white balls. One ball is to be drawn from one or other of the two bags. Find the chance of drawing a red ball. $\left[\textbf{Ans.}\quad \frac{37}{120}\right]$

52. Four students are selected at random from 7 boys and 4 girls. Calculate the probabilities that the selected students include (*i*) two specified boys (*ii*) exactly two boys (*iii*) at least two boys.

53. A bag contains 4 white, 5 red and 6 black balls. Three are drawn at random. Find the prob that

(*a*) no ball drawn is black.

(*b*) exactly two are black.

(*c*) all are of the same colour

54. A has '3' shares in a lottery in which there are '3' prizes and '6' blanks ; B has '1' share in a lottery in which there is '1' prize and '2' blanks. Show that A's chance of success is to B's as 16 : 7.

55. If p is the prob that a man aged x years will die in a year. Find the prob. that out of 'm' men $A_1, A_2, \ldots\ldots A_m$, each aged x, A_1 will die in a year and be the first to die.

$$\left[\textbf{Ans.}\quad \frac{1}{m}\left\{1-(1-p)^m\right\}\right]$$

56. It is 8 · 5 against a person who is now 40 years old living till he is 70 and 4 · 3 against a person now 50 living till he is 80. Find the prob. that at least one of these persons will be alive 30 years hence. $\left[\textbf{Ans.}\quad \frac{59}{91}\right]$

57. The prob that a 50 years old man will be alive at 60 is 0·83 and the prob. that a 45 years old woman will be alive at 55 is 0·87. What is the prob that a man who is 50 and his wife who is 45 will both be alive 10 years hence? [**Ans.** 0·7221]

58. Suppose that it is 9 : 7 against a person A who is now 35 years of age living till he is 65 and 3 : 2 against a person B now 45 living till he is 75 ; find the chance that one at least of these persons will be alive 30 years hence. $\left[\textbf{Ans. } \frac{53}{80}\right]$

59. A number consists of 7 digits whose sum is 59 ; prove that the chance of its being divisible by 11 is $\frac{4}{21}$.

60. If two coins are tossed 5 times, what is the chance that there will be 5 heads and 5 tails. $\left[\textbf{Ans. } \frac{63}{256}\right]$

61. Find the chance of obtaining at least one six in a throw of four dice.

62. Show that the chance of throwing at least one ace in a single throw with two dice is $\frac{11}{36}$.

63. What is the probability of getting 9 cards of the same suit in one hand at a game of bridge ? $\left[\textbf{Ans. } \frac{{}^{13}C_9 \cdot {}^{39}C_4 \cdot {}^4C_1}{{}^{52}C_{13}}\right]$

64. In three throws with a pair of dice, find the chance of throwing doublets at least once. $\left[\textbf{Ans. } \frac{91}{216}\right]$

65. One bag contains 3 white balls and 2 black balls, another contains 5 white and 3 black balls. If a bag is chosen at random and a ball is drawn from it, what is the chance that it is white ? $\left[\textbf{Ans. } \frac{49}{80}\right]$

66. Four cards are drawn without replacement. What is the prob. that these are all aces ? $\left[\textbf{Ans. } \frac{1}{270725}\right]$

67. Find prob. in Ex. 29 if the balls are not replaced before the second trial. $\left[\textbf{Ans. } \frac{7}{429}\right]$

68. If the war breaks out on the average once in 25 years, find the prob. that in 50 years at a stretch there will be no war

[**Ans.** e^{-2}]

69. Three newspapers A, B, C are published in a certain city It is estimated from a survey that of the adult population : 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 40% read both B and C, 2% read all three. (i) What percentage read at least one of the papers ? (ii) of

those that read at least one, what percentage read both A and B? [**Ans.** 35%, 28%]

70. There are three boxes containing respectively 1 white, 2 red, 3 black balls; 2 white, 3 red, 1 black ball; 3 white, 1 red, 2 black balls. A box is chosen at random and from it two balls are drawn at random. The two balls are one red and one white. What is the prob. that they come from the (*i*) 1st box (*ii*) 2nd box (*iii*) 3rd box? $\left[\textbf{Ans.}\ \frac{2}{11}, \frac{6}{11}, \frac{3}{11}\right]$

71. The probability that a person can hit a target is $\frac{3}{5}$ and the prob. that another person can hit the same target is $\frac{2}{5}$. But the first person can fire 4 shots in the time the second person fires 5 shots. They fire together. What is the prob. that the second person shoots the target? $\left[\textbf{Ans.}\ \frac{5}{11}\right]$

8

Mathematical Expectation

8·1. Stochastic Variate. *The variate which can take certain values depending on chance is called* **chance variate** *or* **stochastic variate** *or* **random variate** *e.g. In rolling a die the variate corresponding to the number obtained is a stochastic variate.*

Probability Distribution. *The dist obtained by taking the possible values of a chance variate together with their respective probabilities is called prob. dist.*

Expected value of a chance variate.

Let x be the chance variate with prob dist

$$\begin{matrix} x \rightarrow \\ p \rightarrow \end{matrix} \begin{pmatrix} x_1 \ x_2 \ldots\ldots\ldots x_n \\ p_1 \ p_2 \ldots\ldots\ldots p_n \end{pmatrix}$$

Then expected value of x is defined to be $x_1p_1+x_2p_2+\ldots+x_np_n$ and is denoted by $E(x)$. Thus $E(x)=x_1p_1+x_2p_2+\ldots\ldots+x_np_n$.

Ex. 8·1. *Let x denote the profit that a man makes in a business. He may earn 2,800 with probability 0·5 ; he may lose Rs. 5,500 with probability 0.3 and he may neither earn nor lose with probability 0·2. Calculate the mathematical expectation of x.*

Sol. The given prob dist is

$$\begin{matrix} x \rightarrow \\ p \rightarrow \end{matrix} \begin{pmatrix} -5500 & 0 & 2800 \\ 0{\cdot}3 & 0{\cdot}2 & 0{\cdot}5 \end{pmatrix}$$

$$\therefore \quad E(x)=(-5500)(0{\cdot}3)+(2800)(0{\cdot}5)$$
$$=-1650+1400=-250.$$

Ex. 8.2. *Find the expected value of the number of points that will be obtained in a single throw with an ordinary die.*

Sol. Let x be the number of points obtained in a single throw with an ordinary dice. Then x can take values 1, 2, 3, 4, 5, 6.

Also prob. of getting any number with a single die

$$=\frac{1}{6}$$

Therefore, expected value of x

$$=\frac{1}{6}\{1+2+3+4+5+6\}$$

$$=\frac{21}{6}$$

$$=\frac{7}{2}.$$

Ex. 8·3. *From a bag containing 2 '20 P.' coins and 3 '25 P.' coins, a person is allowed to draw 2 coins indiscriminately. Find the value of his expectation.*

Sol. Prob. of drawing 2 '20 P.' coins

$$=\frac{{}^2c_2}{{}^5c_2}=\frac{1}{10}$$

Prob. of drawing 1 '20 P.' coin and 1 '25 P.' coin

$$=\frac{{}^2c_1 . {}^3c_1}{{}^5c_2}=\frac{3}{5}$$

Prob of drawing 2 '25 P.' coins

$$=\frac{{}^3c_2}{{}^5c_2}=\frac{3}{10}.$$

The person gets 40 P., 45 P. and 50 P. in three cases respectively.

$\therefore$ Expectation of the person$=40.\frac{1}{10}+45.\frac{3}{5}+50.\frac{3}{10}$

$$=4+27+15=46 \text{ P.}$$

Ex. 8·4. *A person draws 2 balls from a bag containing 3 white and 4 red balls. If he is to receive 10 np. for every white ball which he draws and 20 P. for each red ball. Find his expectation.*

Sol. Three different possibilities are :

(*i*) The person draws 2 white balls. In this case he gets 20 P. and the prob. of this happening

$$=\frac{{}^3c_2}{{}^7c_2}=\frac{3}{21}$$

(*ii*) The person draws 1 white and 1 red balls. In this case he gets 30 P. and the prob. of this happening

$$=\frac{{}^3c_1 \times {}^4c_1}{{}^7c_2}=\frac{12}{21}.$$

(*iii*) The person draws 2 red balls. In this case he gets 40 P. and the prob. of this happening

$$=\frac{{}^4c_2}{{}^7c_2}=\frac{6}{21}$$

$$\therefore \qquad \text{Expectation}=\frac{1}{21}\ \{20.3+30.12+40.6\}$$

$$=\frac{1}{21}\ \{60+360+240\}$$

$$=\frac{1}{21}\ (660)$$

$$=\frac{220}{7}=31\ \frac{3}{7}\text{ P.} \simeq 31\text{ P.}$$

Ex. 8-5. *Three urns contain respectively 3 green and 2 white balls, 5 green and 6 white balls and 2 green and 4 white balls. One ball is drawn from each urn. Find the expected number of white balls drawn out.*

Sol. Let x be the number of white balls drawn. Then possible values of x are 0, 1, 2 and 3.

Let p_0, p_1, p_2 and p_3 be the probabilities of x taking these value respectively.

Now p_0=prob of drawing all the three green balls.

$$=\frac{3}{5}\cdot\frac{5}{11}\cdot\frac{2}{6}=\frac{1}{11}.$$

Different possibilities of drawing 1 white and 2 green balls are :

1st urn	2nd urn	3rd urn
W	*G*	*G*
G	*W*	*G*
G	*G*	*W*

where '*G*' denotes the green ball and '*W*' the white ball.

$$\therefore\ p_1=\frac{2}{5}\cdot\frac{5}{11}\cdot\frac{2}{6}+\frac{3}{5}\cdot\frac{6}{11}\cdot\frac{2}{6}+\frac{3}{5}\cdot\frac{5}{11}\cdot\frac{4}{6}=\frac{58}{165}$$

$$\text{Similarly}\quad p_2=\frac{3}{5}\cdot\frac{6}{11}\cdot\frac{4}{6}+\frac{2}{5}\cdot\frac{5}{11}\cdot\frac{4}{6}+\frac{2}{5}\cdot\frac{6}{11}\cdot\frac{2}{6}$$

$$=\frac{68}{165}$$

$$p_3=\frac{2}{5}\cdot\frac{6}{11}\cdot\frac{4}{6}=\frac{8}{55}$$

$$\therefore \qquad E(x)=0.\ \frac{1}{11}+1.\frac{58}{165}+2.\frac{68}{165}+3.\frac{8}{55}=\frac{266}{165}.$$

Ex. 8·6. *What is the expectation of the number of failures preceding the first success in an indefinite series of independent trials with constant probability p of success?*

Sol. Let x be the number of failures preceding the first success. Then x can take values

$$0, 1, 2 \ldots\ldots\ldots n \ldots\ldots$$

with respective probabilities

$$p, qp, q^2p, \ldots\ldots q^n p \ldots\ldots\ldots$$

$$\therefore \quad E(x) = 0.p + 1.qp + 2.q^2p + \ldots\ldots + n.q^np + \ldots\ldots$$
$$= pq(1 + 2q + \ldots\ldots + nq^{n-1} + \ldots\ldots)$$
$$= pq(1-q)^{-2}$$
$$= \frac{pq}{p^2} = q/p.$$

Ex. 8·7. *A makes a bet with B of Rs. 5 to Rs. 2 that in a single throw with two dice he will throw 7 before B throws 4. Each has a pair of dice and they throw simultaneously until one of them wins, equal throws being disregarded. Find B's expectation.*

Sol. Prob. of getting 7 in a single throw with two dice

$$= \frac{1}{6}$$

and prob. of getting 4 $= \frac{1}{12}$.

Since total prob. is unity, prob. of throwing neither 7 nor 4

$$= 1 - \frac{1}{6} - \frac{1}{12} = \frac{3}{4}.$$

Now A wins if he throws 7 but B does not throw 7 or 4 and B wins if throws 4 but A does not throw 7 or 4.

$\therefore$ Prob. of A winning in first trial

$$= \frac{1}{6} \cdot \frac{3}{4} = \frac{1}{8}$$

and prob. of B winning in first trial

$$= \frac{1}{12} \cdot \frac{3}{4} = \frac{1}{16}$$

$\therefore$ Prob. of none winning in the first trial

$$= 1 - \frac{1}{8} - \frac{1}{16} = \frac{13}{16}$$

Now A wins in second trial if in first trial none wins and he throws 7 but B throws neither 7 nor 4 in second trial.

$\therefore$ Prob. of A winning in second trial

$$=\frac{13}{16}\cdot\frac{1}{6}\cdot\frac{3}{4}=\frac{13}{16}\cdot\frac{1}{8}$$

Similarly prob. of B winning in second trial

$$=\frac{13}{16}\cdot\frac{1}{12}\cdot\frac{3}{4}=\frac{13}{16}\cdot\frac{1}{16}$$

prob. of A winning in third throw

$$=\left(\frac{13}{16}\right)^2\cdot\frac{1}{8}$$

prob. of B winning in third throw

$$=\left(\frac{13}{16}\right)^2\cdot\frac{1}{16}$$

and so on.

$\therefore$ A's chance of winning

$$=\frac{1}{8}\left\{1+\frac{13}{16}+\left(\frac{13}{16}\right)^2+\ldots\ldots\right\}$$

$$=\frac{1}{8}\cdot\frac{1}{1-\frac{13}{16}}=\frac{2}{3}$$

and B's chance of winning

$$=\frac{1}{16}\left\{1+\frac{13}{16}+\left(\frac{13}{16}\right)^2+\ldots\ldots\right\}$$

$$=\frac{1}{16}\cdot\frac{1}{1-\frac{13}{16}}=\frac{1}{3}$$

Now A gets Rs. $2.\frac{2}{3}=\frac{4}{3}$ if he wins and pays Rs. $5.\frac{1}{3}=\frac{5}{3}$ if he loses.

$\therefore$ B's expectation$=\frac{5}{3}-\frac{4}{3}=$Rs. $\frac{1}{3}$.

Ex. 8-8. *A coin is tossed until a head appears. What is the expectation of the number of tosses.*

Sol. Prob. of getting a head in a toss$=\frac{1}{2}=$Prob. of getting a tail in a toss.

Let x be the number of tosses until a head appears. Then x can take values

$$1, 2, 3, \ldots\ldots\ldots\ldots$$

When x takes value 1, head appears in very first trial and the prob. for this is $\frac{1}{2}$. When x takes value 2, first trial results in tail and second in head. So by compound prob. theorem, prob. that x takes value $2=\left(\frac{1}{2}\right)^2$. Similarly prob. that x takes value 3 $=\left(\frac{1}{2}\right)^3$ and so on.

Therefore expected value of x

$$=\frac{1}{2}.1+\left(\frac{1}{2}\right)^2.2+\left(\frac{1}{2}\right)^3.3+\ldots\ldots$$

$$=\frac{1}{2}\left\{1+2.\frac{1}{2}+3.\left(\frac{1}{2}\right)^2+\ldots\ldots\right\}$$

$$=\frac{1}{2}.\left(1-\frac{1}{2}\right)^{-2}$$

$$=\frac{\frac{1}{2}}{\frac{1}{4}}=2.$$

Ex. 8-9. *A and B throw with one die for a prize of Rs. 11 which is to be won by the player who first throws 6. If A has the first throw, what are their respective expectations?*

Sol. A can win in 1st, 3rd, 5th, ... trials with respective chances $\frac{1}{6}, \left(\frac{5}{6}\right)^2\frac{1}{6}, \left(\frac{5}{6}\right)^4\frac{1}{6}, \ldots\ldots$

$\therefore$ A's chance of success

$$=\frac{1}{6}\left\{1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\ldots\ldots\right\}$$

$$=\frac{1}{6}\left\{\frac{1}{1-\frac{25}{36}}\right\}=\frac{6}{11}$$

Since there are only two players, and total probability is unity. B's chance of success

$$=1-\frac{6}{11}=\frac{5}{11}$$

$\therefore$ A's expectation $=\frac{6}{11}\times 11=$ Rs. 6

and B's expectation $=\frac{5}{11}\times 11=$ Rs. 5

8.2. Laws of Expectation. *Two basic laws of expectation are*

(1) $E(x+y)=E(x)+E(y)$

(2) $E(xy)=E(x)E(y)$

provided x and y are independent.

Proof. 1. Let x and y be two stochastic variates with probability distributions

$$\begin{matrix} x\to \\ p\to \end{matrix}\begin{pmatrix} x_1 & x_2 & \ldots\ldots x_m \\ p_1 & p_2 & \ldots\ldots p_m \end{pmatrix} \text{ and } \begin{matrix} y\to \\ p\to \end{matrix}\begin{pmatrix} y_1 & y_2 & \ldots\ldots y_n \\ P_1 & P_2 & \ldots\ldots P_n \end{pmatrix}$$

Let $$z=x+y$$

Then z will also be a stochastic variate.

Let $$z_{ij}=x_i+y_j$$

Let p_{ij} be the probability of z taking a value z_{ij}.

Let A_i be the event that x takes the value x_i and A_{ij} be the event that z takes the value z_{ij}.

Then $(A_i)=(A_{i1}+A_{i2}+\ldots\ldots+A_{in})$

$\therefore\ P(A_i)=P(A_{i1}+A_{i2}+\ldots\ldots+A_{in})$

Since z can take only one value at a time the events $A_{i1}, A_{i2},\ldots A_{in}$ are mutually exclusive.

$\therefore$ By total probability theorem,

$$P(A_i)=P(A_{i1})+P(A_{i2})+\ldots\ldots+P(A_{in})$$

i.e., $$p_i=p_{i1}+p_{i2}+\ldots\ldots+p_{in}.$$
$$i=1, 2, \ldots\ldots m$$

Similarly $\quad P_j=P_{1j}+p_{2j}+\ldots\ldots+p_{mj} \qquad j=1, 2,\ldots\ldots n$

Now $E(x+y)=E(z)=\sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}\,p_{ij}$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i+y_j)p_{ij}=\sum_{i=1}^{m}\sum_{j=1}^{n}x_ip_{ij}+\sum_{i=1}^{m}\sum_{j=1}^{n}y_j\,p_{ij}$$

$$=\sum_{i=1}^{m}x_i\{p_{i1}+p_{i2}+\ldots+p_{in}\}+\sum_{j=1}^{n}y_j\{p_{1j}+p_{2j}+\ldots\ldots\ldots+p_{mj}\}$$

$$=\sum_{i=1}^{m}x_ip_i+\sum_{j=1}^{n}y_jP_j$$

$$=E(x)+E(y).$$

(2) **Def.** Two stochastic variates are said to be independent if the probability of either taking a particular value does not depend on what value the other variate takes.

Let x and y be two stochastic variates with probability distributions.

$$x \rightarrow \begin{pmatrix} x_1 \; x_2 \ldots\ldots x_m \\ p_1 \; p_2 \ldots\ldots p_m \end{pmatrix} \text{ and } y \rightarrow \begin{pmatrix} y_1 \; y_2 \ldots\ldots y_n \\ P_1 \; P_2 \ldots\ldots P_n \end{pmatrix}$$

Let $z = xy$

Then z will also be a stochastic variate.

Let $z_{ij} = x_i y_j$.

Let p_{ij} be the probability of z taking a value z_{ij}. Then since x and y are independent, by compound probability theorem

$$p_{ij} = p_i P_j$$

$$\therefore \quad E(xy) = E(z) = \sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} p_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i y_j p_i P_j)$$

$$= \left(\sum_{i=1}^{m} x_i p_i\right)\left(\sum_{j=1}^{n} y_j P_j\right)$$

$$= E(x) E(y).$$

Ex. 8-10. *Find expected value of the product of points obtained on rolling n dice together.*

Sol. Let x_i be the number of points obtained on ith die. Then product of points on n dice $= x_1 . x_2 . x_3 \ldots\ldots x_n$.

Therefore expected value of the product of points obtained

$=$ product of the expected values of x_i

This is because $x_1, x_2, \ldots\ldots x_n$ are independent as number obtained on one die is independent of the number obtained on other dice.

But expected value of $x_i = \dfrac{7}{2}$ (See Ex. 8-2)

Therefore expected value of the product of points obtained

$$= \left(\frac{7}{2}\right)^n$$

Ex. 8-11. *Find the mathematical expectation of the sum of points obtained on rolling n dice together.*

Sol. Let x_i be the number of points obtained on ith die. Then sum of points on n dice will be

$$s=x_1+x_2+\dots\dots+x_n$$

Therefore expected value of s

$$=\text{sum of the expected values of } x_1, x_2, \dots\dots x_n$$

Now Expected value of $x_i=\dfrac{7}{2}$ (See Ex. 8-2)

Therefore expected value of s

$$=\left(\frac{7}{2}\right)n$$

$$=\frac{7n}{2}.$$

Ex. 8-12. *If p_i be the probability of success for ith trial, find the expectation of the number of successes in n independent trials.*

Sol. Associate with every trial a variable which has the value '1' in case of success and the value '0' in case of failure. If x_1, x_2, ... x_n be the variables attached to trials 1, 2, ... n, the number of successes in n trials is given by

$$m=x_1+x_2+\dots\dots+x_n$$

$$\therefore \quad E(m)=E(x_1)+E(x_2)+\dots\dots+E(x_n)$$

Since x_i can take only two values '1' and '0' with respective probabilities p_i and $1-p_i$. its expectation is given by

$$E(x_i)=1.p_i+0.(1-p_i)$$
$$=p_i$$

$$\therefore \quad E(m)=p_1+p_2+\dots\dots p_n$$

Ex. 8-13. *Find the expectation of the number of white balls among c balls drawn from an urn containing a white and b black balls.*

Sol. Associate with every ball a variable which has the value '1' if it is white and the value '0' otherwise. If x_1, $x_2 \dots\dots x_c$ be the variables attached to c balls drawn, the number of white balls is given by

$$m=x_1+x_2+\dots\dots+x_c$$

$$\therefore \quad E(m)=E(x_1)+F(x_2)+\dots\dots+E(x_c)$$

Now the probability that the ith ball drawn will be white when nothing is known of the other balls

$$=\frac{a}{a+b}$$

$$\therefore \qquad E(x_i)=1\cdot\frac{a}{a+b}+0.\left\{1-\frac{a}{a+b}\right\}$$

$$=\frac{a}{a+b} \text{ for all } i$$

$$\therefore \qquad E(m)=\frac{ca}{a+b}$$

Ex. 8·14. *Balls are taken one by one out of an urn containing a white and b black balls until the first white ball is drawn. Show that the expectation of the number of black balls preceding the first white ball is* $\frac{b}{a+1}$.

Sol. Let x be the number of black balls drawn before first white ball. The possible values of x are 0, 1, 2, ... b

Prob. of x taking the value '0'

$$=\text{prob. of drawing a white ball in first draw}=\frac{a}{a+b}$$

Prob. of x taking the value '1'

=Prob. of drawing a black ball in first draw and a white

$$\text{ball in second draw}=\frac{b}{a+b}\cdot\frac{a}{a+b-1}$$

Prob. of x taking the value '2'

=prob. of black balls in first two draws and a white ball in

$$\text{third draw}=\frac{b}{a+b}\cdot\frac{b-1}{a+b-1}\cdot\frac{a}{a+b-2}$$

and so on. In general, prob. of x taking the value 'r'

$$=\frac{b}{a+b}\cdot\frac{b-1}{a+b-1}\cdots\cdots\frac{b-\overline{r-1}}{a+b-\overline{r-1}}\cdot\frac{a}{a+b-r}$$

$\therefore$ Expected value of x

$$=0.\frac{a}{a+b}+1.\frac{b}{a+b}\cdot\frac{a}{a+b-1}+2.\frac{b(b-1)}{(a+b)(a+b-1)}\cdot\frac{a}{(a+b-2)}$$

$$+\cdots\cdots r\frac{b(b-1)\cdots\cdots(b-\overline{r-1})}{(a+b)(a+b-1)\cdots\cdots\cdots(a+b-\overline{r-1})}\cdot\frac{a}{a+b-r}+\cdots\cdots$$

$$=\frac{ab}{a+b}\left[\frac{1}{a+b-1}+2\frac{(b-1)}{(a+b-1)(a+b-2)}+\cdots\cdots\right.$$

$$\left.+r\frac{(b-1)(b-2)\cdots\cdots(b-\overline{r-1})}{(a+b-1)(a+b-2)\cdots\cdots\cdots(a+b-r)}+\cdots\cdots\right]$$

Let $U_r = \dfrac{r(b-1)(b-2)\ldots\ldots b-\overline{r-1}}{(a+b-1)(a+b-2)\ldots\ldots\ldots(a+b-r)}$

$$= \frac{[A(r-1)+B](b-1)(b-2)\ldots\ldots(b-\overline{r-1})}{(a+b-1)(a+b-2)\ldots\ldots(a+b-\overline{r-1})} - \frac{[Ar+B](b-1)(b-2)\ldots\ldots(b-r)}{(a+b-1)(a+b-2)\ldots\ldots(a+b-r)}$$

$\therefore \quad r = [A(r-1)+B](a+b-r) - [Ar+B](b-r)$

Equating co-efficients of r

$$1 = A(a+b) + A - A.b$$

or $\quad A = \dfrac{1}{a+1}$

Equating terms independent of r

$$0 = (B-A)(a+b) - Bb$$

or $\quad B = \dfrac{a+b}{a}\, A$

$$\therefore \quad U_r = \frac{A\left[\overline{r-1} + \dfrac{a+b}{a}\right](b-1)(b-2)\ldots\ldots(b-\overline{r-1})}{(a+b-1)(a+b-2)\ldots\ldots(a+b-\overline{r-1})} - \frac{A\left[r + \dfrac{a+b}{a}\right](b-1)(b-2)\ldots\ldots(b-r)}{(a+b-1)(a+b-2)\ldots\ldots(a+b-r)}$$

$\therefore \quad U_2 + U_3 + \ldots\ldots + U_b$

$$= \frac{A\left[1 + \dfrac{a+b}{a}\right](b-1)}{(a+b-1)} = \frac{\left[1 + \dfrac{a+b}{a}\right](b-1)}{(a+1)(a+b-1)}$$

$\therefore$ Expected value of x

$$= \frac{ab}{a+b}\left[\frac{1}{a+b-1} + \frac{\left[1+\dfrac{a+b}{a}\right](b-1)}{(a+1)(a+b-1)}\right]$$

$$= \frac{ab}{a+b}\left[\frac{a+1+b-1+\dfrac{(a+b)(b-1)}{a}}{(a+1)(a+b-1)}\right]$$

$$= \frac{b}{a+1}.$$

Ex. 8-15. *A bag contains a coin of value M and a number of other coins whose aggregate value is m. A person draws one at a time till he draws the coin M. Find the value of his expectations.*

Sol. Let the total number of coins in a bag$=n$.

Then the average value of coins other than that of value M

$$=\frac{m}{n-1}$$

Now prob. of drawing a coin in first draw

$$=\frac{1}{n}$$

Since in first draw any coin may appear, expectation from first draw

$$=\left\{M+\frac{m}{n-1}+\frac{m}{n-1}+\dots+(n-1)\text{ times}\right\}.\ \frac{1}{n}$$

$$=\frac{M+m}{n}.$$

If the coin M does not appear in first draw, second draw is to be made.

$\therefore$ Chance of second draw$=$Chance of not drawing the coin M in first draw

$$=1-\frac{1}{n}=\frac{n-1}{n}$$

Since there will be $(n-1)$ coins before second draw, chance of drawing a coin in second draw, when it is known that second draw is to be made

$$=\frac{1}{n-1}$$

$\therefore$ By compound prob theorem, prob of drawing a coin in 2nd draw

$$=\frac{n-1}{n}\cdot\frac{1}{n-1}$$

$$=\frac{1}{n}$$

$\therefore$ Expectation from second draw

$$=\frac{1}{n}\left\{M+\frac{m}{n-1}+\dots\dots+(n-2)\text{ times}\right\}$$

$$=\frac{1}{n}\left\{M+\frac{(n-2)m}{n-1}\right\}.$$

Similarly expectation from third draw

$$=\frac{n-1}{n}\cdot\frac{n-2}{n-1}\cdot\frac{1}{n-2}\left\{M+\frac{n-3}{n-1}m\right\}$$

$$=\frac{1}{n}\left\{M+\frac{n-3}{n-1}m\right\}$$

and so on.

Finally expectation from last draw

$$=\frac{M}{n}$$

$\therefore$ Total expectation

$$=\frac{1}{n}\left[(M+m)+\left\{M+\frac{n-2}{n-1}m\right\}+\left\{M+\frac{n-3}{n-1}m\right\}+\ldots\ldots+M\right]$$

$$=M+\frac{1}{n}.m\left\{1+\frac{n-2}{n-1}+\frac{n-3}{n-1}+\ldots\ldots+\frac{1}{n-1}\right\}$$

$$=M+\frac{1}{n(n-1)}m\{(n-1)+(n-2)+\ldots\ldots+1\}$$

$$=M+\frac{1}{n(n-1)}m.\frac{n(n-1)}{2}$$

$$=M+\frac{m}{2}$$

Ex. 8-16. *Show that* $E\mid x\mid \geqslant \mid E(x)\mid$

Sol. $\mid E(x)\mid=\mid\Sigma px\mid$

$$\leqslant\Sigma\mid px\mid=\Sigma p\mid x\mid=E\mid x\mid$$

Ex. 8-17. *Show by an example that the mathematical expectation need not be finite.*

Sol. Consider the prob dist.

$$P(x)=\frac{e^{-1}}{x\,!},\ x=0,\ 1,\ 2\ldots\ldots\infty$$

Here $$E(x\,!)=\sum_{x=0}^{\infty}\frac{e^{-1}}{x\,!}.\,x\,!=e^{-1}\sum_{x=0}^{\infty}1$$

which is not finite.

Ex. 8-18. *If 'a' is constant, show that*

(*i*) $E(a)=a$

(*ii*) $E(ax)=aE(x)$

(*iii*) $Var(ax)=a^2\,Var(x)$

Ex. 8-19. *Show that* $E(x^2) \geqslant \{E(x)\}^2$.

Sol. We have

$$E(x-\bar{x})^2 = \Sigma p(x-\bar{x})^2 \geqslant 0$$

i.e., $E(x^2+\bar{x}^2-2x\bar{x}) \geqslant 0.$

i.e., $E(x^2)+\bar{x}^2-2\bar{x}E(x) \geqslant 0$

i.e., $E(x^2)+\bar{x}^2-2\bar{x}^2 \geqslant 0$

$\therefore$ $E(x^2) \geqslant \bar{x}^2 = \{E(x)\}^2$

8·3. Moment Generating Function. *The moment generating function (m.g.f.) of the chance variate x about the point 'a' is defined to be* $E\left\{ e^{t(x-a)} \right\}$ *and is denoted by* $M_a(t)$, *where t is the real parameter.*

Cumulative Function. *The cumulative function about* $x=a$ *is defined by* $K_a(t)=\log M_a(t)$. *If* $K_a(t)$ *can be expanded as a convergent series in powers of t viz.*

$$K(t) \equiv K_1 t + K_2 \frac{t^2}{2\,!} + K_3 \frac{t^3}{3\,!} + \ldots\ldots$$

the co-efficients K_1. K_2, *etc,, are called first cumulent, second cumulent etc. of the dist.*

Ex. 8-20. (*i*) *Show that* $M_a(t)=e^{-at} M_0(t)$.

(*ii*) *Discuss the effect of change of origin and scale on M.G.F.*

(*iii*) *Show that the m.g.f of the sum of 'n' independent variates is the product of their moment generating functions.*

(*iv*) *Show that*

$$M_a(t) = \sum_{r=0}^{\infty} \mu_r'(a) \frac{t^r}{r\,!}$$

Sol. (*i*) $M_a(t) = E\{e^{t(x-a)}\}$

$$= E\{e^{tx} \cdot e^{-at}\}$$

$$= e^{-at} E\{e^{tx}\}$$

$$= e^{-at} M_0(t)$$

(*ii*) The transformation corresponding to change of origin and scale is

$$X = \frac{x-a}{h}$$

where a corresponds to change of origin and h to change of scale. Both a and h are constants. X is a new variate to which x transforms.

$\therefore \quad x = a + hX$

$$\therefore \quad M_0(t) \text{ of } x = E\{e^{tx}\}$$
$$= E\{e^{t(a+hX)}\}$$
$$= e^{at}\, E\{e^{(th)X}\}$$
$$= e^{at}\{M_0(th) \text{ of } X\}$$

(*iii*) Let $x_1, x_2, \ldots x_n$ be n independent chance variates.

Let $X = x_1 + \ldots\ldots + x_n$.

$$M_0(t) \text{ of } X = E\{e^{tx}\}$$
$$= E\{e^{t(x_1 + \ldots\ldots + x_n)}\}$$
$$= E\{e^{tx_1} \cdot e^{tx_2} \ldots\ldots e^{tx_n}\}$$
$$= E\{e^{tx_1}\} \cdot E\{e^{tx_2}\} \ldots\ldots E\{e^{tx_n}\}$$

($\because$ x_i's are independent)

$$= \{M_0(t) \text{ of } x_1\} \ldots\ldots \{M_0(t) \text{ of } x_n\}$$

(*iv*) $\quad M_a(t) = E\{e^{t(x-a)}\}$

$$= E\left\{1 + t(x-a) + \frac{t^2}{2!}(x-a)^2 + \ldots\ldots\right\}$$
$$= 1 + t\,E(x-a) + \frac{t^2}{2!}\,E(x-a)^2 + \ldots\ldots$$
$$= 1 + t\mu_1'(a) + \frac{t^2}{2!}\,\mu_2'(a) + \ldots\ldots$$
$$= \sum_{r=0}^{\infty} \mu_r'(a)\,\frac{t^r}{r!}$$

Remark. Since from the function $M_a(t)$, moments can be generated, it is called moment generating function.

Ex. 8·21. (*i*) *Discuss the effect of change of origin and scale on cumulants.*

(*ii*) *Prove that the r-th cumulant of the sum of independent chance variates is the sum of the r-th cumulants of the variates.*

(*iii*) *Show that* $k_1 = \mu_1'$, $k_2 = \mu_2$, $k_3 = \mu_3$ and $k_4 = \mu_4 - 3\mu_2^2$.

Sol. (*i*) By Ex. 8·20 (*ii*)

$$K_0(t) \text{ of } x = \log\{M_0(t)_{\text{of } x}\}$$
$$= at + \log\{M_0(th) \text{ of } X\}$$
$$= at + K_0(th) \text{ of } X \qquad \ldots(1)$$

Let $K_1, K_2, \ldots\ldots$ and $K_1', K_2' \ldots\ldots$ be the cummulants for x and X respectively.

Then (1) $\Rightarrow$

$$K_1 t + K_2 \frac{t^2}{2!} + \ldots\ldots = at + K_1'(th) + K_2' \frac{(th)^2}{2!} + \ldots\ldots$$

$$\therefore \qquad K_1 = a + hK_1'$$
$$K_r = h^r K_r'. \qquad r \geqslant 2$$

$\therefore$ Except first all cummulants are independent of origin but all cummulants (including first) depend on scale.

(*ii*) In Ex. 8-20 (*iii*) taking log

$$K_0(t) \text{ of } X = \sum_{i=1}^{n} \{K_0(t) \text{ of } x_i\}$$

Let $K_1, K_2 \ldots\ldots$ and $K_1'\ K_2' \ldots\ldots$ be the cummulants of X and x_i respectively ($i=1, 2 \ldots\ldots n$)

$$\therefore \quad \sum_{r=1} K_r \frac{t^r}{r!} = \sum_{i=1}^{n} \left\{ \sum_{r=1} K_r' \frac{t^r}{r!} \right\}$$

$$= \sum_{r=1} \frac{t^r}{r!} \left\{ \sum_{i=1}^{n} K_r^i \right\}$$

$$\Rightarrow \qquad K_r = \sum_{i=1}^{n} K_r^i$$

(*iii*) By def,

$$K_{\bar{x}}(t) = \log M_{\bar{x}}(t)$$

$$= \log \left\{ 1 + \left(t\mu_1 + \frac{t^2}{2!}\mu_2 + \ldots\ldots \right) \right\}$$

$$= \log \left\{ 1 + \left(\frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \frac{t_4}{4!}\mu_4 \ldots\ldots \right) \right\}$$

$$(\because \quad \mu_1 = 0)$$

$$= \left(\frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \frac{t^4}{4!}\mu_4 + \ldots \right) - \frac{1}{2}\left(\frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \ldots \right)^2 + \ldots$$

$$=\mu_2\frac{t^2}{2\,!}\times\mu_3\frac{t^3}{3\,!}+(\mu_4-3\mu_2^2)\frac{t^4}{4\,!}+\ldots$$

$$\therefore \qquad K_1(\bar{x})=0 \;\Rightarrow\; K_1(0)-\bar{x}=0$$
$$\Rightarrow\; K_1(0)=\bar{x}$$
$$K_2=\mu_2$$
$$K_3=\mu_3$$
and
$$K_4=\mu_4-3\mu_2^2$$

EXERCISE

1. If X is a random variable which assumes values 1, 2, ..., with probability.

$$P(X=k)=q^{k-1}p,\ (q+p=1)$$

Find $E(X)$

$$\left[\textbf{Ans.}\ \frac{1}{p}\right]$$

2. A and B in turn toss an ordinary die for a prize of Rs. 44. The first to toss a 'six' wins. If A has first throw, what is his expectation ? **[Ans.** 24 ; 20]

9

Continuous Distributions

9-1. Continuous Variable. *It is the variable which can take all possible values between certain limits. In the following x will be taken as continuous variable.*

Probability Density Function. *A continuous function $f(x)$. s.t. the probability of the variate value lying in infinitesimal interval $x-\frac{dx}{2}$ to $x+\frac{dx}{2}$ can be expressed in the form $f(x)\,dx$, is called probability density function or simply the density function.*

The density function $f(x)$ has the following properties :

(i) $f(x)\geqslant 0 \quad \forall x.$

(ii) $\int f(x)=1$ *where the integration is being extended to the entire range of the variable x.*

In the following the density function will be denoted by $f(x)$.

Probability Differential. *'f x)dx' is called probability differential. The probability for a variate to lie in the interval (a, b) is given by*

$$P(a\leqslant x\leqslant b)=\int_a^b f(x)dx$$

Probability Curve. *The continuous curve $y=f(x)$ is called the probability density curve or simply probability curve.*

Distribution Function. *The function $F(x)$ defined by*

$$F(x)=\int_{-\infty}^{x} f(x)dx$$

is called the cumulative distribution function (c.d.f.) or simply the distribution function of x. The c.d.f. F(x) has the following properties :

(i) $F'(x)=f(x)\geq 0 \Rightarrow F(x)$ *is non-decreasing function.*

(ii) $F(-\infty)=0$.

(iii) $F(\infty)=\int_{-\infty}^{\infty} f(x)dx=1$.

Any continuous differentiable function F(x) with the above properties may be regarded as c.d.f. of x. Then the denity function of x is F'(x).

Mean Moments etc. *All the quantities e.g., mean, moments are defined as for discrete distribution with the difference that* $\frac{f_i}{N}$ *is replaced by f(x)dx and summation by integration over the range of the variate.*

Median. *Median 'a' is given by*

$$\int_{-\infty}^{a} f(x)dx=\int_{a}^{\infty} f(x)dx$$

Quartiles. *The lower and upper quartiles Q_1 and Q_3 are given by*

$$\int_{-\infty}^{Q_1} f(x)dx=\frac{1}{4}=\int_{Q_3}^{\infty} f(x)dx$$

Mode. *Mode is that value of x for which f(x) is maximum i.e., model value x is s.t.*

$$f'(x)=0$$
$$f''(x)<0$$

provided that the solution of $f'(x)=0$ lies within the permissible range of x.

Ex. 9-1. *Show that for the rectangular population*

$$df=dx \qquad 0\leq x\leq 1$$

$$\mu_1'(0)=\frac{1}{2} \text{ and } \mu_2=\frac{1}{12}$$

Sol. By def. $\mu_1'(0)=\int_0^1 xdx=\frac{1}{2}$

$$\mu_2'(0)=\int_0^1 x^2 dx=\frac{1}{3}$$

$$\therefore \quad \mu_2=\mu_2'(0)-\{\mu_1'(0)\}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

Ex. 9-2. *For the rectangular distribution* $y=\frac{1}{2a}$, $-a<x<a$ *show that* $M_0(t)=\frac{1}{at}$ *sinh at and* $\mu_{2n}=\frac{a^{2n}}{2n+1}$

Sol. By def.

$$M_0(t)=\int_{-a}^{a} e^{tx}\frac{1}{2a}\,dx=\frac{1}{2at}\left\{e^{tx}\right\}_{-a}^{a}$$

$$=\frac{1}{2at}\left\{e^{at}+e^{-at}\right\}=\frac{\sinh at}{at}$$

Also $$\mu_{2n}=\int_{-a}^{a}(x-\bar{x})^{2n}\frac{1}{2a}\,dx$$

where $$\bar{x}=A.M.=\int_{-a}^{a} x\frac{1}{2a}\,dx=0$$

$$\therefore \quad \mu_{2n}=\frac{1}{2a}\int_{-a}^{a} x^{2n}\,dx=\frac{1}{a}\int_0^a x^{2n}\,dx=\frac{a^{2n}}{2n+1}.$$

Ex. 9-3. *Calculate* β_1 *for the dist.* $dF=kxe^{-x}\,dx$, $0<x<\infty$

Sol. k is given by

$$k\int_0^\infty xe^{-x}\,dx=1$$

or $$k\left\{\Big|-e^{-x}.x\Big|_0^\infty+\int_0^\infty e^{-x}dx\right\}=1$$

or $$k\left\{-e^{-x}\right\}_0^\infty=1 \qquad \therefore \quad k=1$$

$$\bar{x}=\int_0^\infty x^2e^{-x}dx=\left\{\left|-x^2e^{-x}\right|_0^\infty+2\int_0^\infty xe^{-x}\,dx\right\}$$

$$=2\left\{\left|-xe^{-x}\right|_0^\infty+\int_0^\infty e^{-x}ax\right\}=2$$

$$\mu_2'(0)=\int_0^\infty x^3e^{-x}dx=\left|-e^{-x}x^3\right|_0^\infty+3\int_0^\infty x^2e^{-x}dx=6$$

$$\mu_3'(0)=\int_0^\infty x^4e^{-x}\,dx=4\int_0^\infty x^3e^{-x}\,dx=24$$

$\therefore \qquad \mu_2=\mu_2'(0)-\bar{x}^2=6-4=2$

$\qquad \mu_3=\mu_3'(0)-3\mu_2'(0)\bar{x}+2\bar{x}^3$

$\qquad =24-36+16=4$

$\therefore \qquad \beta_1=\mu_3^2/\mu_2^3=2$

Ex. 9-4. *Find the s.d., harmonic mean, the mode and the median of the dist given by*

$$f(x)=6(x-x^2),\ 0\leqslant x\leqslant 1$$

Sol. $$\bar{x}=6\int_0^1 x(x-x^2)dx=6\left\{\frac{1}{3}-\frac{1}{4}\right\}=\frac{1}{2}$$

$$\mu_2'(0)=6\int_0^1 x^2(x-x^2)dx=6\left\{\frac{1}{4}-\frac{1}{5}\right\}=\frac{3}{10}$$

$\therefore \qquad \mu_2=\dfrac{3}{10}-\dfrac{1}{4}=\dfrac{1}{20}$

$\therefore \qquad s.d.=\sqrt{\dfrac{1}{20}}$

H.M. is given by

$$\frac{1}{H}=6\int_0^1 \frac{1}{x}\,(x-x^2)dx=6\int_0^1 (1-x)dx$$

$$=6\left(1-\frac{1}{2}\right)=3$$

$\therefore \qquad H=\frac{1}{3}$

To find mode put $f'(x)=0$

i.e., $\qquad 1-2x=0$

or $\qquad x=\frac{1}{2}$

Since $f''(x)=-12<0$, $x=\frac{1}{2}$ is the mode. Let a be the median. Then

$$6\int_0^a (x-x^2)dx=\frac{1}{2}$$

or $$\frac{a^2}{2}-\frac{a^3}{3}=\frac{1}{12}$$

or $$4a^3-6a^2+1=0$$

$\therefore$ $$a=\frac{1}{2}$$

Ex. 9·5. *For the dist*

$$dF=y_0e^{-|x|}\,dx,\ -\infty<x<\infty$$

show that $y_0=\frac{1}{2}$, $\mu_1'(0)=0$, $\sigma=\sqrt{2}$ *and mean deviation about mean*$=1$.

Sol. y_0 is given by.

$$y_0\int_{-\infty}^{\infty} e^{-|x|}\,dx=1$$

or $$2y_0 \int_0^\infty e^{-|x|}\,dx = 1 \quad \left\{ \because \; e^{-|x|} \text{ is an even } f^n \text{ of } x \right\}$$

$\therefore$ $$2y_0 \int_0^\infty e^{-x}\,dx = 1 \qquad (\because \; |x| = x \text{ as } x \geqslant 0)$$

$\therefore$ $$y_0 = \frac{1}{2}$$

$$\mu_1'(0) = \frac{1}{2}\int_{-\infty}^{\infty} xe^{-|x|}\,dx = 0$$

$\therefore$ $$\mu_2 = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}\,dx = \int_0^\infty x^2 e^{-x}\,dx$$

$$= \left| -x^2 e^{-x} \right|_0^\infty + 2\int_0^\infty xe^{-x}\,dx = 2\int_0^\infty xe^{-x}\,dx$$

$$= 2\left\{ \left| -xe^{-x} \right|_0^\infty + \int_0^\infty e^{-x}\,dx \right\} = 2$$

$\therefore$ $$\sigma = \sqrt{2}$$

Mean deviation about mean $= \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} |x-0|\, e^{-|x|}\,dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty} |x|\, e^{-|x|}\,dx = \int_0^\infty xe^{-x}\,dx = 1$$

Ex. 9-6. *Show that for the dist*

$$f(x) = \frac{2a}{\pi}\left(\frac{1}{a^2+x^2}\right) \qquad -a \leqslant x \leqslant a$$

$$\mu_2 = a^2\,\frac{(4-\pi)}{\pi}, \qquad \mu_4 = a^4\left(1 - \frac{8}{3\pi}\right)$$

Sol. $$\int_{-a}^{a} f(x)dx=\frac{2a}{\pi}\int_{-a}^{a}\frac{1}{a^2+x^2}\,dx=\frac{2a}{\pi}\left|\frac{1}{a}\tan^{-1}\frac{x}{a}\right|_{-a}^{a}=1$$

$$\mu'_1(0)=\frac{2a}{\pi}\int_{-a}^{a} x\,\frac{1}{a^2+x^2}\,dx=0$$

$$\mu_2=\frac{2a}{\pi}\int_{-a}^{a}x^2\,\frac{1}{a^2+x^2}\,dx=\frac{4a}{\pi}\int_{0}^{a}\frac{x^2+a^2-a^2}{a^2+x^2}\,dx$$

$$=\frac{4a}{\pi}\int_{0}^{a}\left\{1-\frac{a^2}{a^2+x^2}\right\}dx=\frac{4a}{\pi}\left\{a-a.\frac{\pi}{4}\right\}$$

$$=\frac{a^2}{\pi}(4-\pi)$$

$$\mu_4=\frac{2a}{\pi}\int_{-a}^{a}x^4.\frac{1}{a^2+x^2}\,dx=\frac{4a}{\pi}\int_{0}^{a}\left\{x^2-a^2+\frac{a^4}{x^2+a^2}\right\}dx$$

$$=\frac{4a}{\pi}\left\{\frac{a^3}{3}-a^3+a^3\,\frac{\pi}{4}\right\}=a^4\left\{1-\frac{8}{3\pi}\right\}$$

Ex. 9-7. *For a continuous distribution whose relative frequency density is given by*

$$f(x)=\frac{3x(2-x)}{4},\ 0\leqslant x\leqslant 2$$

Find the first three moments about the origin. Hence or otherwise show that the dist is symmetrical about the mean with variance $=\frac{1}{5}$.

Sol, $$\mu_1'(0)=\frac{3}{4}\int_{0}^{2}x^2(2-x)dx=\frac{3}{4}\left\{\frac{2}{3}x^3-\frac{x^4}{4}\right\}_{0}^{2}=1$$

$$\mu_2'(0)=\frac{3}{4}\int_{0}^{2}x^3(2-x)dx=\frac{3}{4}\left\{\frac{1}{2}x^4-\frac{x^5}{5}\right\}_{0}^{2}=\frac{6}{5}$$

$$\mu_3'(0)=\frac{3}{4}\int_0^2 x^4(2-x)dx=\frac{3}{4}\left\{\frac{2}{5}x^5-\frac{x^6}{6}\right\}_0^2=\frac{8}{5}$$

$\therefore \qquad \mu_2=\mu_2'(0)-\{\mu_1'(0)\}^2=\frac{6}{5}-1=\frac{1}{5}$

Let 'a' be the median

Then $\qquad \displaystyle\int_0^a f(x)dx=\frac{1}{2}$

or $\qquad \displaystyle\frac{3}{4}\int_0^a x(2-x)dx=\frac{1}{2}$

or $\qquad a^2-\frac{a^3}{3}=\frac{2}{3}$

or $\qquad a^3-3a^2+2=0$

or $\qquad (a-1)(a^2-2a-2)=0$

$\therefore \qquad a=1$

$\therefore$ Median=Mean

$\therefore$ Dist. is symmetrical about mean.

Ex. 9-8. *Find the mean deviation, s.d and skewness of the dist given by* $f(x)=\frac{3}{4}x(2-x),\ 0\leqslant x\leqslant 2.$

Sol. From last example, Dist is symmetrical about mean.

$\therefore \qquad$ Skewness=0

$$\text{Mean deviation}=\frac{3}{4}\int_0^2 |x-1|\,x(2-x)dx$$

$$=\frac{3}{4}\int_0^1 x(1-x)(2-x)dx+\frac{3}{4}\int_1^2 x(x-1)(2-x)dx$$

$$=\frac{3}{4}\int_0^1 \{x^3-3x^2+2x\}dx+\frac{3}{4}\int_1^2\{-x^3+3x^2-2x\}dx$$

$$=\frac{3}{4}\left\{\frac{1}{4}-1+1\right\}+\frac{3}{4}\left\{\frac{1}{4}\right\}=\frac{3}{8}$$

For *s.d.* see last example.

Ex. 9-9. *In ex. 9-7 calculate $\mu_3'(0)$ and $\mu_4'(0)$ and deduce β_2.*

Sol. $$\mu_4'(0)=\frac{3}{4}\int_0^2 x^5(2-x)dx=\frac{3}{4}\left\{\frac{x^6}{3}-\frac{x^7}{7}\right\}_0^2=\frac{16}{7}$$

$$\therefore\quad \mu_4=\mu_4'(0)-4\mu_3'(0)\mu_1'(0)+6\mu_2'(0)\{\mu_1'(0)\}^2-3\{\mu_1'(0)\}^4$$

$$=\frac{16}{7}-4.\frac{8}{5}+6\ \frac{6}{5}\ -3=\frac{3}{35}$$

$$\therefore\quad \beta_2=\frac{\mu_4}{\mu_2^2}=\frac{\frac{3}{35}}{\frac{1}{25}}=\frac{15}{7}.$$

Ex. 9-10. *Show that for the dist $dF=y_0e^{-x/\sigma}\,dx,\ 0\leqslant x\leqslant\infty$ the mean and s d are equal to σ and the interquartile range is $\sigma\log_e 3$.*

Sol. The constant y_0 is given by

$$y_0\int_0^\infty e^{-x/\sigma}dx=1 \quad\text{or}\quad y_0=\frac{1}{\sigma}$$

$$\therefore\quad \bar{x}=y_0\int_0^\infty xe^{-x/\sigma}\,dx=\frac{1}{\sigma}\int_0^\infty xe^{-x/\sigma}\,dx=\sigma\int_0^\infty ye^{-y}\,dy$$

where $y=\dfrac{x}{\sigma}$

$$=\sigma\overline{|2}=\sigma$$

$$\mu_2'(0)=y_0\int_0^\infty x^2e^{-x/\sigma}dx=\sigma^2\int_0^\infty y^2e^{-y}dy=\sigma^2\overline{|3}=\sigma^2\,2\,!$$

$$=2\sigma^2$$

$$\therefore\qquad \mu_2=2\sigma^2-\sigma^2=\sigma^2\qquad\therefore\ s.d.=\sigma$$

Let Q_1 and Q_3 be the quartiles

Then $$y_0\int_0^{Q_1} e^{-x/\sigma}\,dx=\frac{1}{4}\ \text{and}\ y_0\int_0^{Q_3} e^{-x/\sigma}\,dx=\frac{3}{4}$$

or $$\frac{1}{\sigma}\left\{-\sigma e^{-x/\sigma}\right\}_0^{Q_1}=\frac{1}{4} \text{ and } \frac{1}{\sigma}\left\{-\sigma e^{-x/\sigma}\right\}_0^{Q_3}=\frac{3}{4}$$

or $$1-e^{-Q_1/\sigma}=\frac{1}{4} \text{ and } 1-e^{-Q_3/\sigma}=\frac{3}{4}$$

or $$Q_1=\sigma \log_e \frac{4}{3} \text{ and } Q_3=\sigma \log_e 4$$

$$\therefore \quad Q_3-Q_1=\sigma \log_e 3.$$

Ex. 9-11. *Prove that the geometric mean G of the dist*

$$dF=6(2-x)(x-1)dx,\ 1\leqslant x\leqslant 2$$

is given by 6 log (16G)=19.

Sol. $$\log G=6\int_1^2 \log x.(2-x)(x-1)dx$$

$$=6\int_0^1 y(-y+1)\log(1+y)dy \qquad \text{where } x-1=y$$

$$=6\left[\left|\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\log(y+1)\right|_0^1-\int_0^1\frac{1}{y+1}\left\{\frac{y^2}{2}-\frac{1}{3}y^3\right\}dy\right]$$

$$=\log 2-3\int_0^1\left\{y-1+\frac{1}{y+1}\right\}dy+2\int_0^1\left(y^2-y+1-\frac{1}{y+1}\right)dy$$

$$=\log 2-3\left\{\frac{y^2}{2}-y\right\}_0^1+2\left\{\frac{y^3}{3}-\frac{y^2}{2}+y\right\}_0^1-5\left|\log(y+1)\right|_0^1$$

$$=\frac{19}{6}-4\log 2$$

$$\therefore \quad 6\log(16G)=19.$$

Ex. 9-12. *The elementary probability law of a continuous random variable x is* $p(x)=y_0e^{-b(x-a)}$, $a\leqslant x<\infty$, *where* a, b, y_0 *are constants. Show that* $y_0=b=\frac{1}{\sigma}$ *and* $a=m-\sigma$ *where* m, σ *are*

respectively the mean and the s.d. of the dist. Show also that $\beta_1=4$ *and* $\beta_2=9$.

Sol. y_0 **is given by**

$$y_0\int_a^\infty e^{-b(x-a)}\,dx=1$$

or $$y_0\left\{\frac{e^{-b(x-a)}}{-b}\right\}_a^\infty=1 \quad \therefore \quad y_0=b$$

$$m=y_0\int_a^\infty xe^{-b(x-a)}\,dx=y_0\left\{\left|\frac{xe^{-b(x-a)}}{-b}\right|_a^\infty+\frac{1}{b}\int_a^\infty e^{-b(x-a)}\,dx\right\}$$

$$=y_0\,\frac{a}{b}+\frac{y_0}{b^2}\left\{-e^{-b(x-a)}\right\}_a^\infty=a+\frac{1}{b}$$

$$\mu_2'(0)=y_0\int_a^\infty x^2e^{-b(x-a)}\,dx$$

$$=y_0\left\{\left|-\frac{x^2}{b}\,e^{-b(x-a)}\right|_a^\infty+\frac{2}{b}\int_a^\infty xe^{-b(x-a)}\,dx\right\}$$

$$=y_0\,\frac{a^2}{b}+\frac{2}{b}\,y_0\int_a^\infty xe^{-b(x-a)}\,dx$$

$$=a^2+\frac{2}{b}\left(a+\frac{1}{b}\right)$$

$$\therefore \quad \mu_2=a^2+\frac{2}{b}\left(a+\frac{1}{b}\right)-\left(a+\frac{1}{b}\right)^2=\frac{1}{b^2}$$

$$\therefore \quad \sigma^2=\frac{1}{b^2}$$

$$\therefore \quad y_0=b=\frac{1}{\sigma}$$

and $$m-\sigma=a$$

$$\mu_3'(0)=y_0\int_a^\infty x^3e^{-b(x-a)}\,dx=y_0\left\{\left|-\frac{x^3}{b}\;e^{-b(x-a)}\right|_a^\infty\right.$$

$$\left.+\frac{3}{b}\int_a^\infty x^2e^{-b(x-a)}\,dx\right\}$$

$$=a^3+\frac{3}{b}\left\{a^2+\frac{2}{b}\left(a+\frac{1}{b}\right)\right\}$$

$\therefore \quad \mu_3=\mu_3'(0)-3\mu_2'(0)\mu_1'(0)+2\{\mu_1'(0)\}^3$

$$=a^3+\frac{3a^2}{b}+\frac{6}{b^2}\left(a+\frac{1}{b}\right)-3\left\{a^2+\frac{2}{b}\left(a+\frac{1}{b}\right)\right\}\left(a+\frac{1}{b}\right)$$

$$+2\left(a+\frac{1}{b}\right)^3=\frac{2}{b^3}$$

$\therefore \quad \beta_1=\dfrac{\mu_3^2}{\mu_2^3}=4$

$$\mu_4'(0)=y_0\int_a^\infty x^4e^{-b(x-a)}\,dx=y_0\left\{\left|-\frac{x^4}{b}\;e^{-b(x-a)}\right|_a^\infty\right.$$

$$\left.+\frac{4}{b}\int_a^\infty x^3e^{-b(x-a)}\,dx\right\}$$

$$=a^4+\frac{4}{b}\left\{a^3+3\frac{a^2}{b}+\frac{6}{b^2}\left(a+\frac{1}{b}\right)\right\}$$

$\therefore \quad \mu_4=\mu_4'(0)-4\mu_3'(0)\mu_1'(0)+6\mu_2'(0)\{\mu_1'(0)\}^2-3\{\mu_1'(0)\}^4$

$$=a^4+\frac{4}{b}\left\{a^3+\frac{3a^2}{b}+\frac{6a}{b^2}+\frac{6}{b^3}\right\}-4\left\{a^3+\frac{3a^2}{b}+\frac{6a}{b^2}+\frac{6}{b^3}\right\}$$

$$\cdot\left(a+\frac{1}{b}\right)+6\left\{a^2+\frac{2a}{b}+\frac{2}{b^2}\right\}\left(a+\frac{1}{b}\right)^2-3\left(a+\frac{1}{b}\right)^4$$

$$=\frac{9}{b^4}$$

$\therefore \quad \beta_2=\dfrac{\mu_4}{\mu_2^2}=9.$

Ex. 9-13. *For the dist*

$$dF=\frac{1}{\beta(m,n)}\;x^{n-1}(1-x)^{m-1}\,dx,\;0<x<1$$

find the mean, s.d. and the Harmonic mean.

Sol. $\bar{x}=\dfrac{1}{\beta(m, n)}\displaystyle\int_0^1 x^{n+1-1}(1-x)^{m-1}\,dx=\dfrac{\beta(n+1, m)}{\beta(n, m)}$

$$=\frac{\left|\overline{n+1}\right.\left|\overline{m}\right.}{\left|\overline{n+m+1}\right.}\cdot\frac{\left|\overline{n+m}\right.}{\left|\overline{n}\right.\left|\overline{m}\right.}=\frac{n}{n+m}$$

$$\mu_2'(0)=\frac{1}{\beta(m, n)}\int_0^1 x^{n+2-1}(1-x)^{m-1}\,dx=\frac{\beta(n+2, m)}{\beta(n, m)}$$

$$=\frac{\left|\overline{n+2}\right.\left|\overline{m}\right.}{\left|\overline{m+n+2}\right.}\cdot\frac{\left|\overline{m+n}\right.}{\left|\overline{m}\right.\left|\overline{n}\right.}=\frac{n(n+1)}{(m+n+1)(m+n)}$$

$\therefore \quad \mu_2=\mu_2'(0)-\bar{x}^2=\dfrac{n(n+1)}{(m+n+1)(m+n)}-\dfrac{n^2}{(m+n)^2}$

$$=n\left[\frac{(n+1)(m+n)-n(m+n+1)}{(m+n)^2(m+n+1)}\right]$$

$$=\frac{mn}{(m+n)^2(m+n+1)}$$

$\therefore \quad s.d.=\sqrt{\dfrac{mn}{m+n+1}}\cdot\dfrac{1}{m+n}$

H.M. is given by

$$\frac{1}{H}=\frac{1}{\beta(m, n)}\int_0^1 x^{n-2}(1-x)^{m-1}\,dx=\frac{\beta(n-1, m)}{\beta(n, m)}$$

$$=\frac{\left|\overline{n-1}\right.\left|\overline{m}\right.}{\left|\overline{n+m-1}\right.}\cdot\frac{\left|\overline{m+n}\right.}{\left|\overline{n}\right.\left|\overline{m}\right.}=\frac{(m+n-1)}{n-1}$$

$\therefore\ H=\dfrac{n-1}{m+n-1}.$

Ex. 9-14. *Prove that for the dist*

$$dP=\frac{1}{\beta(m, n)}\ \frac{x^{m-1}}{(1+x)^{m+n}}\,dx,\ 0\leqslant x<\infty\ ;\ n>2$$

variance is $\dfrac{m(m+n-1)}{(n-1)^2(n-2)}$. *Find also the mode and moment of rth order about the origin.*

Sol. $$\bar{x}=\frac{1}{\beta(m,n)}\int_0^\infty \frac{x^{m+1-1}}{(1+x)^{m+n}}\,dx=\frac{\beta(m+1,n-1)}{\beta(m,n)}$$

$$=\frac{|\overline{m+1}\;|\overline{n-1}}{|\overline{m+n}}\cdot\frac{|\overline{m+n}}{|\overline{m}\;|\overline{n}}=\frac{m}{n-1}$$

$$\mu_2'(0)=\frac{1}{\beta(m,n)}\int_0^\infty \frac{x^{m+2-1}}{(1+x)^{m+n}}\,dx=\frac{\beta(m+2,n-2)}{\beta(m,n)}$$

$$=\frac{|\overline{m+2}\;|\overline{n-2}}{|\overline{m+n}}\cdot\frac{|\overline{m+n}}{|\overline{m}\;|\overline{n}}=\frac{m(m+1)}{(n-1)(n-2)}$$

$$\therefore\quad \mu_2=\frac{m(m+1)}{(n-1)(n-2)}-\frac{m^2}{(n-1)^2}$$

$$=\frac{m}{(n-1)^2(n-2)}\{(m+1)(n-1)-m(n-2)\}$$

$$=\frac{m(m+n-1)}{(n-1)^2(n-2)}$$

$$\mu_r'(0)=\frac{1}{\beta(m,n)}\int_0^\infty \frac{x^{m+r-1}}{(1+x)^{m+n}}\,dx=\frac{\beta(m+r,n-r)}{\beta(m,n)}$$

$$=\frac{|\overline{m+r}\;|\overline{n-r}}{|\overline{m}\;|\overline{n}}=\frac{(m+r-1)(m+r-2)\ldots\ldots m}{(n-1)(n-2)\ldots\ldots(n-r)}\qquad (r<n)$$

Mode is that value of x for which

$$f(x)=\frac{1}{\beta(m,n)}\cdot\frac{x^{m-1}}{(1+x)^{m+n}}$$

is maximum.

$\therefore$ Modal value x is *s.t.*

$$f'(x)=0 \text{ and } f''(x)<0$$

Now $f'(x)=0$ gives

$$x^{m-2}(1+x)^{m+n-1}\{(m-1)(1+x)-(m+n)x\}=0$$

$$\therefore\quad x=0,\,-1,\,\frac{m-1}{n+1}$$

At $x=0$, $f(x)=0$ which is the least value of $f(x)$. $x=-1$ do not belong to the range of x.

$\therefore$ At $x=\frac{m-1}{n+1}$, $f(x)$ is maximum.

$\therefore \quad x=\frac{m-1}{n+1}$ is the mode. $(m > 1)$

Ex. 9-15. *Show that for the gama dist*

$$dF=\frac{1}{m\,!}\,e^{-x}.x^{m-1}, \qquad 0\leqslant x\leqslant \infty,\ m>0$$

mean=variance=m. Find $\mu_r'(0)$ and harmonic mean.

Sol. $$\bar{x}=\frac{1}{\lfloor m}\int_0^\infty x^{m+1-1}e^{-x}dx=\frac{\lfloor m+1}{\lfloor m}=m$$

$$\mu_2'(0)=\frac{1}{\lfloor m}\int_0^\infty x^{m+2-1}e^{-x}dx=\frac{\lfloor m+2}{\lfloor m}=m(m+1)$$

$\therefore \quad \mu_2=\mu_2'(0)-\bar{x}^2=m^2+m-m^2=m$

$$\mu_r'(0)=\frac{1}{\lfloor m}\int_0^\infty x^{r+m-1}e^{-x}dx=\frac{\lfloor m+r}{\lfloor m}=m(m+1)\ldots\ldots(m+r-1)$$

H.M. is given by

$$\frac{1}{H}=\frac{1}{m\,!}\int_0^\infty x^{m-1-1}e^{-x}dx=\frac{\lfloor m-1}{\lfloor m}=\frac{1}{m-1}$$

$\therefore \quad H=m-1.$

Ex. 9-16. *For a continuous dist*

$$dF=y_0e^{-\frac{x^2}{2}}\,x^{n-1}dx,\ 0\leqslant x\leqslant \infty$$

show that $\mu_1'(0)=\sqrt{2}\,\dfrac{\left\lfloor\frac{n+1}{2}\right.}{\left\lfloor\frac{n}{2}\right.}$

and $\qquad \mu_2'(0)=n$

9.16

Sol. y_0 is given by

$$y_0\int_0^\infty e^{-\frac{x^2}{2}}x^{n-1}dx=1$$

Put $$\frac{x^2}{2}=t$$

$\therefore$ $$y_0\int_0^\infty e^{-t}(2t)^{\frac{n-2}{2}}dt=1$$

or $$2^{\frac{n-2}{2}}.y_0\int_0^\infty e^{-t}.t^{\frac{n}{2}-1}dt=1$$

or $$2^{\frac{n}{2}-1}.y_0\Gamma\left(\frac{n}{2}\right)=1$$

$\therefore$ $$y_0=\frac{1}{\Gamma\left(\frac{n}{2}\right).2^{\frac{n}{2}-1}}$$

Also by def.

$$\mu_r'(0)=y_0\int_0^\infty x^r e^{-\frac{x^2}{2}}.x^{n-1}dx=y_0.\int_0^\infty e^{-\frac{x^2}{2}}x^{n-r-1}dx$$

Put $$\frac{x^2}{2}=t.$$

$\therefore$ $$\mu_r'(0)=2^{\frac{n+r-2}{2}}.y_0\int_0^\infty e^{-t}.t^{\frac{n+r}{2}-1}dt$$

$$=2^{\frac{n+r-2}{2}}.\frac{1}{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}.\Gamma\left(\frac{n+r}{2}\right)=\frac{2^{r/2}}{\Gamma\left(\frac{n}{2}\right)}.\Gamma\left(\frac{n+r}{2}\right)$$

$$\therefore \quad \mu_1'(0)=\frac{\sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \text{ and } \mu_2'(0)=2.\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}\right)}=n.$$

Ex. 9-17. *A frequency f^n in the range $(-3, 3)$ is defined by*

$$f(x)=\frac{1}{16}(3+x)^2 \qquad -3\leqslant x\leqslant -1$$

$$=\frac{1}{16}(\ -2x^2) \qquad -1\leqslant x\leqslant 1$$

$$=\frac{1}{16}(3-x)^2 \qquad 1\leqslant x\leqslant 3$$

Find the mean and the s.d. of the dist.

Sol. Total frequency $=\displaystyle\int_{-3}^{3} f(x)\,dx=\frac{1}{16}\int_{-3}^{-1}(3+x)^2dx$

$$+\frac{1}{16}\int_{1}^{1}(6-2x^2)dx+\frac{1}{16}\int_{-1}^{3}(3-x)^2dx$$

$$=\frac{1}{8}\int_{1}^{3}(3-x)^2dx+\frac{1}{8}\int_{0}^{1}(6-2x^2)dx$$

$$=\frac{1}{8}\left[-\frac{(3-x)^3}{3}\right]_1^3+\frac{1}{8}\left(6-\frac{2}{3}\right)=1$$

$$\bar{x}=\int_{-3}^{3} xf(x)dx=\frac{1}{16}\int_{-3}^{-1}x(3+x)^2dx+\frac{1}{16}\int_{-1}^{1}x(6-2x^2)dx$$

$$+\frac{1}{16}\int_{1}^{3}x(3-x)^2\,dx$$

For first integral change x to $-x$

$$\therefore \quad \bar{x}=-\frac{1}{16}\int_{1}^{3} x(3-x)^2dx+\frac{1}{16}\int_{-1}^{1} x(6-2x^2)dx$$

$$+\frac{1}{16}\int_{1}^{3} x(3-x)^2\, dx$$

$$-\frac{1}{16}\int_{-1}^{1} x(6-2x^2)dx=0$$

$$\therefore \quad \mu_2=\int_{-3}^{3} x^2f(x)dx=\frac{1}{16}\int_{-3}^{-1} x^2(3+x)^2dx+\frac{1}{16}\int_{-1}^{1}x^2(6-2x^2)dx$$

$$+\frac{1}{16}\int_{1}^{3} x^2(3-x)^2dx$$

For first integral change x to $-x$

$$\therefore \quad \mu_2=\frac{1}{8}\int_{1}^{3} x^2(3-x)^2dx+\frac{1}{8}\int_{0}^{1} x^2(6-2x^2)dx.$$

$$=\frac{1}{8}\int_{1}^{3} x^2(x^2-6x+9)dx+\frac{1}{8}\int_{0}^{1} (6x^2-2x^4)dx$$

$$=\frac{1}{8}\left|\frac{x^5}{5}-\frac{3}{2}\,x^4+3x^3\right|_{1}^{3}+\frac{1}{8}\left|2x^3-\frac{2}{5}\,x^5\right|_{0}^{1}$$

$$=\frac{4}{5}+\frac{1}{5}=1$$

$\therefore$ *s.d.* $=1$

Ex. 9-18. *The dist. of a variate x in the range (0, 2) is defined by*

$$(x)=x^3 \qquad 0\leqslant x<1f$$
$$=(2-x)^3 \qquad 1\leqslant x<2$$

Calculate the mean, s.d. and the mean deviation about the mean of the above dist.

Sol. Total Freq.$=\int_0^2 f(x)dx=\int_0^1 x^3dx+\int_1^2 (2-x)^3\, dx$

$$=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$\therefore \qquad \mu_1'(0)=2\int_0^2 xf(x)dx=2\int_0^1 x^4\, dx+2\int_1^2 x(2-x)^3\, dx$

$$=\frac{2}{5}+2\left\{\left|-\frac{(2-x)^4}{4}.x\right|_1^2+\frac{1}{4}\int_1^2 (2-x)^4\, dx\right\}$$

$$=\frac{2}{5}+2\left\{\frac{1}{4}+\frac{1}{20}\right\}=1$$

$$\mu_2'(0)=2\int_0^2 x^2 f(x)dx=2\int_0^1 x^5\, dx+2\int_1^2 x^2(2-x)^3\, dx$$

$$=\frac{1}{3}+2\left\{\left|-\frac{(2-x)^4}{4}.x^2\right|_1^2+\frac{1}{2}\int_1^2 x(2-x)^4\, dx\right\}$$

$$=\frac{1}{3}+\frac{1}{2}+\left\{\left|-\frac{(2-x)^5}{5}.x\right|_1^2+\frac{1}{5}\int_1^2 (2-x)^5\, dx\right\}$$

$$=\frac{5}{6}+\frac{1}{5}+\frac{1}{30}=\frac{16}{15}$$

$\therefore \qquad \mu_2=\frac{16}{15}-1=\frac{1}{15}$

$\therefore \qquad s.d.=\frac{1}{\sqrt{15}}$

Mean deviation about mean$=\int_0^2 |\,x-1\,|\,f(x)dx$

$$=2\int_0^1 (1-x)x^3\,dx+2\int_1^2 (x-1)(2-x)^3\,dx$$

$$=\frac{1}{10}+2\left\{\left|-\frac{(2-x)^4}{4}(x-1)\right|_1^2+\frac{1}{4}\int_1^2 (2-x)^4\,dx\right\}$$

$$=\frac{1}{10}+\frac{1}{10}=\frac{1}{5}$$

Ex. 9-19. *Find y_0, μ_2, μ_3 and μ_4 for the dist.*

$$dF=y_0\left(1+\frac{\alpha}{2}x\right)^{\left(\frac{4}{\alpha^2}-1\right)} e^{-\frac{2x}{\alpha}}\,dx \quad -\frac{2}{\alpha}<x<\infty$$

Sol. y_0 is given by

$$y_0\int_{-2/\alpha}^{\infty}\left(1+\frac{\alpha}{2}x\right)^{\left(\frac{4}{\alpha^2}-1\right)} e^{-\frac{2x}{\alpha}}\,dx=1$$

Put $\quad 1+\frac{\alpha}{2}x=y$

$$\therefore \quad 1=\frac{2y_0}{\alpha}\int_0^\infty y^{\frac{4}{\alpha^2}-1}\, e^{-\frac{4}{\alpha^2}(y-1)}\,dy$$

Put $\frac{2}{\alpha}=\beta$

$$=\beta y_0\int_0^\infty y^{\beta^2-1}\, e^{-\beta^2(y-1)}\,dy$$

Put $\quad y\beta^2=t$

$$=\frac{e^{\beta^2}y_0}{\beta^{2\beta^2-1}}\int_0^\infty t^{\beta^2-1}\, e^{-t}\,dt=\frac{e^{\beta^2}y_0}{\beta^{2\beta^2-1}}\left|\overline{\beta^2}\right.$$

$$\therefore \quad y_0=\frac{\beta^{2\beta^2-1}}{e^{\beta^2}\left|\overline{\beta^2}\right.}$$

$$\mu_r'(-\beta)=y_0\int_{-\beta}^{\infty}(x+\beta)^r\left(1+\frac{x}{\beta}\right)^{\beta^2-1}e^{-x\beta}dx$$

$$=\frac{y_0}{\beta^{\beta^2-1}}\int_{-\beta}^{\infty}(x+\beta)^{\beta^2+r-1}\ e^{-x\beta}dx$$

Put $\qquad x+\beta=\frac{y}{\beta}$

$$=\frac{y_0}{\beta^{\beta^2-1}}\int_0^{\infty}\left(\frac{y}{\beta}\right)^{\beta^2+r-1}\ e^{-y+\beta^2}\left(\frac{dy}{\beta}\right)$$

$$=\frac{y_0e^{\beta^2}}{\beta^{2\beta^2+r-1}}\int_0^{\infty}y^{\beta^2+r-1}e^{-y}\,dy$$

$$=\frac{1}{\lfloor\beta^2.\beta^r}\cdot\lfloor\beta^2+r$$

$$\therefore\ \mu_1'(-\beta)=\frac{\lfloor\beta^2+1}{\beta\lfloor\beta^2}=\beta,\ \mu_2'(-\beta)=\frac{\lfloor\beta^2+2}{\beta^2\lfloor\beta^2}=(\beta^2+1)$$

$$\mu_3'(-\beta)=\frac{\lfloor\beta^2+3}{\lfloor\beta^2.\beta^3}=\frac{(\beta^2+2)(\beta^2+1)}{\beta}$$

and $$\mu_4=\frac{\lfloor\beta^2+4}{\lfloor\beta^2.\beta^4}=\frac{(\beta^2+3)(\beta^2+2)(\beta^2+1)}{\beta^2}$$

$$\mu_2=\mu_2'-\{\mu_1'\}^2=\beta^2+1-\beta^2=1$$

$$\mu_3=\mu_3'-3\mu_2'\mu_1'+2\{\mu_1'\}^3=\frac{(\beta^2+2)(\beta^2+1)}{\beta}-3\beta(\beta^2+1)+2\beta^3$$

$$=\frac{2}{\beta}=\alpha$$

$$\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'\{\mu_1'\}^2-3\{\mu_1'\}^4$$

$$=\frac{(\beta^2+3)(\beta^2+2)(\beta^2+1)}{\beta^2}-4(\beta^2+2)(\beta^2+1)+6\beta^2(\beta^2+1)-3\beta^4$$

$$=3+\frac{6}{\beta^2}=3+\frac{3}{2}\alpha^2$$

Ex. 9-20. *(a) Show that for the dist*

$$dF=y_0\left(1-\frac{x^2}{a^2}\right)^{-p}dx,\ -a\leq x\leq a,\ 0<p<1$$

$$\mu_r=\frac{(r-1)a^2}{r+1-2p}\ \mu_{r-2}$$

(b) Express 'a' and 'p' in terms of σ *and* β_2.

Sol. (a) By def.,

$$\mu_1(0)=y_0\int_{-a}^{a}x\left(1-\frac{x^2}{a^2}\right)^{-p}dx=0$$

$$\therefore\ \mu_r=\mu_r'(0)=y_0\int_{-a}^{a}x^r\left(1-\frac{x^2}{a^2}\right)^{-p}dx$$

$$=y_0\,a^2\int_{-a}^{a}x^{r-2}\left(\frac{x^2}{a^2}-1+1\right)\left(1-\frac{x^2}{a^2}\right)^{-p}dx$$

$$=a^2.y_0\int_{-a}^{a}x^{r-2}\left(1-\frac{x^2}{a^2}\right)^{-p}dx-y_0.a^2\int_{-a}^{a}x^{r-2}\left(1-\frac{x^2}{a^2}\right)^{1-p}dx$$

$$=a^2\,\mu_{r-2}-y_0a^2\left\{\left|\frac{x^{r-1}}{r-1}\left(1-\frac{x^2}{a^2}\right)^{1-p}\right|_{-a}^{a}\right.$$

$$\left.+\frac{2(1-p)}{a^2}\int_{-a}^{a}\frac{x^r}{r-1}\left(1-\frac{x^2}{a^2}\right)^{-p}dx\right\}$$

$$=a^2\,\mu_{r-2}-2\,\frac{(1-p)}{r-1}\ \mu_r$$

$$\therefore\ \mu_r=\frac{(r-1)a^2}{r+1-2p}\ \mu_{r-2}$$

(b) Put $r=2,\ 4$

$$\therefore\quad \sigma^2=\mu_2=\frac{a^2}{3-2p}\ \mu_0=\frac{a^2}{3-2p}$$

$$\therefore \quad a^2=(3-2p)\,\sigma^2$$

and $$\mu_4=\frac{3a^2}{5-2p}\,\mu_2=\frac{3(3-2p)}{5-2p}\,\sigma^4$$

$$\therefore \quad \beta_2=\frac{\mu_4}{\mu_2^2}=\frac{3(3-2p)}{5-2p}$$

$$\therefore \quad p=\frac{9-5\beta_2}{2(3-\beta_2)}$$

Ex. 9-21. *For $\beta_1(l, m)$ variate show that*

$$\log G=\frac{\partial}{\partial l}\left\{\log \overline{|l} +\log \overline{|m}-\log \overline{|l+m}\right\}$$

Sol. For $\beta_1(l, m)$ variate x,

$$dF=\frac{1}{\beta(l, m)}\,x^{l-1}(1-x)^{m-1}\,dx \quad 0<x<1,\ l,\ m>0$$

$$\therefore \quad \log G=\frac{1}{\beta(l, m)}\int_0^1 \log x . x^{l-1}(1-x)^{m-1}\,dx$$

$$=\frac{1}{\beta(l, m)}\,\frac{\partial}{\partial l}\left\{\int_0^1 x^{l-1}(1-x)^{m-1}\,dx\right\}$$

$$=\frac{1}{\beta(l, m)}\,\frac{\partial}{\partial l}\{\beta(l, m)\}=\frac{\partial}{\partial l}\{\log \beta(l, m)\}$$

$$=\frac{\partial}{\partial l}\left\{\log \frac{\overline{|l}\ \overline{|m}}{\overline{|l+m}}\right\}$$

$$=\frac{\partial}{\partial l}\left\{\log \overline{|l} +\log \overline{|m}-\log \overline{|l+m}\right\}$$

Ex. 9-22. *Show that for the dist*

$$dF=y_0\left\{1-\frac{|x-b|}{a}\right\}dx,\ b-a<x<b+a$$

$y_0=\dfrac{1}{a}$, *mean*$=b$ *and variance*$=\dfrac{a^2}{6}$.

Sol. y_0 is given by

$$y_0\int_{b-a}^{b+a}\left\{1-\frac{|x-b|}{a}\right\}dx=1$$

Put $\quad x-b=y$

$$\therefore \quad y_0\int_{-a}^{a}\left\{1-\frac{|y|}{a}\right\}dy=1$$

or $$2y_0\int_{0}^{a}\left(1-\frac{y}{a}\right)dy=1$$

or $$2y_0\left\{y-\frac{y^2}{2a}\right\}_0^a=1$$

$$\therefore \quad y_0=\frac{1}{a}$$

$$\text{Mean}=y_0\int_{b-a}^{b+a}x\left\{1-\frac{|x-b|}{a}\right\}dx$$

$$=y_0\int_{-a}^{a}(y+b)\left\{1-\frac{|y|}{a}\right\}dy=2by_0\int_{0}^{a}\left(1-\frac{y}{a}\right)dy$$

$$=by_0a=b$$

$$\mu_2=y_0\int_{b-a}^{b+a}(x-b)^2\left\{1-\frac{|x-b|}{a}\right\}dx=y_0\int_{-a}^{a}y^2\left\{1-\frac{|y|}{a}\right\}dy$$

$$=2y_0\int_{0}^{a}y^2\left(1-\frac{y}{a}\right)dy=2y_0\ \frac{a^3}{12}=\frac{a^2}{6}$$

Ex. 9-23. *Show that the f*ⁿ

$$F(x)=\begin{cases}0 & x<-a\\ \frac{1}{2}\left(\frac{x}{a}+1\right) & -a\leqslant x\leqslant a\\ 1 & x>a\end{cases}$$

is a distribution function.

Sol. Evidently

$$F(x=\infty)=1, \qquad F(x=-\infty)=0$$

and

$$F'(x)=\begin{cases} 0 & x<-a \\ \dfrac{1}{2a} & -a\leqslant x\leqslant a \\ 0 & x>a \end{cases}$$

Evidenty $F'(x)\geqslant 0$

$\therefore$ $F(x)$ is a distribution function.

Ex. 9-24. *For the distribution with density*

$$f(x)=\frac{1}{\pi}\,\frac{1}{1+x^2},\ -\infty<x<\infty$$

Find mean, mode, median, variance, first and third quartiles and distribution function.

Sol. Distribution function is given by

$$F(x)=\frac{1}{\pi}\int_{-\infty}^{x}\frac{dx}{1+x^2}=\frac{1}{\pi}\left\{\tan^{-1}x+\frac{\pi}{2}\right\}$$

$$=\frac{1}{\pi}\tan^{-1}x+\frac{1}{2}$$

For median, $\quad F(x)=\dfrac{1}{2}$

$\therefore \qquad x=0$

For first quartile, $F(x)=\dfrac{1}{4}$

or $\quad \dfrac{1}{\pi}\tan^{-1}x+\dfrac{1}{2}=\dfrac{1}{4}$

or $\qquad \tan^{-1}x=-\dfrac{\pi}{4}$

$\therefore \qquad x=-1$

For third quartile, $F(x)=\dfrac{3}{4}$

$\therefore \qquad x=1$

$$\text{Mean}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,dx=0$$

$$\therefore \quad \mu_2=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}\,dx=\frac{2}{\pi}\int_0^{\infty}\left\{1-\frac{1}{1+x^2}\right\}dx$$

$$=\frac{2}{\pi}\left\{\Big|x\Big|_0^{\infty}-\frac{\pi}{2}\right\}$$

Evidently μ_2 does not exist.

For modal value $x, f'(x)=0$

$$\therefore \qquad \frac{1}{\pi}\frac{-2x}{(1+x^2)^2}=0$$

$\therefore$ Modal value$=0$

Ex. 9·25. *The prob. density function of coded measurements of pitch diameter of threads of a fitting is given by*

$$f(x)=\frac{1}{(1+x)^2},\ 0<x<\infty$$

Find the distribution function of the dist. Hence obtain (i) $P(x>2)$, (ii) the median and the quartiles of the distribution. Investigate whether the mean and the variance of the dist exists.

Sol. Distribution function is given by

$$F(x)=\int_0^x\frac{1}{(1+x)^2}\,dx=\left|-\frac{1}{1+x}\right|_0^x$$

$$=1-\frac{1}{1+x}=\frac{x}{1+x} \text{ for } x>0$$

and $\qquad F(x)=0$ for $x<0$

(i) $\therefore \quad P(x>2)=1-P(x<2)=1-F(2)=1-\dfrac{2}{1+2}=\dfrac{1}{3}$

(ii) For median value x, $F(x)=\dfrac{1}{2}$

or $\qquad \dfrac{x}{1+x}=\dfrac{1}{2}$

$\therefore \qquad x=1$

For first quartile value x, $F(x)=\dfrac{1}{4}$

or $\qquad \dfrac{x}{1+x}=\dfrac{1}{4}$

$$\therefore \qquad x=\frac{1}{3}$$

And for third quartile value x, $F(x)=\frac{3}{4}$

$$\therefore \qquad \frac{x}{1+x}=\frac{3}{4}$$

$$\therefore \qquad x=3$$

$$\text{Mean}=\int_0^\infty \frac{x}{(1+x)^2}dx=\int_0^\infty \frac{1}{1+x}dx-\int_0^\infty \frac{dx}{(1+x)^2}$$

$$=\Big|\log(1+x)\Big|_0^\infty-1$$

Since as $x\to\infty$, $\log(1+x)\to\infty$, mean does not exist. Hence variance will also not exist.

Ex. 9-26. *A bombing plane carrying three bombs flies directly above a railroad track. If a bomb falls within 40 feet of track, the track will be sufficiently damaged to disrupt the trafic. With a certain bomb-sight the points of impact of a bomb have the prob. density function*

$$f(x)=\frac{100+x}{10^4} \qquad \text{when } -100\leqslant x\leqslant 0$$

$$=\frac{100-x}{10^4} \qquad \text{when } 0\leqslant x\leqslant 100$$

$$=0 \qquad \text{elsewhere}$$

where x represents the vertical deviation (in feet) from the aiming point, which is the track in this case. Find the distribution function. If all the three bombs are used, what is the prob. that the track will be damaged?

Sol. Let $F(x)$ be distribution function.

Then $$F(x)=\int_{-100}^{x} f(x)dx=\int_{-100}^{x}\frac{100+x}{10^4}dx \text{ if } -100\leqslant x\leqslant 0$$

$$=\left[\frac{100x+(x^2/2)}{10^4}\right]_{-100}^{x}=\frac{1}{10^4}\left[100x+\frac{x^2}{2}+\frac{10^4}{2}\right]$$

$$F(x)=\int_{-100}^{0}\frac{100+x}{10^4}\,dx+\int_{0}^{x}\frac{100-x}{10^4}\,dx \qquad \text{if } 0\leqslant x\leqslant 100$$

$$=\frac{1}{10^4}\left[100x+\frac{x^2}{2}\right]_{-100}^{0}+\frac{1}{10^4}\left[100x-\frac{x^2}{2}\right]$$

$$=\frac{1}{10^4}\left[100x-\frac{x^2}{2}+\frac{10^4}{2}\right]$$

$$F(x)=0 \qquad \text{if } x<-100$$

and $$F(x)=1 \qquad \text{if } x>100$$

Now Prob. for a bomb to fall within 40 feet of the track

$$=\int_{-40}^{0} f(x)dx+\int_{0}^{40} f(x)dx$$

$$=\int_{-40}^{0}\frac{100+x}{10^4}\,dx+\int_{0}^{40}\frac{100-x}{10^4}\,dx$$

$$=\frac{2}{10^4}\left[100x-\frac{x^2}{2}\right]_{0}^{40}=\frac{2}{10^4}\{4000-800\}=\frac{16}{25}$$

$\therefore$ Prob. for a bomb not to fall within 40 feet of the track

$$=1-\frac{16}{25}=\frac{9}{25}$$

$\therefore$ Prob. for all the three bombs not to fall within 40 feet of the track

$$=\left(\frac{9}{25}\right)^3$$

$\therefore$ Prob. of at least one bomb falling within 40 feet of the track

$$=1-\left(\frac{9}{25}\right)^3$$

$\therefore$ Prob. of the track being damaged $=$ Prob. of at least one bomb falling within 40 feet of the track

$$=1-\left(\frac{9}{25}\right)^3$$

Ex. 9·27. *Suppose the life in hours of a certain kind of radio tube has the density function*

$$f(x)=\frac{100}{x^2} \quad \text{when } x \geqslant 100$$
$$=0 \quad \text{otherwise}$$

Find the prob. that none of the three such tubes in a given radioset will have to be replaced during the first 150 hours of operation ? What is the prob. that all three of the original tubes will have been replaced during the first 150 hours ?

Sol. Let x hours be the life of the tube.

Then $$P\{x \leqslant 150\}=\int_0^{150} f(x)dx=100\int_{100}^{150}\frac{1}{x^2}\,dx=\frac{1}{3}$$

$\therefore$ By compound prob. theorem, prob. that all the three tubes will have to be replaced during the first 150 hours of operation

$$=\left(\frac{1}{3}\right)^3=\frac{1}{27}$$

Also $P(x>150)=1-\frac{1}{3}=\frac{2}{3}$

$\therefore$ Prob. that none of the three tubes will have to be replaced during first 150 hours of operation

$$=\left(\frac{2}{3}\right)^3=\frac{8}{27}$$

Ex. 9·28. *Assuming $F(x)$ to be cumulative probability distribution function for a variable x and given that $F(x_1)=0{\cdot}5$, $F(x_2)=0{\cdot}7$ and $F(x_3)=0{\cdot}8$, find the prob. that the variable will lie between x_1 and x_2 ; x_2 and x_3. Calculate the prob. that two independent observations X_1 and X_2 will lie between $-\infty$ to x_1 and x_3 to ∞. It may be assumed that x takes values from $-\infty$ to ∞.*

Sol. Let $f(x)$ be the density function.

Then $$P\{x_1 \leqslant x \leqslant x_2\}=\int_{x_1}^{x_2} f(x)dx$$

$$=\int_{x_1}^{-\infty} f(x)dx+\int_{-\infty}^{x_2} f(x)dx$$

$$=\int_{-\infty}^{x_2} f(x)dx-\int_{-\infty}^{x_1} f(x)dx$$

$$=F(x_2)-F(x_1)=0{\cdot}7-0{\cdot}5=0{\cdot}2$$

$$P\{x_2 \leqslant x \leqslant x_3\}=\int_{x_2}^{x_3} f(x)dx=\int_{-\infty}^{x_3} f(x)dx-\int_{-\infty}^{x_2} f(x)dx$$

$$=F(x_3)-F(x_2)=0{\cdot}8-0{\cdot}7=0{\cdot}1$$

Now $P\{-\infty < X_1 \leqslant x_1\}=F(x_1)=0{\cdot}5$

$$P\{x_3 \leqslant X_2 < \infty\}=1-F\{-\infty < X_2 \leqslant x_3\}$$

$$=1-F(x_3)=1-0{\cdot}8=0{\cdot}2$$

Since X_1 and X_2 are independent observations, reqd. prob.

$$=(0{\cdot}5)(0{\cdot}2)=0{\cdot}1$$

Ex. 9-29. *A distribution function is defined as follows*

$$F(x)=\begin{cases} 0 & x \leqslant 1 \\ \dfrac{1}{16}(x-1)^4 & 1 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$$

Find the density function $f(x)$. Find the mean of x and the median.

Sol. Density function is given by

$$F'(x)=f(x)=\frac{1}{4}(x-1)^3 \qquad 1 \leqslant x \leqslant 3$$

$$=0 \qquad \text{otherwise}$$

$$\therefore \qquad \text{Mean}=\frac{1}{4}\int_1^3 x(x-1)^3\, dx=\frac{1}{4}\left[\left|\frac{(x-1)^4}{4}x\right|_1^3-\frac{1}{4}\int_1^3 (x-1)^4\, dx\right]$$

$$=3-\frac{1}{80}\Big|(x-1)^5\Big|_1^3=3-\frac{2}{5}=2{\cdot}6$$

For median $F(x)=\dfrac{1}{2}$

$$\therefore \quad \frac{1}{16}(x-1)^4=\frac{1}{2}$$

$$\therefore \quad x=1+(8)^{\frac{1}{4}}.$$

Ex. 9-30. *Determine m so that the following function represents the density function*

$$f(x)=\begin{cases} 0 & x\leq -1 \\ m(x+1) & -1<x\leq 3 \\ 4m & 3<x\leq 4 \\ 0 & x>4 \end{cases}$$

Find the value of x about which the mean deviation of this dist is least.

Sol. m is given by

$$\int_{-1}^{4} f(x)\,dx=1$$

or $$\int_{-1}^{3} m(x+1)dx+\int_{3}^{4} 4m\,dx=1$$

or $$m\left|\frac{(x+1)^2}{2}\right|_{-1}^{3}+4m\left|x\right|_{3}^{4}=1$$

or $$8m+4m=1$$

or $$m=\frac{1}{12}$$

Since the mean deviation is least about median, it is required to find median. Let it be 'a'.

Then $$\int_{-1}^{a} f(x)dx=\frac{1}{2}.$$

Let if possible $a>3$

Then $$m\int_{-1}^{3}(x+1)dx+4m\int_{3}^{a}dx=\frac{1}{2}$$

or $$8m+4m(a-3)=\frac{1}{2}$$

or $$\frac{1}{3}(a-3)=\frac{1}{2}-8m=\frac{1}{2}-\frac{2}{3}=-\frac{1}{6}$$

which is not possible.

$\therefore \quad a \not> 3 \quad$ *i.e.*, $a<3$.

$$\therefore \quad m\int_{-1}^{a}(x+1)dx=\frac{1}{2}$$

$$\therefore \quad (a+1)^2=12$$

$$\therefore \quad a=2\sqrt{3}-1.$$

Ex. 9-31. *A country filling station is supplied with gasoline once a week. If its weekly volume x of sales in thousands of gallons is distributed by $f(x)=5(1-x)^4$, $0<x<1$, what must be the capacity of its tank in order that the prob. that its supply will be exhausted in a given week shall be 0·01 ?*

Sol. Let V be the volume of capacity of the tank in thousands of gallons.

Then prob. that the supply will be exhausted in a given week

$$=P(x \geqslant V)$$

$$\therefore \quad P(x>V)=0\cdot01$$

$$\therefore \quad P(x<V)=0\cdot99$$

$$\therefore \quad 5\int_{0}^{V}(1-x)^4dx=0\cdot99$$

or $$1-(1-V)^5=0\cdot99$$

or $$(1-V)^5=0\cdot01$$

$$\therefore \quad V=0\cdot602$$

$\therefore$ Tank capacity$=602$ gallons.

Ex. 9-32. *For continuous variable, show that the mean deviation is least when measured from the median.*

Sol. Let x be a continuous variable with density function $f(x)$. Now by def., mean deviation about an arbitrary point 'a' is given by

$$F(a)=E\,|\,x-a\,|=\int_{-\infty}^{\infty}|\,x-a\,|\,f(x)dx$$

$$=\int_{-\infty}^{a}(a-x)f(x)dx+\int_{a}^{\infty}(x-a)f(x)dx$$

Differentiating *w.r.t.* 'a' under the sign of integration

$$F'(a)=\int_{-\infty}^{a}f(x)dx-\int_{a}^{\infty}f(x)dx$$

and $$F''(a)=f(a)+f(a)=2f(a)$$

For $F(a)$ to be minimum, 'a' is given by

$$F'(a)=0 \text{ i.e., } \int_{-\infty}^{a}f(x)dx=\int_{a}^{\infty}f(x)dx$$

which implies that 'a' is the median.

If $f(a)\neq 0$. $F''(a)>0$ ($\because$ $f(a)>0$)

$\therefore$ $F(a)$ is minimum when 'a' is median.

If $f(a)=0$. $f(x)$ is minimum for $x=a$ ($\because$ $f(x) \nless 0$).

First derivative of $f(x)$ which is not zero for $x=a$ is of even order and is positive.

$\therefore$ First derivative of $F(a)$ which is not zero when 'a' is the median is of even order and is positive.

$\therefore$ $F(a)$ is minimum when x is median.

9·2. Tchebycheff's inequality.

Let x be a continuous variate with density function $f(x)$ and expected value zero. The variance of x is given by

$$\sigma_x^2=\int_{-\infty}^{\infty}x^2f(x)dx=\int_{-\infty}^{-\sigma_x t}x^2f(x)dx+\int_{-\sigma_x t}^{\sigma_x t}x^2f(x)dx+\int_{\sigma_x t}^{\infty}x^2f(x)dx$$

Now $$\int_{-\sigma_x t}^{\sigma_x t}x^2f(x)dx\geqslant 0$$

$\therefore$ $$\sigma_x^2\geqslant\int_{-\infty}^{-\sigma_x t}x^2f(x)dx+\int_{\sigma_x t}^{\infty}x^2f(x)dx$$

$$\geqslant \sigma_x^2 t^2 \left\{ \int_{-\infty}^{-\sigma_x t} f(x)dx + \int_{\sigma_x t}^{\infty} f(x)dx \right\}$$

$$= \sigma_x^2 t^2 \; P\{ \,|\, x \,|\, > \sigma_x t \}$$

$$\therefore \qquad \frac{1}{t^2} > P\{ \,|\, x \,|\, > \sigma_x t \}$$

Put $\qquad x = y - \bar{y}$

Then $\qquad E(x) = 0$ and $\sigma_x^2 = E(y - \bar{y})^2 = \sigma_y^2$

$$\therefore \qquad \frac{1}{t^2} > P\{ \,|\, y - \bar{y} \,|\, > \sigma_y t \}$$

which is Tchebycheff's inequality.

9-3. State and prove Weak law of large numbers.

Sol. Let $x_1, x_2, \ldots x_n$ be n independent random variables distributed in the same form with mean m and $\bar{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$

Then for any fixed $\epsilon > 0$,

$$\underset{n \to \infty}{\text{Lt}} \; P\{ \,|\, \bar{x} - m \,|\, > \epsilon \} = 0$$

Assume that the variances of $x_1, x_2, \ldots x_n$ exist and are equal to σ^2.

Then $\quad \text{Var}(\bar{x}) = E\left\{ \dfrac{x_1 + x_2 + \ldots + x_n}{n} - m \right\}^2$

$$= \frac{1}{n^2} E[(x_1 - m) + (x_2 - m) + \ldots\ldots + (x_n - m)]^2$$

$$= \frac{1}{n^2} \{ E(x_1 - m)^2 + E(x_2 - m)^2 + \ldots\ldots + E(x_n - m)^2 \}$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$\therefore$ From Tchebycheff's inequality

$$\frac{1}{t^2} > P\left[\,|\, \bar{x} - m \,|\, > \frac{\sigma}{\sqrt{n}} t \right[$$

Let $\qquad t \dfrac{\sigma}{\sqrt{n}} = > \epsilon$

$$\therefore \qquad t > \frac{\sqrt{n}\,\epsilon}{\sigma}$$

$$\therefore \quad \frac{\sigma^2}{n\epsilon^2} > P\{\,|\,\bar{x}-m\,|\,>\epsilon\}$$

which implies that for given $\epsilon>0$, the prob. can be made as small as we please by increasing n.

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \; P\{\,|\,\bar{x}-m\,|\,>\epsilon\}=0$$

Note. (1) The above law remains true even if we discard the requirement that σ^2 exists.

(2) The weak law of large numbers states a limiting property of sums of random variables. In fact, given any positive numbers ϵ and δ there is an N *s.t.*

$$P\{\,|\,\bar{x}-m\,|\,\geqslant \epsilon\} < \delta,\; n > N$$

The weak law states that $|\,\bar{x}-m\,|$ is ultimately small but not that every value is small; it might be that for some n it was large, although such cases could only occur infrequently. The strong law says that the prob. of such happening is extremely small. The law is true for variables which are identically distributed under the sole condition that μ exists; in other cases further conditions must be added.

Ex. 9-33. *Define stochastic convergence of the variate and show that in a infinite series of Bernoullian trials, the proportion of successes converges stochastically to the prob. of success in each trial as the number of trials increases indefinitely.*

Sol. Def. A variate x_n is said to converge stochastically (or in probability sense) to parameter θ if given any positive numbers ϵ and δ there is N *s.t.*

$$P\{\,|\,x_n-\theta\,|\,>\epsilon\} < \delta \text{ for } n \geqslant N$$

Let $x_1, x_2, \ldots\ldots x_n$ be the variates *s.t.*

$$\begin{aligned} x_i &= 1 \quad \text{if } i\text{th trial results in success} \\ &= 0 \quad \text{if } i\text{th trial results in failure} \end{aligned}$$

Then number of successes is given by

$$m = x_1 + x_2 + \ldots + x_n$$

and $\quad E(x_i) = 1.p + 0(1-p) = p,\; E(x_i^2) = p$

$$\therefore \quad \text{Var}(x_i) = p - p^2 = pq$$

where p is the prob. of success in each trial.

$$\therefore \quad E(m) = np \text{ and Var }(m) = npq$$

$$\therefore \quad E\left(\frac{m}{n}\right) = p$$

Also $$\text{Var}\left(\frac{m}{n}\right) = \frac{1}{n^2}\,\text{Var}(m) = \frac{pq}{n}$$

$\therefore$ From Tchebycheff's inequality

$$P\left\{\left|\frac{m}{n}-p\right|\geqslant\sqrt{\frac{pq}{n}}\;t\right\}\leqslant\frac{1}{t^2}$$

Let $$\sqrt{\frac{pq}{n}}.t=\epsilon$$

$$\therefore\quad P\left\{\left|\frac{m}{n}-p\right|\geqslant\epsilon\right\}\leqslant\frac{pq}{n\epsilon^2}$$

or $$P\left\{\left|\frac{m}{n}-p\right|\geqslant\epsilon\right\}<\delta$$

when $$n>\frac{qq}{\epsilon^2\delta}=N$$

which implies the result.

EXERCISE

1. Find the variance of the distribution

$$dF=\frac{1}{\pi}\;x\sin x \qquad 0\leqslant x\leqslant\pi$$

$$\left[\text{Ans. } 2-\frac{16}{\pi^2}\right]$$

2. If $f(x)=be^{-bx}$, $0<x<\infty$

where b is a positive constant. Find mean, μ_2 and μ_3.

$$\left[\text{Ans. } \frac{1}{b},\ \frac{1}{b^2},\ \frac{2}{b^3}\right]$$

3. Find μ_2, μ_3 and μ_4 for the distribution

$$dF=\frac{dx}{2a} \qquad -a\leqslant x\leqslant a \qquad \left[\text{Ans. } \frac{a^2}{3}, 0, \frac{a^4}{5}\right]$$

4. For the distribution

$$dF=x^m\frac{e^{-x}}{m\,!}\;m>0,\ 0\leqslant x\leqslant\infty$$

show that H.M. is m.

5. Find mean, mode and median of the distribution

$$dF=\sin x\,dx \qquad 0\leqslant x\leqslant\frac{x}{2}. \quad \left[\text{Ans. } 1\ ;\ \frac{\pi}{2}\ ;\ \frac{\pi}{3}\right]$$

6. Find the mode and the median of the curve

$$y=\frac{abx^{a-1}}{(1+bx^a)^2} \qquad b>0,\ a>1,\ 0\leqslant x\leqslant\infty$$

$$\left[\text{Ans. } \left\{\frac{a-1}{b(a+1)}\right\}^{1/a};\ \left(\frac{1}{b}\right)^{1/a}\right]$$

10

Theoretical Distribution

10.1. Binomial Distribution (B.D.)

Binomial Probability Distribution. *The B.P.D. of the variate x is*

$$P(x)={}^nc_x p^x q^{n-x},\ x=0,\ 1,\ 2,\ \ldots\ n$$

The variate x is called **Binomial Variate** (*B.V.*) *and n and p are called parameters of the distribution.*

Binomial Frequency Distribution. *The B.F.D. of the variate x is*

$$F(x)=N.{}^nc_x p^x q^{n-x},\ x=0,\ 1,\ 2,\ \ldots n$$

where N is the total frequency.

Derivation

Let there be N sets of n independent trials. Assume that the chance of success of each trial be p and of failure is q. Then

$$p+q=1.$$

Let us first calculate the chances of obtaining 0, 1, 2,...... successes in one set of n trials. Let us find the probability of obtaining x successes and $(n-x)$ failures in n trials. By the theorem of compound probability, the probability that first x trials are successes

$$=p\times p\times p\ldots x \text{ times}=p^x$$

and the probability that the remaining $(n-x)$ trials are failures

$$=q^{n-x}$$

$\therefore$ The probability of jointly getting first x trials successes and the remaining $(n-x)$ trials failures

$$=p^x q^{n-x}$$

Clearly this is also the probability for the x successes and $(n-x)$ failures to occur in any particular definite specified order. Since we are interested in any x trials being successes and x trials can be

chosen out of n in ${}^nc_x$ (mutually exclusive) ways, by the theorem of total probability, the probability of x successes in a series of n trials is given by

$$P(x)={}^nc_x p^x q^{n-x}$$

The chance of getting x successes in one set of n trials is ${}^nc_x p^x q^{n-x}$ means that out of one set ${}^nc_x p^x q^{n-x}$ sets will have r successes.

$\therefore$ Out of N sets $N\,{}^nc_x p^x q^{n-x}$ sets will have x successes.

$\therefore$ The frequencies of getting 0, 1, 2... successes in N sets of n trials each are Nq^n, $N\,{}^nc_1 pq^{n-1}$, $N\,{}^nc_2 p^2q^{n-2}$,.........

10.1.1. First four Moments About Mean

Binomial Probability distribution is

$$P(x)={}^nc_x p^x q^{n-x},\ x=0,\ 1,\ \ldots\ n$$

$$\begin{aligned}\mu_1'(0)&=\sum_{x=0}^{n} xP(x)=\sum_{x=0}^{n} x\,{}^nc_x p^x q^{n-x}\\&={}^nc_1pq^{n-1}+2\,{}^nc_2p^2q^{n-2}+\ldots+n\,{}^nc_np^n\\&=np\{q^{n-1}+(n-1)pq^{n-2}+\ldots\ldots+p^{n-1}\}\\&=np(q+p)^{n-1}\\&=np\end{aligned}$$

$$\begin{aligned}\mu_2'(0)&=\sum_{x=0}^{n} x^2P(x)=\sum_{x=0}^{n}\{x(x-1)+x\}\,{}^nc_xp^xq^{n-x}\\&=\sum_{x=0}^{n} x(x-1)\,{}^nc_xp^xq^{n-x}+\sum_{x=0}^{n} x\,{}^nc_xp^xq^{n-x}\\&=2.1\,{}^nc_2p^2q^{n-2}+3.2\,{}^nc_3p^3q^{n-3}+\ldots+n(n-1)p^n+np\\&=n(n-1)p^2(q^{n-2}+(n-2)pq^{n-3}+\ldots+p^{n-2})+np\\&=n(n-1)p^2(q+p)^{n-2}+np\\&=n(n-1)p^2+np\end{aligned}$$

$$\begin{aligned}\mu_2&=\mu_2'(0)-\{\mu_1'(0)\}^2=n(n-1)p^2+np-n^2p^2\\&=np(1-p)=npq\end{aligned}$$

$$\mu_3'(0)=\sum_{x=0}^{n} x^3P(x)$$

Put $\qquad x^3\equiv Ax+Bx(x-1)+Cx(x-1)(x-2)$

Equating co-efficients of x^3, x^2 and x

$$C=1$$

$$-3C+B=0 \quad \text{or} \quad B=3$$

and $\qquad A-B+2C=0 \quad \text{or} \quad A=1$

$\therefore \qquad x^3\equiv x+3x(x-1)+x(x-1)(x-2)$

$$\therefore \qquad \mu_3'(0)=\sum_{x=0}^{n}\{x+3x(x-1)+x(x-1)(x-2)\}{}^nc_xp^xq^{n-x}$$

$$=np+3n(n-1)p^2+n(n-1)(n-2)p^3$$

$$\therefore \qquad \mu_3=\mu_3'(0)-3\mu_2'(0)\mu_1'(0)+2\{\mu_1'(0)\}^3$$

$$=np+3n(n-1)p^2+n(n-1)(n-2)p^3$$
$$-3\{n(n-1)p^2+np\}np+2n^3p^3$$

$$=np+3n^2p^2-3np^2+n^3p^3-3n^2p^3+2np^3-3n^3p^3$$
$$+3n^2p^3-3n^2p^2+2n^3p^3$$

$$=np(1-3p+2p^2)$$

$$=np(1-p)(1-2p)$$

$$=npq(q-p)$$

$$\mu_4'(0)=\sum_{x=0}^{n}x^4P(x)$$

$$x_4\equiv x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7x(x-1)+x$$

$$\therefore \qquad \mu_4'(0)=\sum_{x=0}^{n}\{x(x-1)(x-2)(x-3)+6x(x-1)(x-2)$$
$$+7x(x-1)+x\}.{}^nc_xp^xq^{n-x}$$

$$=n(n-1)(n-2)(n-3)p^4+6n(n-1)(n-2)p^3+7n(n-1)p^2+np$$

$$\therefore \quad \mu_4=\mu_4'(0)-4\mu_3'(0)\mu_1'(0)+6\mu_2'(0)\{\mu_1'(0)\}^2-3\{\mu_1'(0)\}^4$$

$$=\{n(n-1)(n-2)(n-3)p^4+6n(n-1)(n-2)p^3+7n(n-1)p^2+np\}$$
$$-4\{n(n-1)(n-2)p^3+3n(n-1)p^2+np\}np$$
$$+6\{n(n-1)p^2+np\}n^2p^2-3n^4p^4$$

$$=(n^4-6n^3+11n^2-6n)p^4+6(n^3-3n^2+2n)p^3+7(n^2-n)p^2+np$$
$$-4(n^4-3n^3+2n^2)p^4-12(n^3-n^2)p^3-4n^2p^2$$
$$+6(n^4-n^3)p^4+6n^3p^3-3n^4p^4$$

$$=3n^2p^4-6np^4-6n^2p^3+12np^3+3n^2p^2-7np^2+np$$

$$=np\{(1-7p+12p^2-6p^3)+3np(1-2p+p^2)\}$$

$$=np\{(1-p)(1-6p+6p^2)+3np(1-p)^2\}$$

$$=npq\{(1-6p(1-p)+3npq\}$$

$$=npq\{1+3pq(n-2)\}$$

Ex. 10-1. *For binomial distribution show that*

$$\mu_{r+1}=pq\left\{nr.\mu_{r-1}+\frac{d\mu_r}{dp}\right\}$$

and deduce the values of μ_2, μ_3 *and* μ_4.

Sol. Binomial distribution is

$$P(x)={}^nc_xp^xq^{n-x},\ x=0,\ 1,\ 2,\ \ldots\ldots n$$

By def. $\mu_r = E(x-np)^r$

$$= \sum_{x=0}^{n} {}^nc_x p^x q^{n-x}(x-np)^r$$

$$\therefore \quad \frac{d\mu_r}{dp} = \sum_{x=0}^{n} {}^nc_x \{xp^{x-1}q^{n-x}(x-np)^r$$

$$+p^x(n-x)q^{n-x-1}\frac{dq}{dp}\ (x-np)^r + p^x q^{n-x} r(x-np)^{r-1}(-n)\}$$

$$=\sum_{x=0}^{n} {}^nc_x p^{x-1}q^{n-x-1}(x-np)^r\{xq-p(n-x)\}$$

$$-nr\sum_{x=0}^{n} {}^nc_x p^x q^{n-x}(x-np)^{r-1} \quad \left(\because \quad \frac{dq}{dp}=-1\right)$$

$$=\frac{1}{pq}\sum_{x=0}^{n} {}^nc_x p^x p^{n-x}(x-np)^{r+1} - nr\mu_{r-1}$$

$$=\frac{1}{pq}\ \mu_{r+1} - nr\mu_{r-1}$$

$$\therefore \quad \mu_{r+1} = pq\left\{ nrd_{r-1} + \frac{d\mu_r}{dp} \right\}$$

Put $r=1$, 2, and 3

$$\mu_2 = pq\left\{n\mu_0 + \frac{d\mu_1}{dp}\right\} = npq \qquad (\therefore \quad \mu_0 = 1,\ \mu_1 = 0)$$

$$\mu_3 = pq\left\{2n\mu_1 + \frac{d\mu_2}{dp}\right\} = npq\frac{d}{dp}\ \{pq\}$$

$$= npq(q-p)$$

$$\mu_4 = pq\left\{3n\mu_2 + \ \frac{d\mu_3}{dp}\right\} = npq\left[3npq + \frac{d}{dp}\ \ \{pq(q-p)\}\right]$$

$$= npq[3npq + (q-p)^2 - 2pq]$$

$$= npq[(q+p)^2 + 3pq(n-2)]$$

$$= npq[1+3pq(n-2)]$$

Ex. 10-2. *For a binomial variate x with parameters n and p show that*

$$\mu'_{r+1} = np\mu_r' + pq\frac{d\mu_r'}{dp}$$

where $\mu_r' = E(x^r)$ *and r is a non-negative integer.*

Sol. By def.

$$\mu_r' = E(x^r)$$

$$= \sum_{x=0}^{n} x^r \, {}^nc_x p^x q^{n-x}$$

$$\therefore \quad \frac{d\mu_r'}{dp} = \sum_{x=0}^{n} x^r \, {}^nc_x \left[x p^{x-1} q^{n-x} - (n-x)\, q^{n-x-1}\, p^x \right]$$

$$\left(\because \frac{dq}{dp} = -1 \right)$$

$$= \sum_{x=0}^{n} x^r \, {}^nc_x p^{x-1}\, q^{n-x-1} \,[xq-(n-x)p]$$

$$= \frac{1}{pq} \sum_{x=0}^{n} x^r \, {}^nc_x \, p^x q^{n-x} \,(x-np)$$

$$= \frac{1}{pq} \left\{ \sum_{x=0}^{n} x^{r+1} \, {}^nc_x p^x q^{n-x} - np \sum_{x=0}^{n} x^r \, {}^nc_x p^x q^{n-x} \right\}$$

$$= \frac{1}{pq} \left\{ \mu_{r+1}' - np\mu_r' \right\}$$

$$\mu_{r+1}' = np\, \mu_r' + pq \frac{d\mu_r'}{dp}$$

10-1.2. Measures of skewness and Kurtosis

Sol. Measure of skewness

$$= \gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}}$$

$$= \frac{\mu_3}{\mu_2^{3/2}}$$

$$= \frac{npq\,(q-p)}{(npq)^{3/2}} = \frac{q-p}{\sqrt{npq}}$$

Measure of kurtosis $= \beta_2 = \dfrac{\mu_4}{\mu_2^2}$

$$= \frac{npq\,\{1+3pq\,(n-2)\}}{(npq)^2}$$

$$= \frac{1+3pq\,(n-2)}{npq} = 3 + \frac{1-6pq}{npq}$$

10.1.3. Mean deviation about mean

Mean deviation about mean is given by

$$\eta = E\,|\,x-np\,|$$

$$= \sum_{x=0}^{n} |\,x-np\,|\ {}^nc_x p^x q^{n-x}$$

$$= \sum_{x>np} (x-np)\ {}^nc_x p^x q^{n-x} + \sum_{x<np} (np-x)\ {}^nc_x p^x q^{n-x}$$

Now $\qquad \mu_1 = 0$

$$\therefore \qquad \sum_{x=0}^{n} (x-np)\ {}^nc_x\ p^x q^{n-x} = 0$$

$$\Rightarrow \qquad \sum_{x>np} (x-np)\ {}^nc_x p^x q^{n-x} = \sum_{x<np} (np-x)\ {}^nc_x p^x q^{n-x}$$

$$\therefore \qquad \eta = 2\sum_{x>np} (x-np)\ {}^nc_x p^x q^{n-x}$$

$$= 2\sum_{x>np} \{xq-(n-x)p\}\ {}^nc_x p^x q^{n-x}$$

$$= 2\sum_{x>np} \{x\,{}^n c_x p^x q^{n-x+1} - (n-x)\,{}^nc_x\ p^{x+1}\ q^{n-x}\}$$

$$= 2\sum_{x>np} \left\{\frac{n\,!}{(x-1)\,!\,(n-x)\,!} p^x q^{n-x+1} - \frac{n\,!}{x\,!(n-x-1)\,!} p^{x+1} q^{n-x}\right\}$$

$$= 2\sum_{x>np} \{t_{x-1} - t_x\}$$

where $\qquad t_x = \dfrac{n\,!}{x\,!\,(n-x-1)\,!}\ p^{x+1}\ q^{n-x}$

Let μ = greatest integer contained in $np+1$

Then $\qquad \eta = 2\sum_{x=\mu}^{n} \{t_{x-1} - t_x\}$

$$= 2\ \{t_{\mu-1} - t_n\}$$

$$= 2t_{\mu-1} \qquad\qquad (\because\ t_n = 0)$$

$$= \frac{2\,.\,n\,!}{(\mu-1)\,!\,(n-\mu)\,!}\ p^{\mu}\ q^{n-\mu+1}$$

$$= 2npq\,\{{}^{n-1}c_{\mu-1}\ p^{\mu-1}\ q^{n-\mu}\}$$

10.1.4. Mode of the Binomial Distribution.

In binomial distribution the probability of x successes is given by

$$P(x)={}^nc_x p^x q^{n-x},\ x=0,\ 1,\ 2,\ldots\ldots n$$

The values of x together with the corresponding probabilities form binomial distribution. The mode is that value of x for which $P(x)$ is greater than or equal to $P(x-1)$ and $P(x+1)$. *i.e.*,

$$P(x-1) \leqslant P(x) \geqslant P(x+1)$$

Consider $\qquad P(x-1) \leqslant P(x)$

or $\qquad {}^nc_{x-1}\, p^{x-1}\, q^{n-x+1} \leqslant {}^nc_x p^x q^{n-x}$

or $\qquad \dfrac{\underline{|n}}{\underline{|x-1}\ \underline{|n-x+1}}\, q \leqslant \dfrac{\underline{|n}}{\underline{|x}\ \underline{|n-x}}\, p$

or $\qquad xq \leqslant (n+1)p-xp$

or $\qquad x(q+p) \leqslant (n+1)p$

or $\qquad x \leqslant (n+1)p \qquad \ldots(i)$

Similarly other inequality gives

$$x \geqslant (n+1)\, p-1 \qquad \ldots(ii)$$

From (i) and (ii), modal value x satisfies the inequality

$$(n+1)\, p-1 \leqslant x \leqslant (n+1)\, p \qquad \ldots(iii)$$

Case I : If $(n+1)\, p=k$ is an integer, then $(n+1)p-1=k-1$ is also an integer.

Now $\qquad \dfrac{P(x=k)}{P(x=k-1)}=\dfrac{{}^nc_k p^k q^{n-k}}{{}^nc_{k-1}\, p^{k-1}\, q^{n-k+1}}$

$$=\frac{n\,!}{k\,!\,(n-k)\,!}\cdot\frac{(k-1)\,!\,(n-k+1)\,!}{n\,!}\,\frac{p}{q}$$

$$=\frac{(n+1)p-kp}{kq}=\frac{k(1-p)}{kq}=1$$

$\therefore \qquad P(x=k)=P(x=k-1) \qquad \ldots(iv)$

$\therefore$ In this case $P(x)$ increases till $x=k-1$ and then (iv) holds and after that it begins to decrease.

$\therefore$ $x=k$ and $x=k-1$ are two modes.

Case II : If $(n+1)p=k$ is not an integer, let

$$(n+1)p=a \text{ (an integer)}+f \text{ (}a\text{ fraction)}$$

when x takes the value 'a' (which is obviously less than k and greater than $k-1$) from (i) and (ii)

$$P(a-1)<P(a)>P(a+1)$$

$\therefore$ $x=a$ (greatest integer less than k) is the mode.

Ex. 10-3. *If np be a whole number, the mean of the binomial disbutrition coincides with the greatest term.*

Sol. If np is a whole number, then since p is a fraction, np is the greatest integer less than $np+p=k$.

$\therefore$ From case II, mode$=np=$mean.

Ex. 10-3(a). *If x is the unique mode of the B.D., show that*

$$(n+1)\,p-1 < x < (n+1)p$$

10.1.5. Moment generating function

M.G.F., by def, is given by

$$M_0(t)=E(e^{tx})=\sum_{x=0}^{n} e^{tx}\,{}^nc_x p^x q^{n-x}$$

$$=\sum_{x=0}^{n} {}^nc_x (pe^t)^x q^{n-x}=(q+pe^t)^n$$

M.G.F. about the mean 'np' is given by

$$M_{\bar{x}}\ (t)=E(e^{t(x-np)})=e^{-npt}M_0(t)$$

$$=e^{-npt}(q+pe^t)^n$$

$$=\{qe^{-pt}+pe^{qt}\}^n$$

Deduction of moments about mean

$$M_{\bar{x}}\ (t)=(qe^{-pt}+pe^{-qt})^n$$

$$=\left\{q\left(1-pt+p^2\frac{t^2}{2\,!}-p^3\frac{t^3}{3\,!}+p^4\frac{t^4}{4\,!}+\ldots\ldots\right)\right.$$

$$\left.+p\left(1+qt+q^2\frac{t^2}{2\,!}+q^3\frac{t^3}{3\,!}+q^4\frac{t^4}{4\,!}+\ldots\ldots\right)\right\}$$

$$=\left\{1+pq\frac{t^2}{2\,!}+qp(q^2-p^2)\frac{t^3}{3\,!}+qp(p^3+q^3)\frac{t^4}{4\,!}+\ldots\ldots\right\}^n$$

$$=\left\{1+pq\frac{t^2}{2\,!}+pq(q-p)\frac{t^3}{3\,!}+qp(p^2-pq+q^2)\frac{t^4}{4\,!}+\ldots\ldots\right\}^n$$

$$\therefore\quad 1+\mu_1 t+\mu_2\frac{t^2}{2\,!}+\mu_3\frac{t^3}{3\,!}+\mu_4\frac{t^4}{4\,!}+\ldots\ldots$$

$$=1+npq\frac{t^2}{2\,!}+npq(q-p)\frac{t^3}{3\,!}+[nqp\{p^2-pq+q^2\}$$

$$+3n(n-1)p^2q^2]\frac{t^4}{4\,!}+\ldots\ldots$$

$\therefore\quad \mu_1=0,\quad \mu_2=npq,\quad \mu_3=npq(q-p)$

and $\quad \mu_4=npq\{p^2-pq+q^2+3(n-1)pq\}$

$\qquad =npq\{1+3(n-2)pq\}$

Ex. 10-4. *Is the sum of two independent binomial variates a binomial variate? If not, what are the conditions under which it is so?*

Sol. Let x_1 and x_2 be two independent binomial variates with parameters n_1, p_1 and n_2, p_2 respectively.

Then $M_0(t)$ of $x_1=(q_1+p_1e^t)^{n_1}$

and $M_0(t)$ of $x_2=(q_2+p_2e^t)^{n_2}$

Let $x=x_1+x_2$

Then $M_0(t)$ of $x=\{M_0(t) \text{ of } x_1\}.\{M_0(t) \text{ of } x_2\}$

$$=(q_1+p_1e^t)^{n_1}\,.\,(q_2+p_2e^t)^{n_2}$$

which being not of the form $(q+pe^t)^n$ implies that x is not a binomial variate.

If $p_1=p_2=p$ so that $q_1=q_2=q$

Then $M_0(t)$ of $x=(q+pe^t)^{n_1+n_2}$

which implies that x is a binomial variate with parameters (n_1+n_2) and p. Therefore, the required condition is

$$p_1=p_2$$

10.1.6. Cummulative Function and Cumulants

By def, cumulative f^n is given by

$$K_0(t)=\log M_0(t)=n\log(q+pe^t)$$

$$=n\log\left\{q+p\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\ldots\ldots\right)\right\}$$

$$=n\log\left\{1+pt+p\frac{t^2}{2!}+p\frac{t^3}{3!}+p\frac{t^4}{4!}+\ldots\ldots\right\}$$

$$=n\left[\left\{pt+p\frac{t^2}{2!}+p\frac{t^3}{3!}+p\frac{t^4}{4!}+\ldots\right\}-\frac{1}{2}\left\{pt+p\frac{t^2}{2!}+p\frac{t^3}{3!}\ldots\right\}^2\right.$$
$$\left.+\frac{1}{3}\left\{pt+p\frac{t^2}{2!}+\ldots\right\}^3-\frac{1}{4}\{pt+\ldots\ldots\}^4+\ldots\ldots\right]$$

But $K_0(t)=k_1t+k_2\dfrac{t^2}{2!}+k_3\dfrac{t^3}{3!}+\ldots\ldots$

$\therefore\quad k_1(0)=np,\ k_2=npq,$

$$k_3=n\{p-3p^2+2p^3\}=np\{1-3p+2x^2\}$$
$$=np(1-2p)(1-p)=npq(q-p)$$
$$k_4=n[p-7p^2+12p^3-6p^4]=np\{1-7p+12p^2-6p^3\}$$
$$=np\{(1-p)(1-6p+6p^2)\}$$
$$=npq\{1-6p(1-p)\}$$
$$=npq\{1-6pq\}.$$

Ex. 10-5. *Show that for the binomial dist. with parameters n and p,*

$$k_{r+1}=pq\frac{dk_r}{dp}$$

Hence deduce the values of k_2, k_3 and k_4.

Sol. For B.D., $M_0(t)=(q+pe^t)^n$

$\therefore\quad K_0(t)=\log M_0(t)=n\log\{q+pe^t\}$

$\therefore\quad k_r=n\left[\dfrac{d^r}{dt^r}\{\log(q+pe^t)\}\right]_{t=0}$

$\therefore\ \dfrac{dk_r}{dp}=n\left[\dfrac{d^r}{dt^r}\left\{\dfrac{e^t-1}{q+pe^t}\right\}\right]_{t=0}$

Also $\quad k_{r+1}=n\left[\dfrac{d^{r+1}}{dt^{r+1}}\{\log(q+pe^t)\}\right]_{t=0}$

$$=n\left[\frac{d^r}{dt^r}\left\{\frac{pe^t}{p+pe^t}\right\}\right]_{t=0}$$

$\therefore\quad k_{r+1}-pq\dfrac{dk_r}{dp}=n\left[\dfrac{d^r}{dt^r}\left\{\dfrac{pe^t-pqe^t+pq}{q+pe^t}\right\}\right]_{t=0}$

$$=n\left[\frac{d^r}{dt^r}\left\{\frac{p(pe^t+q)}{pe^t+q}\right\}\right]_{t=0}$$

$$=n\left[\frac{d^r}{dt^r}(p)\right]_{t=0}=0$$

$\therefore$
$$k_{r+1}=pq\frac{dk_r}{dp}$$

Put $\qquad r=1, 2$ and 3

$$k_2=pq\frac{dk_1}{dp}=npq\quad \text{as } k_1=\mu_1'(0)=np$$

$$k_3=pq\frac{dk_2}{dp}=npq(q-p)$$

and
$$k_4=pq\frac{dk_3}{dp}=npq\{q(q-p)-p(q-p)-2pq\}$$
$$=npq\{(q+p)^2-6pq\}$$
$$=npq\{1-6pq\}$$

Ex. 10-6. *If a coin is tossed n times where n is a large even number, show that the probability of exactly $\frac{n}{2}-x$ heads and $\frac{n}{2}+x$ tails is*

$$\left(\frac{2}{\pi n}\right)^{\frac{1}{2}}.e^{\frac{-2x^2}{n}}$$

Sol. Let the occurrence of a head in a toss be called success and p be its probability.

Then $$p=\frac{1}{2}=q$$

$\therefore$ Probability of x successes is given by

$$P(x)={}^{n}c_{x}\left(\frac{1}{2}\right)^{n}$$

Since n is an even number, let

$$n=2k$$

where k is a positive integer.

$$\therefore \quad P(x)={}^{2k}c_{x}\left(\frac{1}{2}\right)^{2k}$$

Now $$\frac{P(k-x)}{P(k)}=\frac{{}^{2k}c_{k-x}\left(\frac{1}{2}\right)^{2k}}{{}^{2k}c_{k}\left(\frac{1}{2}\right)^{2k}}$$

$$=\frac{2k\,!}{(k-x)!\,k+x\,!}\cdot\frac{k\,!\,k\,!}{2k\,!}$$

$$=\frac{k\,!\,k\,!}{(k-x)!\,(k+x)!}$$

By Stirling's formula,

$$k\,!\simeq\sqrt{2\pi}\;e^{-k}\;k^{k+\frac{1}{2}}$$

$$\therefore \quad \frac{P(k-x)}{P(k)}$$

$$\simeq\frac{\sqrt{2\pi}\,e^{-k}\,k^{k+\frac{1}{2}}\cdot\sqrt{2\pi}\,e^{-k}\,k^{k+\frac{1}{2}}}{\sqrt{2\pi}\,e^{-k+x}(k-x)^{k-x+\frac{1}{2}}\sqrt{2\pi}\,e^{-k-x}(k+x)^{k+x+\frac{1}{2}}}$$

$$=\frac{k^{2k+1}}{(k-x)^{k-x+\frac{1}{2}}(k+x)^{k+x+\frac{1}{2}}}$$

$$=\frac{1}{\left(1-\frac{x}{k}\right)^{k-x+\frac{1}{2}}\left(1+\frac{x}{k}\right)^{k+x+\frac{1}{2}}}$$

$$\therefore \quad \log\frac{P(k-x)}{P(k)}\simeq-\left(k-x+\frac{1}{2}\right)\log\left(1-\frac{x}{k}\right)$$
$$-\left(k+x+\frac{1}{2}\right)\log\left(1+\frac{x}{k}\right)$$

$$=\left(k-x+\frac{1}{2}\right)\left(\frac{x}{k}+\frac{1}{2}\frac{x^2}{k^2}+\ldots\ldots\ldots\right)$$

$$-\left(k+x+\frac{1}{2}\right)\left(\frac{x}{k}-\frac{1}{2}\frac{x^2}{k^2}+\ldots\ldots\right)$$

$$\simeq-\frac{x^2}{k}$$

neglecting terms containing $\frac{1}{k^2}$ and higher powers of $\frac{1}{k}$ as k is large.

$$\therefore \quad P(k-x)\simeq P(k)\ e^{-\frac{x^2}{k}}$$

$$={}^{2k}C_k.\left(\frac{1}{2}\right)^{2k}.e^{-\frac{2x^2}{n}}$$

$$=\frac{2k\,!}{k\,!\,k\,!}\left(\frac{1}{2}\right)^{2k}.e^{-\frac{2x^2}{n}}$$

$$=\frac{\sqrt{2\pi}\,e^{-2k}\,.\,(2k)^{2k+\frac{1}{2}}}{\left(\sqrt{2\pi}\,e^{-k}\,.\,k^{k+\frac{1}{2}}\right)^2}\cdot\left(\frac{1}{2}\right)^{2k}.e^{-\frac{2x^2}{n}}$$

$$=\frac{1}{\sqrt{\pi k}}\,e^{-\frac{2x^2}{n}}$$

$$=\sqrt{\frac{2}{\pi n}}\,e^{-\frac{2x^2}{n}}$$

Ex. 10-7. *Six dice are thrown 729 times. How many times do you expect at least three dice to show a 5 or 6 ?*

Sol. Here $N=729$, $n=6$

Let the occurrence of 5 or 6 be regarded as success and p be the probability of success.

Now p=prob. of occurrence of 5 or 6

$$=\frac{2}{6}=\frac{1}{3}$$

$$\therefore \quad q=1-p=\frac{2}{3}$$

$\therefore$ Prob of x successes is given by

$$P(x)={}^6c_x p^x p^{6-x}=\frac{{}^6c_x 2^{6-x}}{3^6}$$

By theorem of total probability, probability of at least three successes

$$=P(3)+P(4)+P(5)+P(6)$$

$$=\frac{1}{729}\{{}^6c_3.2^3+{}^6c_4.2^2+{}^6c_5.2+{}^6c_6\}$$

$$=\frac{1}{729}\left\{\frac{6.5.4}{3.2.1}.8+\frac{6.5.4.3}{4.3.2.1}.4+6.2+1\right\}$$

$$=\frac{1}{729}\{160+60+12+1\}$$

$$=\frac{233}{729}$$

$\therefore$ No. of times at least three successes occur

$$=\frac{233}{729}\,729=233$$

Ex. 10.8. *A perfect cubic die is thrown a large number of times in sets of 8. The occurrence of 5 or 6 is called a success. In what proportion of the sets would you expect 3 successes?*

Sol. Here $n=8$,

p=probability of success

=probability of occurrence of 5 or 6

$$=\frac{2}{6}=\frac{1}{3}$$

$$\therefore \quad q=1-p=1-\frac{1}{3}=\frac{2}{3}$$

$\therefore$ Probability of x successes is given by

$$P(x)={}^8c_x\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{8-x}=\frac{{}^8c_x.2^{8-x}}{3^8}$$

$$\therefore \quad P(x=3)=\frac{{}^8c_3.2^5}{3^8}=\frac{8.7.6}{3.2}\cdot\frac{32}{81\times 81}$$

$$=\frac{1792}{6561}=0{\cdot}2731$$

$\therefore$ **Required proportion=0·2731=27·31%.**

Ex. 10.9. *Assuming that half the population are consumers of rice so that the chance of an individual being a consumer is $\frac{1}{2}$ and assuming that 100 investigators each take 10 individuals to see whether they are consumers, how many investigators would you expect to report that three people or less were consumers ?*

Sol. Here $N=100$, $n=10$, $p=\frac{1}{2}$

$$\therefore \qquad p=\frac{1}{2}$$

$$\therefore \qquad P(x)={}^{10}c_x\left(\frac{1}{2}\right)^{10}$$

$\therefore$ Required number $=100\{P(3)+P(2)+P(1)+P(0)\}$

$$=\frac{100}{2^{10}}\{{}^{10}c_8+{}^{10}c_9+{}^{10}c_1+{}^{10}c_0\}$$

$$=\frac{100}{1024}\left\{\frac{10.9.8}{3.2.1}+\frac{10.9}{2.1}+10+1\right\}$$

$$=\frac{100}{1024}\{120+45+10+1\}$$

$$=\frac{17600}{1024}\simeq 17.$$

Ex. 10-10. *An irregular six-faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws would you expect it to give no even number ?*

Sol. Let p be the probability of an even number and $q=1-p$.

Then the prob. of getting five even numbers

$$={}^{10}c_5.p^5q^5$$

and the prob. of getting four even numbers

$$={}^{10}c_4.p^4q^6$$

By given,

$${}^{10}c_5.p^5q^5=2.{}^{10}c_4.p^4q^6$$

or

$$3p=5q=5-5p$$

$$\therefore \qquad p=\frac{5}{8} \text{ and } q=\frac{3}{8}$$

$$\therefore \text{ Required number} = 10{,}000 \left\{ {}^{10}c_0 \left(\frac{3}{8} \right)^{10} \right\}$$

$$= 10{,}000 \left(\frac{3}{8} \right)^{10}$$

$$= 0{\cdot}549 \simeq 1.$$

Ex. 10-11. *In a precision bombing attack there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target ?*

Sol. Here $p = \frac{1}{2} = q.$

Let n be the required number of bombs. Out of n at least 2 bombs must hit the target in order to destroy it completely.

$$\therefore \qquad P(2) + P(3) + \ldots\ldots + P(n) > 0{\cdot}99$$

where $\qquad P(x) = {}^nc_x \left(\frac{1}{2} \right)^n.$

$$\therefore \qquad 1 - P(0) - P(1) > 0{\cdot}99$$

or $\qquad 0{\cdot}01 > P(0) + P(1) = \frac{1}{2^n} + \frac{{}^nc_1}{2^n}$

or $\qquad 100(n+1) < 2^n$

The value of n is the least positive integer satisfying the inequality.

Putting $\qquad n = 10$

$$1100 < 2^{10} \quad \text{or} \quad 1100 < 1024$$

which is not true.

Putting $\qquad n = 11$

$$1200 < 2^{11} \quad \text{or} \quad 1200 < 2048$$

which is true.

$$\therefore \qquad n = 11.$$

Ex. 10-12. *Show that if two symmetrical binomial distributions* $\left(p = q = \frac{1}{2} \right)$ *of degree n (and of the same number of observations) are so superposed that the rth term of the one coincides with the (r+1)th term of the other, the distribution formed by adding superposed terms is a symmetrical binomial distribution of degree (n+1).*

Sol. Let N be the number of observations. Then the successive terms of the binomial distribution are

$$N.\left(\frac{1}{2} \right)^n, \; N\,{}^nc_1 \left(\frac{1}{2} \right)^n, \ldots\ldots N\,{}^nc_r \left(\frac{1}{2} \right)^n, \ldots\ldots N\,{}^nc_n \left(\frac{1}{2} \right)^n$$

$\therefore$ rth term of the first distribution $= N.{}^nc_{r-1}\ \dfrac{1}{2^n}$

and $(r+1)$th term of the second distribution $= N.{}^nc_r\ \dfrac{1}{2^n}$

$$\therefore \text{ Sum } = \frac{N}{2^n}\{{}^nc_{r-1}+{}^nc_r\}$$

$$= N.{}^{n+1}c_r\frac{1}{2^n}$$

$$= 2N.{}^{n+1}c_r\frac{1}{2^{n+1}}.$$

which is the $(r+1)$th term of the binomial distribution

$$2N.\left(\frac{1}{2}+\frac{1}{2}\right)^{n+1}$$

which is symmetrical binomial distribution of degree $(n+1)$ and total frequency $2N$.

Ex. 10-13. *Eight mice are selected at random and they are divided into two groups of 4 each. Each mouse in group A is given a dose of certain poison 'a' which is expected to kill one in four; each mouse in group B is given a dose of certain poison 'b' which is expected to kill one in two. Find the probability that the deaths in group B are less than in group A.*

Sol. For group A,

$$n=4,\ p=\frac{1}{4},\ q=\frac{3}{4}$$

$\therefore$ Prob. of x successes is given by

$$P(x)={}^4c_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x}$$

$$=\frac{{}^4c_x(3)^{4-x}}{256}$$

For group B,

$$n=4,\ p=\frac{1}{2},\ q=\frac{1}{2}$$

$\therefore$ Prob. of x successes is given by

$$Q(x)={}^4c_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}$$

$$=\frac{{}^4c_x}{16}$$

$$\text{Reqd. prob.} = Q(0)\{P(1)+P(2)+P(3)+P(4)\}+Q(1)\{P(2)+P(3)+P(4)\}+Q(2)\{P(3)+P(4)\}+Q(3)\{P(4)\}$$

$$=\frac{1}{4096}\{{}^4c_0(4c_1.3^3+4c_2.3^2+{}^4c_3.3+{}^4c_4)+{}^4c_1({}^4c_2.3^2+{}^4c_3.3+{}^4c_4)+{}^4c_2({}^4c_3.3+{}^4c_4)+{}^4c_3({}^4c_4)\}$$

$$=\frac{1}{4096}\{(108+54+12+1)+4(54+12+1)+6(12+1)+4\}$$

$$=\frac{525}{4096}.$$

Ex. 10-14. *Find the probability of success if the ratio of the probability of exactly r failures to the probability of exactly (n−r) failures in n trials is independent of n.*

Sol. Let p be the prob. of success and $q=1-p$.

Then prob. of r failures $={}^nc_r q^r p^{n-r}$

and prob. of $(n-r)$ failures $={}^nc_{n-r} q^{n-r} p^r$

$\therefore$ Ratio $=\left(\frac{p}{q}\right)^{n-2r}$

This ratio can be independent of n only when

$$\frac{p}{q}=1$$

or $\quad p=q=1-p$

$\therefore \quad p=\frac{1}{2}.$

Ex. 10-15. *Bring out the fallacy, if any, in the following statement.*

The mean of a binomial distribution is 5 and its s.d. is 3.

Sol. Here $\quad np=5$

and $\quad \sqrt{npq}=3$

$\therefore \quad npq=9$

or $\quad 5q=9$

$\therefore \quad q=1{\cdot}8$

which is not true as probability is to be less than unity.

Ex. 10-16. *If on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessels expected to arrive 4 at least will arrive safely.*

Sol. Let the arrival of a vessel safely be called success,

Then $\quad p=$prob. of a vessel to arrive safely

$=(1-$ prob. of a vessel to be wrecked)

$=1-\dfrac{1}{10}=\dfrac{9}{10}$

$\therefore \quad q=\dfrac{1}{10}$

Here $\quad n=5$

$\therefore \quad P(x)=\dfrac{{}^5c_x.9^x}{10^5}$

Required prob. $=P(4)+P(5)$

$=\dfrac{1}{10^5}\{{}^5c_4.9^4+9^5\}$

$=\dfrac{9^4.(5+9)}{10^5}=\dfrac{(14)9^4}{10^5}=0{\cdot}91854.$

Ex. 10·17. *'m' things are distributed among 'a' men and 'b' women, show that the chance that the number of things received by men is odd, is*

$$\frac{1}{2}\cdot\frac{(b+a)^m-(b-a)^m}{(b+a)^m}$$

Sol. If one thing is distributed, prob. of a man to get it

$=\dfrac{a}{a+b}$

and prob. for a woman to get it

$=\dfrac{b}{a+b}$

$\therefore$ out of m things distributed, prob. for men to receive 'r' things

$$={}^mc_r\left(\frac{a}{a+c}\right)^r\left(\frac{b}{a+b}\right)^{m-r}$$

$\therefore$ Prob. for men to receive odd number of things

$$=\sum {}^mc_r\left(\frac{a}{a+b}\right)\left(\frac{b}{a+b}\right)^{m-r}$$

where summation extends over odd values of r from 0 to m.

$$=\frac{1}{(a+b)^m}\{{}^mc_1ab^{m-1}+{}^mc_3a^3b^{m-3}+{}^mc_5a^5b^{m-5}+\ldots\ldots\}$$

$$=\frac{1}{(a+b)^m}\left\{\frac{(b+a)^m-(b-a)^m}{2}\right\}$$

Ex. 10·18. *Mean of a binomial distribution is 4 and its third moment about mean is 1·92. Find other constants of the distribution.*

Sol. Here $np=4$ ...(*i*)

$\mu_3=npq(q-p)=1{\cdot}92$ ...(*ii*)

$\therefore \quad q(q-p)=0{\cdot}48$

or $\quad q(2q-1)=0{\cdot}48$

or $\quad 2q^2-q-0{\cdot}48=0$

or $\quad (2q+0{\cdot}6)(q-0{\cdot}8)=0$

$\therefore \quad q=0{\cdot}8 \quad \text{as} \quad q\neq-0{\cdot}3$

$\therefore \quad p=0{\cdot}2$

$\therefore$ From (*i*) $\quad n=\dfrac{4}{0{\cdot}2}=20$

$\therefore \quad n=20,\ p=0{\cdot}2,\ q=0{\cdot}8$

Mode = greatest integer less than $(n+1)\,p=4{\cdot}2$

$=4$

Variance $=\mu_2=npq=20(0{\cdot}2)(0{\cdot}8)=3{\cdot}2$

$\therefore \quad s.d.=\sqrt{\mu_2}=\sqrt{3{\cdot}2}$

$$\mu_4=npq\{1+3pq(n-2)\}$$
$$=3{\cdot}2\{1+3(0{\cdot}2)(0{\cdot}8)18\}$$
$$=(3{\cdot}2)(9{\cdot}64)=30{\cdot}848$$
$$\beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{(1{\cdot}92)^2}{(3{\cdot}2)^3}=0{\cdot}1125$$
$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{30{\cdot}848}{(3{\cdot}2)^2}=3{\cdot}0125$$
$$\gamma_1=\sqrt{\beta_1}=0{\cdot}3354$$
$$\gamma_2=\beta_2-3=0{\cdot}0125$$

Ex. 10·19. *The following data are the number of seeds germinating out of 10 on damp filter paper for 80 sets of seeds. Fit a binomial distribution to these data :*

x :	0	1	2	3	4	5	6	7	8	9	10	*Total*
f :	6	20	28	12	8	6	0	0	0	0	0	80

Sol. Here $n=10$, $N=80$ and $\Sigma f=80$

$$\therefore \quad \text{A.M.}=\frac{\Sigma fx}{\Sigma f}=\frac{(20)1+(28)2+(12)3+(8)4+(6)5}{80}$$
$$=\frac{20+56+36+32+30}{80}$$

$$=\frac{174}{80}$$

$$\therefore \qquad np=\frac{174}{80}$$

$$\therefore \qquad p=\frac{1\cdot74}{8}=0\cdot2175$$

$$\therefore \qquad q=1-q=0\cdot7825$$

Hence the binomial distribution to be fitted to the data is

$$80(0\cdot7825+0\cdot2175)^{10}$$

$\therefore$ **Required binomial frequency distribution is**

x :	0	1	2	3	4	5	6	7	8	9	10
f :	6·9	19·1	24·0	17·8	8·6	2·9	0·7	0·1	0	0	0

Ex. 10-20. *For a binomial variate x, find p if n=4 and*

$$P(x=4)=6P(x=2)$$

Sol. The distribution of x is

$$P(x)=4c_x p^x q^{n-x}$$

Now $\qquad P(4)=6P(2)$

$$\therefore \qquad p^4=6\,{}^4c_2 p^2 q^2$$

i.e., $\qquad p^2=36q^2 \qquad$ (assuming $p\neq0$)

$\Rightarrow \qquad p=6q \qquad (\because \; p\nless 0)$

$\Rightarrow \qquad =6-6p$

$$p=\frac{6}{7}$$

Ex. 10-21. *For a binomial distribution the mean is 4 and variance is 2. Find the distribution*

Sol. $np=4, \quad npq=2$

$$\therefore \qquad q=\frac{1}{2}$$

$$\therefore \qquad p=1-\frac{1}{2}=\frac{1}{2}$$

$$\therefore \qquad n=8$$

$\therefore$ Binomial distribution is

$$P(x)=8c_x\left(\frac{1}{2}\right)^8, \; x=0, 1, \ldots\ldots 8$$

Ex. 10·22. *If x and y are binomial variates with $n=10$, $p=\frac{1}{2}$ and $n=5$, $p=\frac{1}{2}$ respectively.*

Find $P(x+y \geqslant 1)$

Sol. Since probabilities of success for x and y are same, $x+y$ is a binomial variate with

$$n=10+5=15, \quad p=\frac{1}{2}$$

Let $$z=x+y$$

Then distribution of z is

$$P(z)={}^{15}c_z\left(\frac{1}{2}\right)^{15},\ z=0,\ 1,\ldots\ldots15$$

$$\therefore \qquad P(z \geqslant 1)=1-P(z<1)$$
$$=1-P(z=0)$$
$$=1-\left(\frac{1}{2}\right)^{15}$$

Ex. 10·23. *Starting with the identity*

$$\sum_{x=0}^{n} {}^nc_x p^x q^{n-x}=(q+p)^n$$

find mean and variance of B.D.

Sol. By given identity

$$\sum_{x=0}^{n} {}^nc_x p^x q^{n-x}=(q+p)^n$$

Differentiating w.r.t. p

$$\sum_{x=0}^{n} {}^nc_x\{xp^{x-1}\,q^{n-1}-(n-x)\,q^{n-x-1}\,p^x\}=n(q+p)^{n-1}(-1+1)$$

$$\left(\because \ \frac{dq}{dp}=-1\right)$$

i.e., $$\sum_{x=0}^{n} {}^nc_x p^{x-1}q^{n-x-1}\,\{xq-(n-x)p\}=0$$

$$\Rightarrow \qquad \sum_{x=0}^{n} {}^nc_x p^x q^{n-x}\,(x-np)=0 \qquad \ldots(1)$$

i.e., $$\sum_{x=0}^{n} x\,.\,{}^nc_x p^x q^{n-x}-np\sum_{x=0}^{n} {}^nc_x p^x q^{n-x}=0$$

i.e., $\bar{x}-np\,(q+p)^n=0$

$\Rightarrow$ $\bar{x}=np$

Differentiating (1) w.r.t. p

$$\sum_{x=0}^{n} {}^nc_x[\{xp^{x-1}q^{n-x}-(n-x)\,p^xq^{n-x-1}\}(x-np)+p^xq^{n-x}(-n)]=0$$

i.e.,
$$\sum_{x=0}^{n} {}^nc_xp^{x-1}q^{n-x-1}\{xq-(n-x)p\}(x-np)$$
$$-n\sum_{x=0}^{n} {}^nc_xp^xq^{n-x}=0$$

$$\Rightarrow \quad \frac{1}{p\cdot q}\sum_{x=0}^{n} {}^nc_xp^xq^{n-x}(x-np)^2-n\,(q+p)^n=0$$

$$\Rightarrow \quad \mu_2=npq.$$

10-2. Poisson Distribution (P.D.)

Poisson Probability Distribution. *The poisson probability dist. of the variate x is*

$$P(x)=e^{-m}\frac{m^x}{x\,!},\ x=0,\ 1,\ldots\infty$$

The variate x is called **Poisson Variate** *and m is called the parameter of the distribution.*

Poisson Frequency Distribution. *The poisson frequency dist. of the variate x is*

$$P(x)=Ne^{-m}\frac{m^x}{x\,!},\ x=0,\ 1,\ldots\ldots\infty$$

where N is the total frequency.

Derivation. Poisson distribution is a probability distribution in which the chances of 0, 1, 2, ... successes are e^{-m}, me^{-m}, $\frac{m^2e^{-m}}{2\,!}$ ... respectively. In this case the number of values of the variate is infinite. It is the limiting form of binomial distribution when n (no. of trials) approaches '∞' and p (prob. of success) approaches zero such that np remains a finite constant m.

In binomial distribution the probability of x successes is given by

$$P(x)={}^nc_xp^xq^{n-x}$$

$$\therefore \lim_{\substack{n\to\infty\\ p\to 0\\ np=m}} P(x) = \lim_{\substack{n\to\infty\\ p\to 0\\ np=m}} {}^nc_x p^x q^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!\,n-x!}\left(\frac{n}{..}\right)^x\left(1-\frac{m}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{x!\,n-x!}\left(\frac{m}{n}\right)^x\left(1-\frac{m}{n}\right)^n\left(1-\frac{m}{n}\right)^{-x}$$

$$= \frac{m^x}{x!} \mathop{\mathrm{L}}_{n\to\infty} \frac{n(n-1)\ldots(n-x+1)}{n^x}\left(1-\frac{m}{n}\right)^n\left(1-\frac{m}{n}\right)^{-x}$$

$$= \frac{m^x}{x!} \lim_{n\to\infty}\left(\frac{n}{n}\right)\cdot\left(1-\frac{1}{n}\right)\ldots\ldots\ldots$$

$$\left(1-\frac{x-1)}{n}\right)\left[\left(1-\frac{m}{n}\right)^{-n/m}\right]^{-m}\left(1-\frac{m}{n}\right)^{-x}$$

$$= \frac{m^x}{x!}e^{-m}\left\{\because \lim_{n\to\infty}\left(1-\frac{m}{n}\right)^{-n/m}=e\right\}$$

$\therefore$ Probability of x successes for Poisson distribution

$$= \frac{m^x}{x!}e^{-m}$$

10.2.1. First four moments about mean

For Poisson distribution the probability of x successes is given by

$$P(x)=e^{-m}\frac{m^x}{x!},\ x=0,\ 1,\ 2,\ \ldots\infty$$

$$\mu_1'(0)=\sum_{x=0}^{\infty} x\,P(x)=\sum_{x=0}^{\infty} x\,e^{-m}\frac{m^x}{x!}$$

$$=e^{-m}\left\{1.m+2.\frac{m^2}{2!}+3\,.\frac{m^3}{3!}+\ldots\ldots\right\}$$

$$=me^{-m}\left\{1+m+\frac{m^2}{2}+\ldots\ldots\right\}$$

$$=me^{-m}.e^m=m$$

$$\mu_2'(0)=\sum_{x=0}^{\infty} x^2\,P(x)=\sum_{x=0}^{\infty}\{x(x-1)+x\}P(x)$$

$$=e^{-m}\sum_{x=0}^{\infty} x(x-1)\frac{m^x}{x!}+\sum_{x=0}^{\infty} x\,P(x)$$

$$=e^{-m}\left\{2.1.\ \frac{m^2}{2!}+3.2.\ \frac{m^3}{3!}+\ldots\ldots\right\}+m$$

$$=e^{-m}\,m^2(1+m+\ldots)+m$$

$$=e^{-m}\,m^2\,e^m+m=m^2+m$$

$\therefore\qquad \mu_2=\mu_2'(0)-\{\mu_1'(0)\}^2=m^2+m-m^2=m$

$$\mu_3'(0)=\sum_{x=0}^{\infty} x^3\,P(x)$$

Writing x^3 as $x(x-1)(x-2)+3x(x-1)+x$

$$\mu_3'(0)=\sum_{x=0}^{\infty}\{x(x-1)(x-2)+3x(x-1)+x\}\,P(x)$$

$$=m^3+3m^2+m$$

$\therefore\qquad \mu_3=\mu_3'(0)-3\mu_2'(0)\mu_1'(0)+2\{\mu_1'(0)\}^3$

$$=m^3+3m^2+m-3(m^2+m)m+2m^3=m$$

$$\mu_4'(0)=\sum_{x=0}^{\infty} x^4\,P(x)$$

But $\quad x^4\equiv x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7x(x-1)+x$

$$\therefore\ \mu_4'(0)=\sum_{x=0}^{\infty}\{x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7x(x-1)+x\}\,P(x)$$

$$=m^4+6m^3+7m^2+m.$$

$\therefore\qquad \mu_4=\mu_4'(0)-4\mu_3'(0)\mu_1'(0)+6\mu_2'(0)\{\mu_1'(0)\}^2-3\{\mu_1'(0)\}^4$

$$=m^4+6m^3+7m^2+m-4(m^3+3m^2+m)m+6(m^2+m)m^2-3m^4$$

$$=3m^2+m.$$

Ex. 10-24. *In a Poisson distribution, $P(x)$ for $x=0$ is 10%. Find the mean.*

Sol. Let m be the mean

Then $\qquad P(x)=e^{-m}\dfrac{m^x}{x!}$

$\therefore\qquad P(0)=e^{-m}$

$$\therefore \qquad e^{-m}=0{\cdot}1$$

$$\therefore \qquad e^{m}=\frac{1}{0{\cdot}1}=10$$

$$\therefore \qquad m=\log_e 10=2{\cdot}3026$$

Ex. 10·25. *Deduce first four moments about mean for Poisson distribution from those of binomial distribution.*

Sol. Poisson distribution is the limiting form of binomial distribution when n(no. of trials) tends to infinity and p (prob. of success) tends to zero such that np remains a finite constant m.

For binomial distribution,

$$\mu_1'(0)=np,\ \mu_2=npq,\ \mu_3=npq(q-p)$$

and

$$\mu_4=npq\{1+3pq(n-2)\}$$

$$\therefore \quad \mu_1'(0) \text{ for } P.D.=\underset{\substack{n\to\infty\\ p\to 0\\ np=m}}{\text{Lt}} \mu_1'(0) \text{ for } B.D.$$

$$=\underset{\substack{n\to\infty\\ p\to 0\\ np=m}}{\text{Lt}} (np)=m$$

$$\mu_2 \text{ for } P.D.=\underset{\substack{n\to\infty\\ p\to 0\\ np=m}}{\text{Lt}} \mu_2 \text{ for } B.D.$$

$$=\underset{\substack{n\to\infty\\ p\to 0\\ np=m}}{\text{Lt}} (npq)$$

$$=\underset{n\to\infty}{\text{Lt}}\ m\left(1-\frac{m}{n}\right)=m$$

$$\mu_3 \text{ for } P.D.=\underset{\substack{n\to\infty\\ p\to 0\\ np=m}}{\text{Lt}} \mu_2 \text{ for } B.D.$$

$$=\underset{\substack{n\to\infty\\ p\to 0\\ np=m}}{\text{Lt}} npq(q-p)$$

$$=\underset{n\to\infty}{\text{Lt}}\ m\left(1-\frac{m}{n}\right)\left(1-\frac{2m}{n}\right)=m$$

and $$\mu_4 \text{ for } P.D. = \underset{\substack{n\to\infty \\ p\to 0 \\ np=m}}{\text{Lt}} \mu_4 \text{ for } B.D.$$

$$= \underset{\substack{n\to\infty \\ p\to 0 \\ np=m}}{\text{Lt}} npq\{1+3pq(n-2)\}$$

$$= \underset{n\to\infty}{\text{Lt}}\ m\left(1-\frac{m}{n}\right)\left\{1+\frac{3m}{n}\left(1-\frac{m}{n}\right)(n-2)\right\}$$

$$= \underset{n\to\infty}{\text{Lt}}\ m\left(1-\frac{m}{n}\right)\left\{1+3m\left(1-\frac{m}{n}\right)\left(1-\frac{2}{n}\right)\right\}$$

$$=3m^2+m.$$

Ex. 10-26. *Let λ and μ_r denote the mean and central rth moment of a Poisson distribution respectively. Obtain the recurrence formula*

$$\mu_{r+1}=r\lambda\mu_{r-1}+\lambda\frac{d\mu_r}{d\lambda}$$

Hence deduce the values of β_1 and β_2

Sol. By def.,

$$\mu_r=\sum_{x=0}^{\infty}(x-\lambda)^r\ e^{-\lambda}\frac{\lambda^x}{x\,!}$$

$$\therefore\quad \frac{d\mu_r}{d\lambda}=\sum_{x=0}^{\infty}\frac{1}{x\,!}\left\{-e^{-\lambda}(x-\lambda)^r\lambda^x+x\lambda^{x-1}e^{-\lambda}(x-\lambda)^r\right.$$
$$\left.-r(x-\lambda)^{r-1}e^{-\lambda}.\lambda^x\right\}$$

$$=\sum_{x=0}^{\infty}\left\{\frac{1}{x\,!}\ e^{-\lambda}.\lambda^{x-1}(x-\lambda)^r(x-\lambda)-r(x-\lambda)^{r-1}e^{-\lambda}.\lambda^x\right\}$$

$$=\frac{1}{\lambda}\sum_{x=0}^{\infty}(x-\lambda)^{r+1}e^{-\lambda}\frac{\lambda^x}{x\,!}-r\sum_{x=0}^{\infty}(x-\lambda)^{r-1}e^{-\lambda}\frac{\lambda^x}{x\,!}$$

$$=\frac{1}{\lambda}\ \mu_{r+1}-r\mu_{r-1}$$

$$\therefore \quad \mu_{r+1}=r\lambda\mu_{r-1}+\lambda\,\frac{d\mu_r}{d\lambda}$$

Put $r=1$, 2 and 3

$$\therefore \quad \mu_2=\lambda\mu_0+\lambda\,\frac{d\mu_1}{d\lambda}=\lambda \text{ as } \mu_0=1 \text{ and } \mu_1=0$$

$$\mu_3=2\lambda\mu_1+\lambda\,\frac{d\mu_2}{d\lambda}=\lambda$$

and $$\mu_4=3\lambda\mu_2+\lambda\,\frac{d\mu_3}{d\lambda}=3\lambda^2+\lambda$$

$$\therefore \quad \beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{1}{\lambda} \text{ and } \beta_2=\frac{\mu_4}{\mu_2^2}=3+\frac{1}{\lambda}$$

Ex. 10.27. *For a Poisson variate x with parameter λ, show that*

$$\mu'_{r+1}=\lambda\mu_r'+\lambda\,\frac{d\mu_r'}{d\lambda}$$

where $\mu_r'=E(x^r)$ and r is a non-negative integer.

Sol. By def.

$$\mu_r'=E(x^r)$$

$$=\sum_{x=0}^{\infty} x^r\, e^{-\lambda}\,\frac{\lambda^x}{x\,!}$$

$$\frac{d\mu_r'}{d\lambda}=\sum_{x=0}^{\infty}\frac{x^r}{x\,!}\left\{-e^{-\lambda}\,\lambda^x+e^{-\lambda}.x\,\lambda^{x-1}\right\}$$

$$=-\sum_{x=0}^{\infty}x^r\,e^{-\lambda}\,\frac{\lambda^x}{x\,!}+\frac{1}{\lambda}.\sum_{x=0}^{\infty}x^{r+1}\,e^{-\lambda}\,.\,\frac{\lambda^x}{x\,!}$$

$$=-\mu_r'+\frac{1}{\lambda}\,\mu'_{r+1}$$

$$\Rightarrow \quad \mu'_{r+1}=\lambda\mu_r'+\lambda\,\frac{d\mu_r'}{d\lambda}$$

10.2.2. Measure of skewness and kurtosis

By def.

$$\gamma_1=\sqrt{\beta_1}=\sqrt{\frac{\mu_3^2}{\mu_2^3}}=\sqrt{\frac{m^2}{m^3}}=\frac{1}{\sqrt{m}}$$

$$\therefore \quad \gamma_1\to 0 \text{ as } m\to\infty$$

$$\gamma_2=\beta_2-3=\frac{\mu_4}{\mu_2^2}-3=\frac{3m^2+m}{m^2}-3=\frac{1}{m}$$

$$\gamma_2\to 0 \text{ as } m\to\infty.$$

Ex. 10.28. *For Poisson distribution show that*

$$m\sigma\gamma_1\gamma_2=1.$$

Sol. L.H.S.$=m\sqrt{m}\,.\dfrac{1}{\sqrt{m}}.\dfrac{1}{m}=1$

10.2.3. Mode of the Poisson Distribution

In Poisson distribution the probability of x successes is given by

$$P(x)=e^{-m}.\ \frac{m^x}{x\,!}$$

The mode is that value of x for which $P(x)$ is greater than or equal to $P(x-1)$ and $P(x+1)$ *i.e.,*

$$P(x-1) \leqslant P(x) \geqslant P(x+1)$$

Consider

$$P(x-1) \leqslant P(x)$$

or
$$e^{-m}\ \frac{m^{x-1}}{(x-1)\,!} \leqslant e^{-m}\ \frac{m^x}{x\,!}$$

or
$$x \leqslant m \qquad \ldots(i)$$

Similarly other inequality gives

$$x \geqslant m-1 \qquad \ldots(ii)$$

From (i) and (ii) modal value x satisfies the inequality

$$m-1 \leqslant x \leqslant m \qquad \ldots(iii)$$

Case I. If m is an integer, then $(m-1)$ is also an integer.

Now $\dfrac{P(x=m)}{P(x=m-1)}=e^{-m}.\ \dfrac{m^m}{m\,!}.\dfrac{(m-1)\,!}{e^{-m}.m^{m-1}}$

$$=\frac{m}{m}=1$$

$\therefore$
$$P(x=m)=P(x=m-1) \qquad \ldots(iv)$$

$\therefore$ In this case $P(x)$ increases till $x=m-1$ and then (iv) holds and after that it begins to decrease.

Case II. If m is not an integer, let

$$m=a(\text{an integer})+f(\text{a fraction})$$

when x takes the value 'a' (which is less than m but greater than $m-1$) from (i) and (ii)

$$P(a-1) < P(a) > P(a+1)$$

$x=a$(greatest integer less than m) is the mode.

10.2.4. Moment Generating Function

By def.,

$$M_0(t)=E(e^{tx})=\sum_{x=0}^{\infty} e^{tx}.e^{-m}.\frac{m^x}{x!}$$

$$=e^{-m}\sum_{x=0}^{\infty}\frac{(me^t)^x}{x!}=e^{-m}.e^{me^t}=e^{m(e^t-1)}$$

Now $\quad M_{\bar{x}}(t)=E\{e^{t(x-m)}\}=e^{-mt}E(e^{tx})$

$$=e^{-mt}.M_0(t)=e^{m(e^t-t-1)}$$

Deduction of Moments

$$M_{\bar{x}}(t)=e^{m(e^t-1-t)}$$

$$=e^{m\left\{\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\ldots\right\}}$$

$$=1+m\left\{\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\ldots\right\}+\frac{m^2}{2!}\left\{\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\right\}^2$$

$$+\frac{m^3}{3!}\left\{\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\ldots\right\}^3+\frac{m^2}{4!}\left\{\frac{t^2}{2!}+\ldots\ldots\right\}^4+\ldots\ldots$$

But $\quad M_{\bar{x}}(t)=1+\mu_1 t+\mu_2\frac{t^2}{2!}+\mu_3\frac{t^3}{3!}+\mu_4\frac{t^4}{4!}+\ldots\ldots$

$\therefore \quad \mu_1=0,\ \mu_2=m,\ \mu_3=m,\ \mu_4=m+3m^2.$

Ex. 10-29. *Show that the sum of two independent Poisson variates is a Poisson variate.*

Sol. Let x_1 and x_2 be two Poisson variates with means m_1 and m_2.

Let $\quad x=x_1+x_2$

Then $\quad M_0(t)$ of $x=\{M_0(t)$ of $x_1\}.\{M_0(t)$ of $x_2\}$

Now $\quad M_0(t)$ of $x_1=e^{m_1(e^t-1)}$

and $\quad M_0(t)$ of $x_2=e^{m_2(e^t-1)}$

$\therefore \quad M_0(t)$ of $x=e^{(m_1+m_2)(e^t-1)}$

which is a M.G.F. of a poisson variate with mean m_1+m_2.

$\therefore \quad x$ is a Poisson variate with mean m_1+m_2.

Ex. 10-30. *Find $M_0(t)$ of the difference of two independent Poisson variates with means m_1 and m_2 and show that it is not a Poisson variate.*

Sol. List $\quad u=x-y$

Then $\quad M_0(t)$ of $u=E(e^{tu})=E(e^{t(x-y)})$

$$=E(e^{tx})E(e^{-ty})$$

$$=e^{m_1(e^t-1)}\ e^{m_2(e^{-t}-1)}$$

$$=\exp\{m_1(e^t-1)+m_2(e^{-t}-1)\}$$

Since $M_0(t)$ of u is not of the form $e^{m(e^t-1)}$, u is not a Poisson variate.

10.2.5. Cummulative Function and Cummulants

By def., cummulative function is given by

$$K_0(t)=\log M_0(t)=\log e^{m(e^t-1)}$$

$$=m(e^t-1)=m\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\ldots\right)$$

$\therefore \quad k_1(0)=m,\ k_2=k_3=\ldots\ldots=m.$

Ex. 10-31. *If x is a Poisson variate such that*

$$P(x=1)=3\ P(x=2)$$

Find mean and variance. Also find $P(x=0)$

Sol. Let λ be the parameter of x.

Then $\quad P(x)=e^{-\lambda}\dfrac{\lambda^x}{x!}$

Now $\quad P(x=1)=2\ P(x=2)$

$\therefore \quad e^{-\lambda}.\lambda=2e^{-\lambda}\dfrac{\lambda^2}{2!}$

$\therefore \quad \lambda=1$

$\therefore \quad$ Mean $=$ Variance $=\lambda=1$ and $P(0)=e^{-1}$

Ex. 10-32. *If x is a Poisson variate with mean m, find*

(*i*) $E(e^{-kx})$.

(*ii*) $E(xe^{-kx})$.

Sol. $\quad$ (*i*) $E(e^{-kx})=\displaystyle\sum_{x=0}^{\infty} e^{-m}\frac{m^x}{x!}e^{-kx}$

$$=e^{-m}\sum_{x=0}^{\infty}\frac{(me^{-k})^x}{x\,!}=e^{-m}.e^{me^{-k}}$$

$$=e^{-m(1-e^{-k})}$$

(ii) $$E(xe^{-kx})=\sum_{x=0}^{\infty}e^{-m}\frac{m^x}{x\,!}xe^{-kx}$$

$$=e^{-m}\sum_{x=1}^{\infty}\frac{m^x}{(x-1)!}e^{-kx}$$

$$=e^{-m}\sum_{x=1}^{\infty}\frac{(me^{-k})^x}{(x-1)\,!}$$

$$=e^{-m}.(me^{-k})\ e^{me^{-k}}$$

$$=me^{-m(1-e^{-k})-k}$$

Ex. 10-33. *Show that in a poisson distribution with unit mean, mean deviation about mean is* $\frac{2}{e}$.

Sol. Since in a poisson distribution with parameter m, mean is m.

$\therefore$ $m=1$

$\therefore$ Poisson distribution is

$$P(x)=\frac{e^{-1}}{x\,!},\ x=0,1,2,\ldots$$

$\therefore$ Mean deviation about mean is given by

$$\text{M.D.}=E\,|\,x-1\,|$$

$$=\sum_{x=0}^{\infty}|\,x-1\,|\,\frac{e^{-1}}{x\,!}$$

$$=e^{-1}+\sum_{x=2}^{\infty}(x-1)\ \frac{e^{-1}}{x\,!}$$

$$=e^{-1}+e^{-1}\sum_{x=2}^{\infty}\left\{\frac{1}{(x-1)!}-\frac{1}{x!}\right\}$$

$$=e^{-1}\left\{1+\frac{1}{1!}\right\}$$

$$=\frac{2}{e}$$

Ex. 10-34. *If x and y are Poisson variates with means m and m′ respectively. Prove that the probability that $(x-y)$ has the value r is the co-efficient of t^r in exp. $\{mt+m't^{-1}-m-m'\}$.*

Sol. $P(x-y=r)$ is required.

Now $x-y$ will take the value r when x takes the value $r+s$ and y takes the value s where $s=0, 1, 2,\ldots\ldots$

By compound prob theorem, prob of x taking the value $r+s$ and y taking the value s is

$$\left(e^{-m}\frac{m^{r+s}}{(r+s)\,!}\right)\left(e^{-m'}\frac{m'^{s}}{s\,!}\right)$$

$\therefore$ By total prob theorem, prob of $(x-y)$ taking the value r

$$=e^{-(m+m')}\sum_{s=0}^{\infty}\frac{m^{r+s}m'^{s}}{s\,!\,r+s\,!}$$

$$=e^{-(m+m')}.\ \text{co-efficient of } t^r \text{ in } e^{mt+m't^{-1}}$$

$$=\text{co-efficient of } t^r \text{ in } e^{mt+m't^{-1}-m-m'}$$

Ex. 10-35. *If x is a Poisson variate with mean m, find M.G.F. of* $z=\dfrac{x-m}{\sqrt{m}}$ *and find its limit when $m\to\infty$,*

Sol. $M_0(t)$ of $z=E(e^{tz})$

$$=E\left\{e^{t\left(\frac{x-m}{\sqrt{m}}\right)}\right\}=e^{-t\sqrt{m}}\,E\left\{e^{\frac{tx}{\sqrt{m}}}\right\}$$

$$=e^{-t\sqrt{m}}\,M_0\left(\frac{t}{\sqrt{m}}\right)\text{ of } x$$

$$=e^{-t\sqrt{m}}\,.e^{m\left(e^{\frac{t}{\sqrt{m}}}-1\right)}$$

$$=e^{m(e^{t/\sqrt{m}}-1)-t\sqrt{m}}$$

$$\therefore \quad \log\{M_0(t) \text{ of } z\} = m\left(e^{\frac{t}{\sqrt{m}}} - 1\right) - t\sqrt{m}$$

$$= m\left\{\frac{t}{\sqrt{m}} + \frac{1}{2!}\left(\frac{t}{\sqrt{m}}\right)^2 + \frac{1}{3!}\left(\frac{t}{\sqrt{m}}\right)^3 + \ldots\ldots\right\} - t\sqrt{m}$$

$$= \frac{1}{2}\, t^2 + \text{terms containing } \frac{t}{\sqrt{m}} \text{ and higher powers}$$

$$\therefore \quad \underset{m\to\infty}{\text{Lt}} \log\{M_0(t) \text{ of } z\} = \frac{1}{2}\, t^2$$

$$\therefore \quad \underset{m\to\infty}{\text{Lt}} M_0(t) \text{ of } z = e^{\frac{1}{2}t^2}$$

Ex. 10-36. *Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience show that 2% of such fuses are defective.*

Sol. Let the presence of a defective fuse in the box be called success.

Then p = prob of success = 0·02

Here $n = 200$.

Since n is large and p is small the distribution can be taken to be Poissonian.

$\therefore \quad m = np = (200)(0{\cdot}02) = 4$

$\therefore \quad e^{-m} = e^{-4} = 0{\cdot}0183$

$\therefore \quad P(x) = e^{-4} \cdot \dfrac{4^x}{x!} = (0{\cdot}0183)\, \dfrac{4^x}{x!}$

Required prob $= P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$$= (0{\cdot}0183)\left\{1 + 4 + 8 + \frac{32}{3} + \frac{32}{3} + \frac{128}{15}\right\} = 0{\cdot}78$$

Ex. 10-37. *In a certain factory turning out razor blades, there is a small chance $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use Poisson's distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 10,000 packets.*

Sol. Here $p = \dfrac{1}{500}$, $n = 10$, $N = 10{,}000$

$\therefore \quad m = np = \dfrac{1}{500} \cdot 10 = \dfrac{1}{50} = 0{\cdot}02$

$e^{m} = e^{-0{\cdot}02} = 0{\cdot}9802$

$\therefore$ Prob. of having x defective blades is given by

$$P(x)=\frac{(0\cdot9802)(0\cdot02)^x}{x\,!}$$

$\because$ No. of packets containing x defective blades

$$=10{,}000\ \frac{(0\cdot9802)(0\cdot02)^x}{x\,!}$$

$\therefore$ No. of packets containing no defective blade

$$=10{,}000(0\cdot9802)=9802$$

No. of packets containing one defective blade

$$=10{,}000(0\cdot9802)(0\cdot02)=196\cdot04$$
$$\simeq 196$$

and No. of packets containing two defective blades

$$=10{,}000(0\cdot9802)\ \frac{(0\cdot02)^2}{2\,!}=1\cdot9604$$
$$\simeq 2.$$

Ex. 10·38. *Fit Poisson's distribution to the following and calculate theoretical frequencies :*

Death	0	1	2	3	4
Frequencies	122	60	15	2	1

Sol. $m=\text{mean}=\dfrac{(122)0+(60)1+(15)2+(2)3+(1)4}{200}$

$$=\frac{60+30+6+4}{200}=0\cdot5$$

$$\therefore\ e^{-m}=e^{-0\cdot5}=1+(-0\cdot5)+\frac{1}{2\,!}(-0\cdot5)^2+\frac{1}{3\,!}(-0\cdot5)^3$$
$$+\frac{1}{4\,!}(-0\cdot5)^4+\frac{1}{5\,!}(-0\cdot5)^5+\ldots\ldots\ldots$$
$$=1-0\cdot5+0\cdot125-0\cdot0208+0\cdot0026-0\cdot00026$$
$$\simeq 0\cdot61\ \text{(nearly)}$$

$\therefore$ Theoretical frequency of x deaths is

$$200.e^{-0\cdot5}\ \frac{(0\cdot5)^x}{x\,!}$$
$$=200.(0\cdot61)\ \frac{(0\cdot5)^x}{x\,!}$$

$\therefore$ Theoretical frequencies are
122, 61, 15, 2 and 0.

Ex. 10·39. *A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1·5. Calculate the proportion of days on which neither car is used and the proportion of days on which some demand is refused* ($e^{-1\cdot5}=0\cdot2231$).

Sol. Let x be the number of demands for a car in a day.

Then dist of x is

$$P(x)=e^{-1\cdot5}\frac{(1\cdot5)^x}{x\,!}$$

Now the proportion of days on which neither car is used

$$=P(\text{of no demand in a day})$$

$$=P(x=0)=e^{-1\cdot5}=0\cdot2231$$

and the proportion of days on which some demand is refused

$$=P(x>2)$$

$$=1-P(x\leqslant 2)$$

$$=1-P(x=0)-P(x=1)-P(x=2)$$

$$=1-e^{-1\cdot5}\{1+1\cdot5+1\cdot125\}$$

$$=1-(0\;2231)(3\cdot625)=0\cdot19126$$

Ex. 10·40. *For a Poisson variate with parameter* λ, *show that*

$$\lambda\{{}^rc_1\mu_{r-1}+{}^rc_2\mu_{r-2}+\ldots\ldots+{}^rc_r\mu_0\}=\mu_{r+1}$$

Sol. By def.

$$\mu_{r+1}=E(x-\lambda)^{r+1}$$

$$=\sum_{x=0}^{\infty}(x-\lambda)^{r+1}\,e^{-\lambda}\,\frac{\lambda^x}{x\,!}$$

$$=\sum_{x=0}^{\infty}(x-\lambda)^r\,e^{-\lambda}\,\frac{\lambda^x}{x\,!}\,(x-\lambda)$$

$$=\sum_{x=0}^{\infty}x(x-\lambda)^r\,e^{-\lambda}\,\frac{\lambda^x}{x\,!}-\lambda\sum_{x=0}^{\infty}(x-\lambda)^r\,e^{-\lambda}\,\frac{\lambda^x}{x\,!}$$

$$=\sum_{x=1}^{\infty}(x-1-\lambda+1)^r\,e^{-\lambda}\,\frac{\lambda^x}{(x-1)!}-\lambda\,\mu_r$$

$$=\lambda \sum_{x=0}^{\infty}(x-\lambda+1)^r e^{-\lambda} \frac{\lambda^x}{x!}-\lambda \mu_r$$

$$=\lambda \sum_{x=0}^{\infty}\left\{(x-\lambda)^r+{}^rc_1(x-\lambda)^{r-1}-\ldots+{}^rc_r\right\} e^{-\lambda} \frac{\lambda^x}{x!}$$

$$-\lambda \mu_r$$

$$=\lambda\,[\mu_r+{}^rc_1\,\mu_{r-1}+\ldots\ldots+{}^rc_r]-\lambda\,\mu_r$$

$$=\lambda\,[{}^rc_1\,\mu_{r-1}+\ldots\ldots+{}^rc_r\,\mu_0]$$

$$(\because\quad \mu_0=1)$$

Ex. 10·41. *If x and y are independent Poisson Variates, show that the conditional distribution of x given x+y is binomial.*

Sol. Let λ, μ be the parameters of x, y respectively.

Then, $z=x+y$ is a P.V. with parameter $\lambda+\mu$. Distributions of x, y, z are

$$P(x)=e^{-\lambda}\frac{\lambda^x}{x!}$$

$$P(y)=e^{-\mu}\frac{\mu^y}{y!}$$

$$P(z)=e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^z}{z!}$$

Now $$P(x=r/z=n)=\frac{P(x=r,\ z=n)}{P(z=n)}$$

$$=\frac{P(x=r,\ x+y=n)}{P(z=n)}$$

$$=\frac{P(x=r,\ y=n-r)}{P(z=n)}$$

$$=\frac{P(x=r)\,P(y=n-r)}{P(z=n)}$$

$$\{\because\ x,\ y \text{ are independent}\}$$

$$=\frac{e^{-\lambda}\cdot\dfrac{\lambda^r}{r!}\cdot e^{-\mu}\cdot\dfrac{\mu^{n-r}}{(n-r)!}}{e^{-(\lambda+\mu)}\dfrac{(\lambda+\mu)^n}{n!}}$$

$$=\frac{n!}{r!\,n-r!}\left(\frac{\lambda}{\lambda+\mu}\right)^r\left(\frac{\mu}{\lambda+\mu}\right)^{n-r}$$

$$= {}^nc_r\, p^r\, q^{n-r}$$

where $p=\dfrac{\lambda}{\lambda+\mu}$, $q=\dfrac{\mu}{\lambda+\mu}$

which gives the conditional distribution of x given $x+y=n$. This is a B.D. with parameters n and $p=\dfrac{\lambda}{\lambda+\mu}$.

Ex. 10·42. *For a Poisson distribution with parameter λ show that*

$$P(x+1)=\frac{\lambda}{x+1}\,P(x)$$

10.3. Normal Distribution

Normal Probability Distribution. *The normal distribution of the variate x with mean m and s.d. σ is*

$$dP=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx,\ -\infty<x<\infty.$$

The variate x is called normal variate.

The curve with equation

$$y=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

is called normal curve.

If N is the total freq., the corresponding normal curve is

$$y=\frac{N}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

(i) To derive normal distribution as a limiting form of binomial distribution.

Sol. Normal distribution can be regarded as the limiting form of the bionomial distribution when n, the number of trials is very large and neither p nor q is very small.

Let $$z=\frac{x-np}{\sqrt{npq}} \qquad \ldots(1)$$

where x is a binomial variate with parameters n and p. Since mean and *s.d.* of x are np and $\sqrt{npq}$, the variate z defined by (1) has zero mean and unit variance. As x takes values from 0 to n,

z takes values from $-\sqrt{\dfrac{np}{q}}$ to $\sqrt{\dfrac{nq}{p}}$ and the jump in the value of z at each stage is $\dfrac{1}{\sqrt{npq}}$. Now as $n \to \infty$ two extreme values of z tends to $-\infty$ and ∞ respectively and the jump at each stage tends to zero. Thus in the limit we expect the distribution of z to be continuous extending from $-\infty$ to ∞ and having zero mean and unit variance.

In binomial dist the prob for the variate x to take value x is given by

$$P(x) = {}^nc_x p^x q^{n-x} = \frac{n\,!}{x\,!\;\overline{n-x}\,!}\, p^x q^{n-x}$$

By Stirling's formula

$$n\,! = \sqrt{2\pi}\; e^{-n}\, n^{\,n+\frac{1}{2}}$$

$\therefore\;\; P(x)$

$$= \frac{\sqrt{2\pi}\, e^{-n}. n^{\,n+\frac{1}{2}}}{\left(\sqrt{2\pi} e^{-x}. x^{\,x+\frac{1}{2}}\right)\left(\sqrt{2\pi}\;\; e^{-n+x}. (n-x)^{n-x+\frac{1}{2}}\right)}\, p^x p^{n-x}$$

$$= \frac{1}{\sqrt{2\pi}}\;\frac{1}{\sqrt{npq}}\left(\frac{np}{x}\right)^{x+\frac{1}{2}} \cdot \left(\frac{nq}{n-x}\right)^{n-x+\frac{1}{2}}$$

Let $\quad N = \left(\dfrac{np}{x}\right)^{x+\frac{1}{2}}\left(\dfrac{nq}{n-x}\right)^{n-x+\frac{1}{2}}$

$$\therefore\;\; \log N = -\left(x+\frac{1}{2}\right)\log\,\frac{x}{np} - \left(\,n-x+\frac{1}{2}\right)\log\,\frac{n-x}{nq}$$

From (1) $x = np + z\sqrt{npq}$

$$\therefore\;\; \log\; N = -\left(np+z\sqrt{npq}+\frac{1}{2}\right)\log\left(1+z\sqrt{\frac{q}{np}}\right)$$

$$-\left(\,nq-z\sqrt{npq}\;+\frac{1}{2}\right)\log\left(\;1-z\sqrt{\frac{p}{nq}}\right)$$

As n is very large and tends to infinity both $z\sqrt{\dfrac{q}{np}}$ and $z\sqrt{\dfrac{p}{nq}}$ can be taken to be less than unity and hence both the logarithms can be expanded in series.

$$\therefore \log N = -\left(np + z\sqrt{npq} + \frac{1}{2}\right)\left[z\sqrt{\frac{q}{np}} - \frac{1}{2}z^2 \frac{q}{np} + \ldots \right]$$

$$+\left(nq - z\sqrt{npq} + \frac{1}{2}\right)\left[z\sqrt{\frac{p}{nq}} + \frac{1}{2} z^2 \frac{p}{nq} + \ldots \right]$$

$$= -\frac{1}{2} z^2 + \text{terms containing } n \text{ in the denominator}$$

$$\therefore \quad \log N \to -\frac{1}{2} z^2 \text{ as } n \to \infty$$

i.e., $\quad N \to e^{-\frac{1}{2}z^2}$ as $n \to \infty$

Since $\frac{1}{\sqrt{npq}}$ is the increment in z at each stage and tends to zero as $n \to \infty$ we denote its limit by dz.

$\therefore$ If dP denote the probability for the variate z to lie in the internal $z - \frac{1}{2} dz$ and $z + \frac{1}{2} dz$ we have

$$dP = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz.$$

This is the required continuous distribution of z and is called Normal distribution.

(*ii*) *To derive normal distribution as a limiting form of Poisson distribution.*

Sol. Normal distribution can also be regarded as the limiting form of the Poisson distribution when its parameter m is large.

Let $$z = \frac{x - m}{\sqrt{m}} \qquad \ldots(1)$$

where x is a Poisson variate with parameter m. Since mean and *s.d.* of x are m and $\sqrt{m}$, the variate z defined by (1) has zero mean and unit variance. As x takes values from 0 to ∞, z takes values from $-\sqrt{m}$ to ∞ and the jump in the value of z at each stage is $\frac{1}{\sqrt{m}}$ Now as $m \to \infty$ two extreme values of z tend to $-\infty$ and ∞ and the jump at each stage tends to zero. Thus in the limit we expect the distribution of z to be continuous extending from $-\infty$ to ∞ and having zero mean and unit variance. In Poisson dist. the prob. for x to take value x is given by

$$P(x) = e^{-m} \frac{m^x}{x!} = e^{-m} \frac{m^x}{\sqrt{2\pi}\, e^{-x} . x^{x+\frac{1}{2}}}$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{m}}e^{x-m}\cdot\left(\frac{m}{x}\right)^{x+\frac{1}{2}}$$

Let $N=e^{x-m}.\left(\frac{m}{x}\right)^{x+\frac{1}{2}}$

$$\therefore \log N=x-m-\left(x+\frac{1}{2}\right)\log\frac{x}{m}$$

$$=x-m-\left(m+z\sqrt{m}+\frac{1}{2}\right)\log\left(1+\frac{z}{\sqrt{m}}\right)\text{[from (1)]}$$

$$=x-m-\left(m+z\sqrt{m}+\frac{1}{2}\right)\left(\frac{z}{\sqrt{m}}-\frac{1}{2}\frac{z^2}{m}+\ldots\ldots\right)$$

$$=-\frac{1}{2}z^2+\text{terms containing } m \text{ in the denominator}$$

$$\therefore \log N\to-\frac{1}{2}z^2 \text{ or } N\to e^{-\frac{1}{2}z^2} \text{ as } m\to\infty$$

Since $\frac{1}{\sqrt{m}}$ is the increment in z at each stage and tends to zero as $m\to\infty$ we denote its limit by dz.

$\therefore$ If dP denotes the probability for the variate z to lie in the interval $z-\frac{1}{2}dz$ and $z+\frac{1}{2}dz$ we have

$$dP=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$

10.3.1. Mean deviation about mean for a normal variate with mean m and s.d. σ.

Sol. Dist of a normal variate x is

$$dP=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx\quad -\infty<x<\infty$$

$\therefore$ Mean deviation from mean

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}|x-m|\,e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\sigma y|\,e^{-\frac{1}{2}y^2}dy$$

where $\quad y=\dfrac{x-m}{\sigma}$

$$=\frac{2\sigma}{\sqrt{2\pi}}\int_0^\infty ye^{-\frac{1}{2}y^2}\,dy=\frac{2\sigma}{\sqrt{2\pi}}\left\{-e^{-\frac{1}{2}y^2}\right\}_0^\infty$$

$$=\sigma\sqrt{\frac{2}{\pi}}\approx\frac{4\sigma}{5}$$

10.3.2. Moments

(i) Odd order moments about mean

For a normal variate with mean m and *s.d.* σ

$$dP=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx\quad -\infty<x<\infty$$

$\therefore\quad \mu_{2n+1}=E\{x-m\}^{2n+1}$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-m)^{2n+1}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$$

$$=\frac{\sigma^{2n+1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}y^{2n+1}e^{-\frac{1}{2}y^2}\,dy$$

where $\quad y=\dfrac{x-m}{\sigma}$

$=0$ (as integrand is an odd f^n).

(ii) Even order moments about mean

For a normal variate x with mean m and *s.d.* σ

$$dP=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx\quad -\infty<x<\infty$$

By def.,

$$\therefore\quad \mu_{2n}=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-m)^{2n}\,e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$$

$$=\frac{\sigma^{2n}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^{2n}e^{-\frac{1}{2}y^2}\,dy$$

where $\quad y=\dfrac{x-m}{\sigma}$

$$=\frac{\sigma^{2n}}{\sqrt{2\pi}}\left\{\left|-e^{-\frac{1}{2}y^2}.y^{2n-1}\right|_0^{\infty}+(2n-1)\int_0^{\infty} y^{2n-2}e^{-\frac{1}{2}y^2}\,dy\right\}$$

$$=\sigma^2(2n-1).\frac{1}{\sqrt{2\pi}}\sigma^{2n-2}\int_0^{\infty} e^{-\frac{1}{2}y^2}.y^{2n-2}\,dy$$

$$=\sigma^2(2n-1)\mu_{2n-2}.$$

Put $\quad n=2$

$\therefore \quad \mu_4=3\sigma^2\mu_2=3\sigma^4.$

Ex. 10·43. *Show that*

$$\mu_{2n}=1.3.5.......(2n-1)\sigma^2$$

Sol. By recurrence formula

$$\mu_{2n}=(2n-1)\,\sigma^2\,\mu^{2n-2}$$

Put $\qquad n=n,\ n-1,\ \ldots,\ 2,\ 1$

$$\mu_{2n}=(2n-1)\,\sigma^2\,\mu_{2n-2}$$
$$\mu_{2n-2}=(2n-3)\,\sigma^2\,\mu_{2n-4}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mu_4=3\sigma^2\mu_2$$
$$\mu_2=1.\sigma^2\mu_0=1.\sigma^2 \qquad (\because \quad \mu_0=1)$$

Multiplying

$$\mu_{2n}=1.3......(2n-1)\sigma^{2n}$$

Ex. 10·44. *Let x be a* $N(m, \sigma)$ {*i.e. a normal variate with mean m and s.d.* σ, *then*

(*i*) $\mu'_{r+2}=2m\mu'_{r+1}+(\sigma^2-\mu^2)\mu'_r+\sigma^3\dfrac{d\mu_r'}{d\sigma}$

where μ_r' *denotes the rth moment about zero.*

(*ii*) $\mu_{2r+2}=\sigma^2\mu_{2r}+\sigma^3\dfrac{d\mu_{2r}}{d\sigma}.$

β_{-1} By def.

$$\mu'_r=E(x^r)$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x^r e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$\therefore \quad \frac{d\mu_r'}{d\sigma}=-\frac{1}{\sigma^2\sqrt{2\pi}}\int_{-\infty}^{\infty} x^r\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$+\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x^r\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \cdot \left\{\frac{(x-m)^2}{\sigma^3}\right\} dx$$

$$=-\frac{\mu_r'}{\sigma}+\frac{1}{\sigma^4\sqrt{2\pi}}\int_{-\infty}^{\infty} x^r(x^2-2xm+m^2)\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$=\mu_r'\left(\frac{m^2}{\sigma^3}-\frac{1}{\sigma}\right)+\frac{\mu'_{r+2}}{\sigma^3}-\frac{2m}{\sigma^3}\mu'_{r+1}$$

$$\therefore \quad \mu'_{r+2}=2m\mu'_{r+1}+(\sigma^2-m^2)\mu_r'+\sigma^3\frac{d\mu_r'}{d\sigma}$$

(*ii*) By def.

$$\mu_{2r}=E(x-m)^{2r}$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} (x-m)^{2r}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$\frac{d\mu_{2r}}{d\sigma}=-\frac{1}{\sigma^2\sqrt{2\pi}}\int_{-\infty}^{\infty} (x-m)^{2r}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$+\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} (x-m)^{2r}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \cdot \left\{\frac{(x-m)}{\sigma^3}\right\} dx$$

$$=-\frac{\mu_{2r}}{\sigma}+\frac{\mu_{2r+2}}{\sigma^3}$$

$$\therefore \quad \mu_{2r+2}=\sigma^2\mu_{2r}+\sigma^3\frac{d\mu_{2r}}{d\sigma}$$

10.3.3. Measures of Skewness and Kurtosis

For a normal variate with mean m and *s.d.* σ,

$$\mu_3=0 \text{ and } \mu_4=3\sigma^4$$

$$\therefore \qquad \beta_1=\frac{\mu_3^2}{\mu_2^3}=0 \quad\text{and}\quad \beta_2=\frac{\mu_4}{\mu_2^2}=3$$

$$\gamma_1=\sqrt{\beta_1}=0 \text{ and } \gamma_2=\beta_2-3=0.$$

10.3.4. Moment Generating Function

By Def.,

$$M_0(t)=E\{e^{tx}\}=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}.\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t(m+\sigma y)}\, e^{-\frac{1}{2}y^2} dy$$

where $\quad y=\dfrac{x-m}{\sigma}$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tm}.e^{-\frac{1}{2}(y-t\sigma)^2+\frac{t^2\sigma^2}{2}} dy$$

$$=e^{tm+\frac{1}{2}t^2\sigma^2}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz$$

where $\quad z=y-t\sigma$

$$=e^{tm+\frac{1}{2}t^2\sigma^2}$$

$\therefore$ M.G.F. about mean m is given by

$$M_{\bar{x}}(t)=E\{e^{t(x-m)}\}=e^{-mt}.E\{e^{tx}\}$$

$$=e^{-mt}.\,e^{mt+\frac{1}{2}t^2\sigma^2}=e^{\frac{1}{2}t^2\sigma^2}.$$

Deductions

$$M_{\bar{x}}(t)=e^{\frac{1}{2}t^2\sigma^2}$$

$$=1+\left(\frac{1}{2}\ t^2\sigma^2\right)+\frac{\left(\frac{1}{2}\ t^2\sigma^2\right)^2}{2\,!}+\ldots+\frac{\left(\frac{1}{2}\ t^2\sigma^2\right)^n}{n\,!}+\ldots$$

$\therefore\ \mu_{2n+1}=0$

and $\dfrac{\mu_{2n}}{(2n)!}=\dfrac{1}{2^n}\cdot\dfrac{\sigma^{2n}}{n\,!}$

$$\Rightarrow\quad \mu_{2n}=\frac{1}{2^n}\cdot\frac{(2n)\,!}{n\,!}2\sigma^{2n}$$

$$=(2n-1)\ldots\ldots3.1\,\sigma^{2n}$$

10.3.5. Cummulative Function and Cummulants

By def., cumulative f^n is given by

$$K_0(t)=\log M_0(t)=\log e^{mt+\frac{1}{2}t^2\sigma^2}=mt+\frac{1}{2}\ t^2\sigma^2$$

But $K_0(t)=k_1t+k_2\dfrac{t^2}{2\,!}+\ldots\ldots\ldots\ldots$

where $k_1, k_2, \ldots\ldots$ are various cumulants.

$\therefore\quad (0)k_1=m,\quad k_2=\sigma^2,\quad k_3=0,\quad k_4=0,\ldots\ldots\ldots$

Thus all cumulants after the second are equal to zero.

Ex. 10-45. *Show that a linear combination of independent normal variates is also a normal variate.*

Sol. Let $x_1, x_2, \ldots x_n$ be normal variates with means $m_1, m_2, \ldots\ldots m_n$ and *s.ds.* $\sigma_1, \sigma_2, \ldots\ldots\sigma_n$.

Let. $u=a_1x_1+a_2x_2+\ldots\ldots a_nx_n$

where *a's* are constants.

Now $M_0(t)_{\text{of } u}=E\{e^{tu}\}=E\left\{e^{t(a_1x_1+a_2x_2\ldots\ldots+a_nx_n)}\right\}$

$$=E(e^{a_1tx_1}).\ E(e^{ta_2x_2})\ldots\ldots E(e^{a_ntx_n})$$

$$\because\ \{x_1, x_2,\ldots\ldots\text{are independent}\}$$

$$=M_0(ta_1)_{\text{of } x_1}\cdot M_0(ta_2)_{\text{of } x_2}\ldots\ldots M_0(a_nt)_{\text{of } x_n}$$

$$=e^{\{ta_1m_1+\frac{1}{2}(ta_1)^2\sigma_1^2\}}\cdot e^{\{(ta_2)m_2+\frac{1}{2}(ta_2)^2\sigma_2^2\}}\ldots\ldots e^{\{(ta_n)m_n+\frac{1}{2}(ta_n)^2\sigma_n^2\}}$$

$$=e^{t\Sigma a_im_i+\frac{1}{2}t^2\Sigma a_i^2\sigma_i^2}$$

which is the M.G.F. of a normal variate with mean Σa_im_i and variance $\Sigma a_i^2\sigma_i^2$.

$\therefore$ u is a normal variate with mean $\Sigma a_i m_i$ and variance $\Sigma a_i^2 \sigma_i^2$.

Ex. 10·46. *If the independent variates x_i $(i=1, 2, \ldots\ldots n)$ are normally distributed about the common mean μ, with a common variance σ^2, show that their mean $\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$ is also normally distributed about the same mean μ but with variance $\frac{\sigma^2}{n}$.*

Sol. Here $m_1=m_2=\ldots\ldots=m_n=\mu$ and $a_1=a_2=\ldots\ldots=a_n=\frac{1}{n}$

$\therefore$ $u=\frac{1}{n}\sum_{i=1}^{n} x_i=\bar{x}$ is a normal variate with mean $\Sigma a_i m_i$

$=\frac{\Sigma\mu}{n}=\mu$ and variance $=\frac{n\sigma^2}{n^2}=\frac{\sigma^2}{n}$.

Ex. 10·47. *Show that for the N.D. mean, mode and median coincide.*

Sol. The density curve for the N.D. is

$$y=\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

Put $\qquad x-m=X$

$$y=\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{X^2}{\sigma^2}}$$

which is evidently symmetrical about the line $X=0$ *i.e.*, $x=m$.

$\therefore$ $x=m$ is the median.

Also evidently y decreases continuously as X increases numerically and is maximum for $X=0$.

$\therefore$ $X=0$ *i.e.*, $x=m$ is the mode.

$\therefore$ Mean$=$Mode$=$Median$=m$.

Ex. 10·48. *Find the points of inflextion of the normal curve.*

Sol. The eq. of the normal curve is

$$y=\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \qquad \ldots(1)$$

At the points of inflextion

$$\frac{d^2y}{dx^2}=0 \text{ and } \frac{d^3y}{dx^3}\neq 0$$

From (1) $$\frac{d^2y}{dx^2}=\frac{1}{\sigma\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}\left\{-\frac{x-m}{\sigma^2}\right\}^2 - \frac{1}{\sigma^2}\, e^{-\frac{1}{2}\left\{\frac{x-m}{\sigma}\right\}^2}\right]$$

and $$\frac{d^3y}{dx^3}=\frac{1}{\sigma\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}\left\{-\frac{(x-m)}{\sigma^2}\right\}^3 + \frac{3(x-m)}{\sigma^4}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}\right]$$

Put $$\frac{d^2y}{dx^2}=0$$

$\therefore$ $$\frac{(x-m)^2}{\sigma^2}-1=0$$

or $$x=m\pm\sigma.$$

At $x=m\pm\sigma$, $$\frac{d^3y}{dx^3}=\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}}\left[\mp\frac{1}{\sigma^3}\pm\frac{3}{\sigma^3}\right]$$

$$=\pm\frac{2}{\sigma^4\sqrt{2\pi}}\, e^{-\frac{1}{2}}\neq 0$$

$\therefore$ At the points of inflextion

$$x=m\pm\sigma$$

and hence from (1) $$y=\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}}$$

Ex. 10·49. *Give chief features of the normal curve.*

Sol. The eq. of the normal curve is

$$y=\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \qquad ...(1)$$

(*i*) Mean, mode and median of the normal curve coincide.

(*ii*) Since y becomes zero when x is numerically infinite, curve touches x-axis both on negative and positive side at infinity *i.e.*, x—axis is asymtote to the curve both on negative and positive sides.

(*iii*) At the points of inflextion

$$x=m\pm\sigma$$

Evidently the points of inflextion are equidistant from $x=m$.

(*iv*) Maximum values of ordinate is

$$y=\frac{1}{\sigma\sqrt{2\pi}}.$$

Ex. 10·50. *Deduce the first four moments about the mean of the normal distribution from those of (i) the Binomial dist (ii) the Poisson distribution.*

Sol. (*i*) For B.D. $\mu_2=npq$

$$\mu_3=npq(q-p)$$

$$\mu_4=npq\{1+3(n-2)pq\}$$

Let $$z=\frac{x-np}{\sqrt{npq}}$$

when x is a binomial variate with parameters n and p.

Then mean of $z=E(z)=\dfrac{E(x-np)}{\sqrt{npq}}=0$

μ_2 for $z=E(z-0)^2=\dfrac{1}{npq}E(x-np)^2=1$

μ_3 for $z=E(z^3)=\dfrac{1}{(npq)^{\frac{3}{2}}}E(x-np)^3=\dfrac{q-p}{\sqrt{npq}}$

μ_4 for $z=E(z^4)=\dfrac{1}{(npq)^2}E(x-np)^4=\dfrac{1+3(n-2)pq}{npq}$

Now as $n\to\infty$, $z\to$ a normal variate

$\therefore$ μ_2 for normal variate$=\underset{n\to\infty}{\text{Lt}}$ μ_2 for $z=1$

μ_3 for normal variate$=\underset{n\to\infty}{\text{Lt}}$ μ_3 for $z=\underset{n\to\infty}{\text{Lt}}\dfrac{q-p}{\sqrt{npq}}=0$

μ_4 for normal variate$=\underset{n\to\infty}{\text{Lt}}$ μ_4 for z

$$=\underset{n\to\infty}{\text{Lt}}\left\{\frac{1-6pp}{npq}+3\right\}=3.$$

(*ii*) For P.D. $\mu_2=m$

$$\mu_3=m$$

$$\mu_4=m+3m^2$$

Let $$z=\frac{x-m}{\sqrt{m}}$$

where x is a Poisson variate with parameter m.

Then mean of $z=\dfrac{E(x-m)}{\sqrt{m}}=0$

μ_2 for $z=E(z)^2=\dfrac{1}{m}E(x-m)^2=1$

μ_3 for $z=\dfrac{1}{m\sqrt{m}}E(x-m)^3=\dfrac{1}{\sqrt{m}}$

μ_4 for $z=\dfrac{1}{m^2}E(x-m)^4=\dfrac{1}{m}+3$

As $m\to\infty$, $z\to a$ normal variate.

$\therefore$ μ_2 for normal variate$=\underset{m\to\infty}{\text{Lt}}\ \mu_2$ for $z=1$

μ_3 for normal variate$=\underset{m\to\infty}{\text{Lt}}\ \mu_3$ for $z=\underset{m\to\infty}{\text{Lt}}\ \dfrac{1}{\sqrt{m}}=0$

μ_4 for normal variate$=\underset{m\to\infty}{\text{Lt}}\left(3+\dfrac{1}{m}\right)=3.$

Ex. 10-51. *For a certain normal distribution the first moment about 10 is 40 and that the 4th moment about 50 is 48, what is the A.M. and s.d. of the dist ?*

Sol. Let m and σ be A.M. and *s.d*

Then $\mu_1'(10)=40$

$\therefore$ $E(x-10)=40$

or $E(x)=50$

$\therefore$ $m=50$

Also $\mu_4=48$

$\therefore$ $3\sigma^4=48$

$\therefore$ $\sigma=2.$

Ex. 10-52. *If X is a normal variate with mean 30 and s.d. 5. Find the probabilities that*

(*i*) $26\leqslant X\leqslant 40$, (*ii*) $|X-30|>5$.

Sol (*i*) $P\{26\leqslant X\leqslant 40\}=P\{26\leqslant X\leqslant 30\}+P\{30\leqslant X\leqslant 40\}$

Put $Z=\dfrac{X-30}{5}$

$=P\{-0{\cdot}8\leqslant Z\leqslant 0\}+p\{0\leqslant Z\leqslant 2\}$

$=P\{0\leqslant Z\leqslant 0{\cdot}8\}+P\{0\leqslant Z\leqslant 2\}$

$=0{\cdot}2881+0{\cdot}4772=0{\cdot}7653.$

(using normal tables)

(ii) $P\{|X-30|>5\}=1-P\{|X-30|\leqslant 5\}$

$$=1-P\{25\leqslant X\leqslant 35\}$$
$$=1-2P\{30\leqslant X\leqslant 35\}$$
$$=1-2P\{0\leqslant Z\leqslant 1\}$$
$$=1-2(0\cdot3413)$$
$$=0\cdot3174.$$

Ex. 10·53. *For a normal distribution with mean 2 and variance 9, find the value x of the variate such that the probability of the variate lying in the interval (2, x) is 0·4115.*

Sol. Let X be normal variate.

Then dist of X is

$$dP=\frac{1}{3\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{X-2}{3}\right)^2}dX$$

$$P(2<X<x)=\frac{1}{3\sqrt{2\pi}}\int_2^x e^{-\frac{1}{2}\left(\frac{X-2}{3}\right)}dX$$

Put $\dfrac{X-2}{3}=z$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{\frac{x-2}{3}} e^{-\frac{1}{2}z^2}\,dz$$

$$\therefore\quad 0\cdot4115=\frac{1}{\sqrt{2\pi}}\int_0^{\frac{x-2}{3}} e^{-\frac{1}{2}(z^2)}dz$$

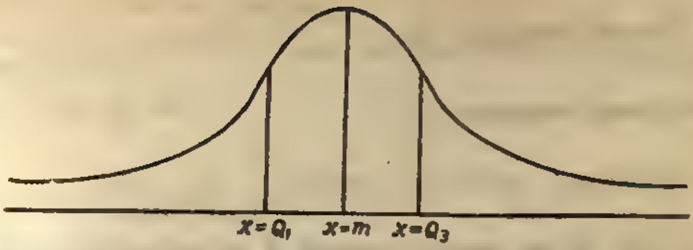
$$\therefore\quad \frac{x-2}{3}=1\cdot35$$

$$\therefore\quad x=2+4\cdot05=6\cdot05.$$

Ex. 10·54. *Prove that, for the normal distribution, the quartile deviation, mean deviation and the s.d. are approximately in the ratio 10 : 12 : 15.*

Sol. Let Q_1 and Q_3 be the Quartiles

Then $P\{x\leqslant Q_1\}=0\cdot25$



$\therefore \qquad P\{Q_1<x<m\}=0{\cdot}5-0{\cdot}25$

$$=0{\cdot}25$$

$$\therefore \qquad \frac{1}{\sigma\sqrt{2\pi}}\int_{Q_1}^{m} e^{-\frac{1}{2}\left(\frac{x-m)}{\sigma}\right)^2} dz=0{\cdot}25$$

Put $\qquad \dfrac{m-x}{\sigma}=y$

$$\therefore \qquad \frac{1}{\sqrt{2\pi}}\int_{0}^{\frac{m-Q_1}{\sigma}} e^{-\frac{1}{2}y^2} dy=0{\cdot}25$$

$$\therefore \qquad \frac{m-Q_1}{\sigma}=0{\cdot}6744 \qquad \text{.. (1)}$$

Also $\qquad P\{x>Q_3\}=0{\cdot}25$

$\therefore \qquad P\{m\leqslant x<Q_3\}=0{\cdot}25$

$$\therefore \qquad \frac{1}{\sigma\sqrt{2\pi}}\int_{m}^{Q_3} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx=0{\cdot}25$$

or

$$\frac{1}{\sqrt{2\pi}}\int_{0}^{\frac{Q_3-m}{\sigma}} e^{-\frac{1}{2}y^2} dy=0{\cdot}25$$

$$\frac{Q_3-m}{\sigma}=0{\cdot}6744 \qquad \text{...(2)}$$

From (1) and (2)

$$\frac{Q_3-Q_1}{2}=0{\cdot}6744\sigma \simeq \frac{2}{3}\,\sigma$$

$\therefore$ Quartile Deviation$=\frac{2}{3}\,\sigma$

Also Mean deviation$=\frac{4}{5}\,\sigma$

$\therefore$ Q.D. : M.D. : S.D. : : 10 : 12 : 15.

Ex. 10.55. *If two normal universes A and B have the same total frequency but the s.d. of universe A is k times that of the universe B, show that maximum frequency of universe A is $\frac{1}{k}$ times that of universe B.*

Sol. Let N be the total frequency and σ_1, σ_2 be the *s.d.* of A and B.

Then $$\sigma_1=k\sigma_2$$

Let m_1 and m_2 be the A.Ms. of A and B.

The frequency functions of A and B are

$$F_A(x)=\frac{N}{\sigma_1\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-m_1}{\sigma_1}\right)^2}$$

and $$F_B(x)=\frac{N}{\sigma_2\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-m_2}{\sigma_2}\right)^2}$$

Evidently $F_A(x)$ is *max* for $x=m_1$.

$$\Big[F_A(x)\Big]_{max}=\frac{N}{\sigma_1\sqrt{2\pi}}$$

Similarly $$\Big[F_B(x)\Big]_{max}=\frac{N}{\sigma_2\sqrt{2\pi}}$$

$$\therefore \quad \frac{[F_A(x)]max}{[F_B(x)]max}=\frac{\sigma_2}{\sigma_1}=\frac{1}{k}$$

$$\therefore \quad [F_A(x)]_{max}=\frac{1}{k}\,[F_B(x)]_{max}$$

Ex. 10-56. *Assume the mean heights of soldiers to be 68·22 inches with a variance of 10·8 (in)². How many soldiers in a regiment of 1000 would you expect to be over 6 feet tall? (Given that the area under the standard normal curve between x=0 and x=0·35 is 0·1368 and between x=0 and x=1·15 is 0·3746).*

Sol. Let x inches be the height.

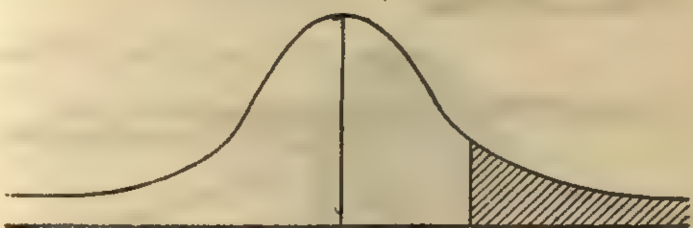
Then x is a normal Variate with mean 68·22 inches and variance 10·8 (in)^2.

$\therefore$ Dist. of x is

$$dP = \frac{1}{\sqrt{10\cdot8}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-68\cdot22}{\sqrt{10\cdot8}}\right)^2} dx$$

$$\therefore \quad P\{x>72\} = 0\cdot5 - \int_{x=68\cdot22}^{72} \frac{1}{\sqrt{10\cdot8}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-68\cdot22}{\sqrt{10\cdot8}}\right)^2} dx$$

Put $\quad z = \dfrac{x-68\cdot22}{\sqrt{10\cdot8}}$



$$\therefore \quad P\{x>72\} = 0\cdot5 - \frac{1}{\sqrt{2\pi}} \int_{0}^{1\cdot15} e^{-\frac{1}{2}z^2} dz$$

$$= 0\cdot5 - (0\cdot3746) \qquad \text{(given)}$$

$$= 0\cdot1254.$$

$\therefore$ In a regiment of 1000, the number of soldiers taller than 6 feet.

$$= 1000 \times 0\cdot1254 = 125\cdot4$$

$$\simeq 125.$$

Ex. 10-57. *If $\log_{10} x$ is normally distributed with mean 4 and variance 4, find the probability of $1\cdot202 < x < 83180000$.*

(Given $\log_{10} 1202 = 3\cdot08$, $\log_{10} 8318 = 3\cdot92$).

(b) $\log_{10} x$ is normally distributed with mean 7 and variance 3. $\log_{10} y$ is normally distributed with mean 3 and unit variance. If the distributions x and y are independent, find the prob of

$$1\cdot202 < \frac{x}{y} < 83180000$$

{Given $\log_{10} 1202 = 3\cdot08$, $\log_{10} 8318 = 3\cdot92$}

10.54

Sol. (*a*) Let $y = \log_{10} x$.

Dist of y is

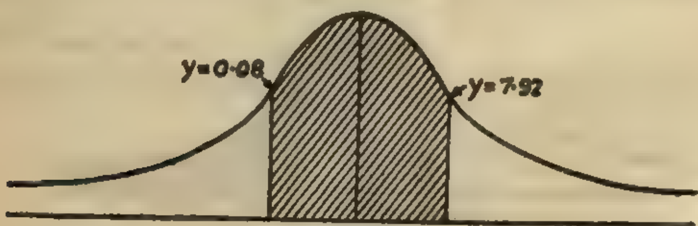
$$dP = \frac{1}{2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left\{\frac{y-4}{2}\right\}^2} dy$$

Now $P\{1{\cdot}202 < x < 83180000\} = P\{0{\cdot}08 < y < 7{\cdot}92\}$

$$= \frac{1}{2\sqrt{2\pi}} \int_{0\cdot08}^{7\cdot22} e^{-\frac{1}{2}\left(\frac{y-4}{2}\right)^2} dy$$

Put $\qquad \dfrac{y-4}{2} = z$

$\therefore \quad P\{1{\cdot}202 < x < 83180000\}$



$$= \frac{1}{\sqrt{2\pi}} \int_{-1\cdot96}^{1\cdot96} e^{-\frac{1}{2} z^2}\, dz$$

$$= 2 .\, \frac{1}{\sqrt{2\pi}} \int_{0}^{1\cdot96} e^{-\frac{1}{2} z^2}\, dz = 2(0{\cdot}4750)$$

(from normal tables)

$= 0{\cdot}95.$

(*b*) Let $z_1 = \log_{10} x$ and $z_2 = \log_{10} y$.

Then $z = z_1 - z_2$ is also a normal variate with mean $7 - 3 = 4$ and variance $3 + 1 = 4$.

$\therefore \quad$ Dist of z is

$$dP = \frac{1}{2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{z-4}{2}\right)^2} dz$$

$$\text{Now } P\left\{1{\cdot}202<\frac{x}{y}<83180000\right\}$$
$$=P\{0{\cdot}08<z<7{\cdot}92\}$$
$$=0{\cdot}95. \qquad \{\text{from } (a)\}$$

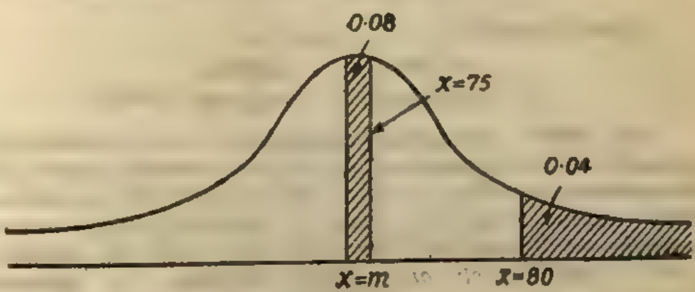
Ex. 10·58. *If the skulls are classified A, B and C according as the length breadth index is under 75, between 75 and 80 and over 80, find approximately (assuming that the dist is normal) the means and s.d. of a series in which A are 58%, B are 38% and C are 4%, being given that if*

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_0^t e^{-\frac{x^2}{2}}\,dx$$

then $f(0{\cdot}20)=0{\cdot}08$ and $f(1{\cdot}75)=0{\cdot}46$)

Sol. Let m and σ be the mean and *s.d.* respectively and x be the length breadth index. Then dist of x is

$$dP=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$$



Now $P\{x<75\}=0{\cdot}58$ which is greater than $0{\cdot}50$ and hence the ordinate $x=75$ is on the right of $x=m$.

From fig., $P\{m<x<75\}=0.08$

$$\therefore \qquad \frac{1}{\sigma\sqrt{2\pi}}\int_m^{75} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx=0{\cdot}08$$

Put $\qquad \dfrac{x-m}{\sigma}=z$

$$\therefore \quad \frac{1}{\sqrt{2\pi}}\int_0^{\frac{75-m}{\sigma}} e^{-\frac{1}{2}z^2}\,dz=0\cdot08$$

$\therefore$ From given,

$$\frac{75-m}{\sigma}=0\cdot20 \qquad \ldots(1)$$

Again $\qquad P(x>80)=0\cdot04$

$\therefore \qquad P\{m<x<80\}=0\cdot46$

$$\therefore \quad \frac{1}{\sigma\sqrt{2\pi}}\int_m^{80} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx=0\cdot46$$

or
$$\frac{1}{\sqrt{2\pi}}\int_0^{\frac{80-m}{\sigma}} e^{-\frac{1}{2}z^2}\,dz=0\cdot46$$

$$\therefore \quad \frac{80-m}{\sigma}=1\cdot75 \qquad \text{(from given)} \quad \ldots(2)$$

From (1) and (2)

$m=74\cdot4$, (approx)

$\sigma=3\cdot2$. (approx)

Ex. 10·59. *One thousand candidates in an examination were grouped in to three classes I, II, III in desceding order of merit. The numbers in the first two classes were 50 and 350 respectively. The highest and lowest marks in class II were 60 and 50 respectively. Assuming the distribution to be normal, prove that the average mark is 48·2 approximately and standard deviation 7·1 approximately. Given that :*

$\frac{x}{\sigma}$	A	$\frac{x}{\sigma}$	A
0·2	0·079	1·5	0·433
0·3	0·118	1·6	0·445
0·4	0·155	1·71	0·455

where the area A is measured from the mean zero to any ordinate x.

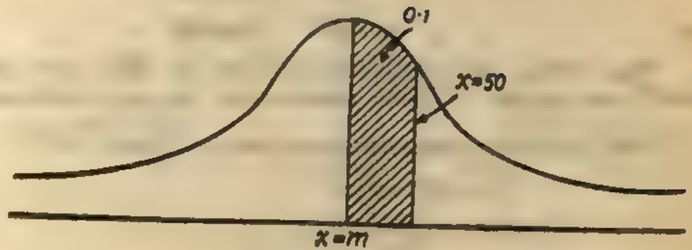
Sol. Number of candidates getting III class

$$=1000-(350+50)=600$$

$\therefore \qquad P\{x<50\}=0\cdot6$

Also $\quad P\{x<m\}=0.5$

$\therefore \quad P\{m<x<50\}=0.1$



Put $$z=\frac{x-m}{\sigma}$$

$$\therefore \quad P\left\{0<z<\frac{50-m}{\sigma}\right\}=0.1$$

From given data

Value of A for $\frac{x}{\sigma}=0.2$ is 0.079

Value of A for $\frac{x}{\sigma}=0.3$ is 0.118

$\therefore$ Increment in A for increment 0.1 in $\frac{x}{\sigma}$

$$=0.039.$$

$\therefore$ Increment in $\frac{x}{\sigma}$ for increment 0.021 in A

$$=\frac{0.1}{0.039}0.021=0.054$$

$\therefore$ Value of $\frac{x}{\sigma}$ (for $A=0.1$) $=0.2+0.054$

$$=0.254$$

$$\therefore \quad \frac{50-m}{\sigma}=0.254 \qquad \dots(1)$$

Also $\quad P\{x>60\}=0.05$

$$P\{m<x<60\}=0.5-0.05=0.45$$

$$\therefore \quad P\left\{0<z<\frac{60-m}{\sigma}\right\}=0.45$$

As above, from given data

$$\frac{60-m}{\sigma}=1.65 \qquad \dots(2)$$

From (1) and (2)

$$\sigma = 7{\cdot}1 \quad \text{(approx)}$$

and
$$m = 48{\cdot}2. \quad \text{(approx)}$$

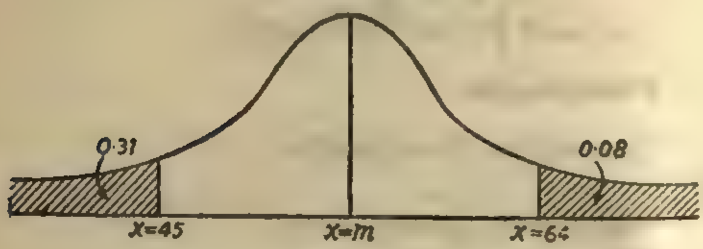
Ex. 10-60. *In a normal dist, 31% of the items are under 45 and 8% are over 64. Find the mean and s.d. of the distribution.*

Sol. Let m be the mean and σ the *s.d.* Then

$$P\{x < 45\} = 0{\cdot}31$$

$$\therefore \quad P\{45 < x < m\} = 0{\cdot}19$$

or
$$P\left\{\frac{45-m}{\sigma} < z < 0\right\} = 0{\cdot}19$$



$$\therefore \quad P\left\{0 < z < \frac{m-45}{\sigma}\right\} = 0{\cdot}19$$

$$\therefore \quad \frac{m-45}{\sigma} = 0{\cdot}496 \qquad \ldots(1)$$

Similarly $P\left\{0 < z < \dfrac{64-m}{\sigma}\right\} = 0{\cdot}42$

$$\therefore \quad \frac{64-m}{\sigma} = 1{\cdot}405 \qquad \ldots(2)$$

From (1) and (2)

$$\sigma = 10 \quad \text{(approx.)}$$

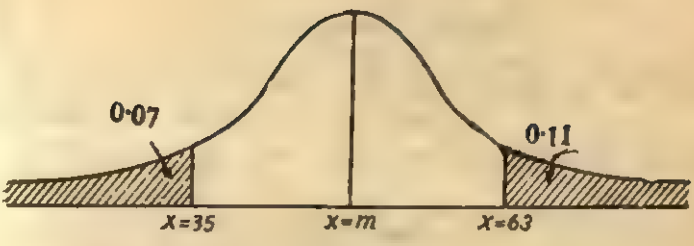
$$m = 50. \quad \text{(approx.)}$$

Ex. 10-61. *In a distribution exactly normal, 7% of the items are under 35 and 89% are under 63. Wh t are the mean and s.d. of the dist ?*

Sol.
$$P\{x < 35\} = 0{\cdot}07$$

$$\therefore \quad P\{35 < x < m\} = 0{\cdot}43$$

$$\therefore \quad P\left\{0 < z < \frac{m-35}{\sigma}\right\} = 0{\cdot}43$$



$$\therefore \qquad \frac{m-35}{\sigma}=1{\cdot}476 \qquad \ldots(1)$$

$$P\{x<63\}=0{\cdot}89$$

$$\therefore \qquad P\{m<x<63\}=0{\cdot}39$$

$$\therefore \qquad P\left\{0<z<\frac{63-m}{\sigma}\right\}=0{\cdot}39$$

$$\therefore \qquad \frac{63-m}{\sigma}=1{\cdot}226 \qquad \ldots(2)$$

From (1) and (2)

$$\sigma=10{\cdot}36 \quad \text{(approx.)}$$

$$m=50{\cdot}29 \quad \text{(approx.)}$$

Ex. 10·62. *Five thousand candidates appeared in a certain examination paper carrying a maximum of 100 marks. It was found that the marks were normally distributed with mean 39·5 and s.d. 12·5. Determine approximately the number of students who secured a first class for which a minimum of 60 marks is necessary you may use the table given below :*

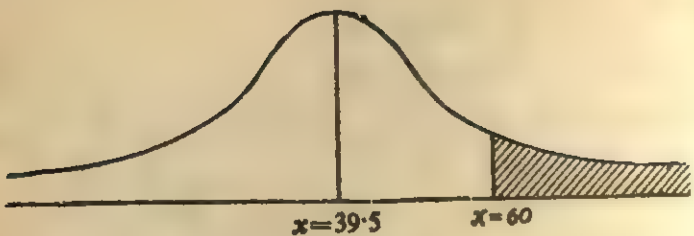
The proportion A of the whole area of the normal curve lying to the left of the ordinate at the deviation $\frac{x}{\sigma}$ *is*

$\frac{x}{\sigma}$:	1·5	1·6	1·7	1·8
A :	0·93319	0·94520	0·95543	0·95407

Sol. $P\{39{\cdot}5<x<60\}$.

$$=P\{0<z<1{\cdot}64\}=0{\cdot}94929-0{\cdot}5$$
$$=0{\cdot}44929$$

$$\therefore \quad P\{x>60\}=0{\cdot}5-0{\cdot}44929$$
$$=0{\cdot}05071$$



$\therefore$ No. of students getting first class
$=253.$

Ex. 10·63. *A minimum height is to be prescribed for eligibility to government services such that 60% of the youngmen will have a fair chance of coming upto that standard. The heights of youngmen are normally distributed with mean 60·6" and s.d. 2·55. Determine the minimum specification.*

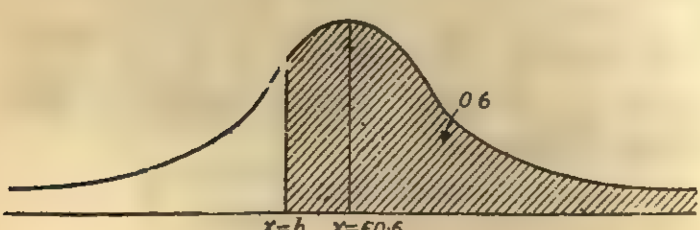
$$\left\{ \textit{From table if } f(t)=\frac{1}{\sqrt{2\pi}}\int_t^{\infty} \exp\left(-\frac{t^2}{2}\right)dt, \textit{ then } f(-0{\cdot}2533)=0{\cdot}6 \right\}.$$

Sol. Let h be the minimum height prescribed.

Then
$$\frac{1}{\sigma\sqrt{2\pi}}\int_h^{\infty} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx=0{\cdot}6$$

Put
$$t=\frac{x-m}{\sigma}$$

$\therefore$
$$\frac{1}{\sqrt{2\pi}}\int_{\frac{h-m}{\sigma}}^{\infty} e^{-\frac{1}{2}t^2}\, dt=0{\cdot}6.$$



$\therefore$
$$\frac{h-m}{\sigma}=-0{\cdot}2533$$

Here $m=60{\cdot}6, \quad \sigma=2{\cdot}55$

$\therefore \quad h=59{\cdot}95 \simeq 60$ (approx.)

Ex. 10·64. *The local authorities in a certain city installed 2,000 electric lamps in streets. If the lamps have an average life of 1,000 burning hours with a s.d. of 200 hours.*

(a) What number of lamps might be expected to fail in first 700 burning hours ?

(b) After what period of burning hours would you expect that 10% of the lamps would have failed ?

Assume that lives of the lamps are normally distributed.

Given that if $F(t)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{t} e^{-\frac{1}{2}t^2}\,dt$

Then $\quad F(1{\cdot}50)=0{\cdot}933$

and $\quad F(1{\cdot}28)=0{\cdot}900$

Sol. Let x hours be the life of a lamp.

(*a*) Since normal curve is symmetrical about

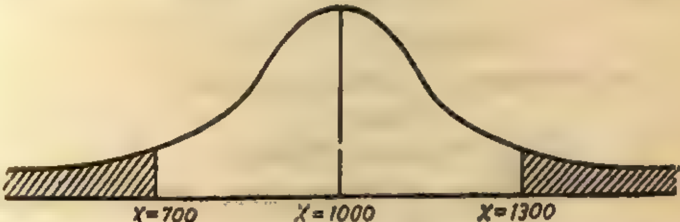
$$x=1000,$$

$$P\{x<700\}=P\{x>1300\}$$

$$=1-P\{x<1300\}$$

$$=1-\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{1300} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}\,dx.$$

Here $\quad m=1000, \quad \sigma=200$



Put $\quad z=\dfrac{x-m}{\sigma}=\dfrac{x-1000}{200}$

$$\therefore \quad P\{x<700\}=1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1{\cdot}5} e^{-\frac{1}{2}z^2}\,dz$$

$$=1-0{\cdot}933=0{\cdot}067$$

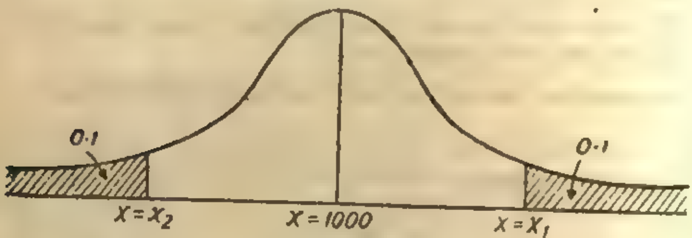
$\therefore$ Number of lamps expected to fail in first 700 hours of burning

$$=2000\times 0{\cdot}067=134$$

(b) Let $x=x_1$ be *s.t.*

$$P\{x\geq x_1\}=0{\cdot}1$$

$\therefore$ $$P\{x<x_1\}=0{\cdot}9$$



or $$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x_1} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx=0{\cdot}9$$

or $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x_1-m}{\sigma}} e^{-\frac{1}{2}z^2}dz=0{\cdot}9$$

$\therefore$ $$\frac{x_1-m}{\sigma}=1{\cdot}28$$

$\therefore$ $$x_1=1000+200(1{\cdot}28)=1256$$

Let $x=x_2$ be *s.t.*

$$P\{x<x_2\}=0{\cdot}1.$$

Then by symmetry of normal curve about

$$x=1000,$$

$$x_2=1000-256=744$$

$\therefore$ After 744 hours of burning, 10% lamps are expected to fail.

Ex. 10·65. *The incomes of a group of 10,000 persons were found to be normally distributed with mean = Rs. 750 p.m. and s.d. = Rs. 50. Show that of this group about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs. 832. What was the lowest income among the richest 100 ?*

Sol. Let x be the variate

Here $m=750,\quad \sigma=50$

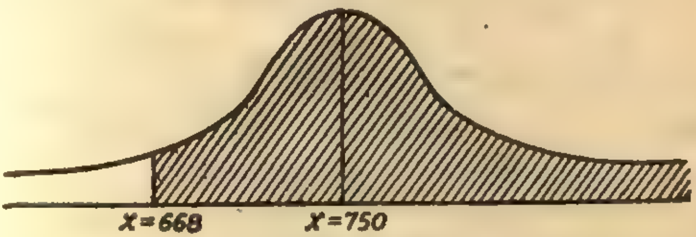
(i) $P\{x>668\}=0{\cdot}5+P\{668<x<750\}$

$=0{\cdot}5+P\{-1{\cdot}64<z<0\}$

$=0{\cdot}5+P\{0<z<1{\cdot}64\}$

$=0{\cdot}5+0{\cdot}4495$

$=0{\cdot}9495$



$\therefore$ Percentage of persons having income exceeding Rs. 668

$=94{\cdot}95\simeq 95\%$.

(ii) $P\{x>832\}=0{\cdot}5-P\{750<x<832\}$

$=0{\cdot}5-P\{0<z<1{\cdot}64\}$

$=0{\cdot}5-0{\cdot}4495=0{\cdot}0505$

$\therefore$ Percentage of persons having income exceeding Rs. 832

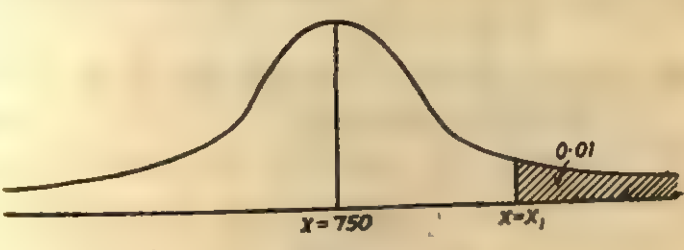
$=5\%$.

(iii) Let $x=x_1$ be *s.t.*

$P\{x>x_1\}=0{\cdot}01$

Then $x=x_1$ is the lowest income among the richest 100.

$\therefore$ $P\{750<x<x_1\}=0{\cdot}49$



or $P\left\{0<z<\dfrac{x_1-750}{50}\right\}=0{\cdot}49.$

$\therefore$ $\dfrac{x_1-750}{50}=2{\cdot}3267$

$\therefore$ $x_1=866{\cdot}34.$

Ex. 10·66. *If x is a $N(2, 3)$, find $P\left(y > \frac{3}{2}\right)$ where $y = x - 1$.*

Sol. Now $P\left(y > \frac{3}{2}\right)$

$$= P\left(x - 1 > \frac{3}{2}\right)$$

$$= P(x > 2{\cdot}5)$$

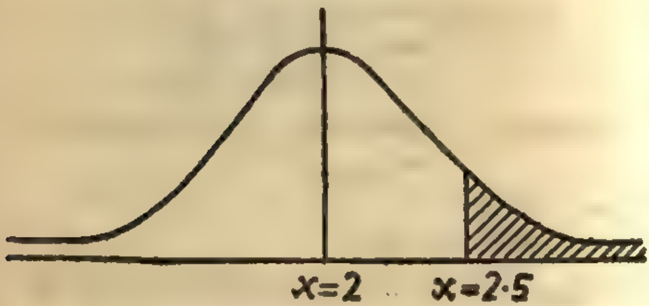
$$= 0{\cdot}5 - P(2 \leq x \leq 2{\cdot}5)$$

Put $$z = \frac{x-2}{3}$$

$\therefore$ $$P\left(y > \frac{3}{2}\right) = 0{\cdot}5 - P(0 \leq z \leq {\cdot}17)$$

$$= 0{\cdot}5 - {\cdot}0675$$

$$= 0{\cdot}4325$$

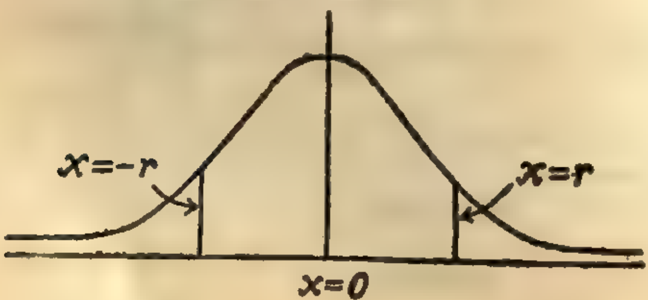


Ex. 10·67. *If $N(r) = P(x \leq r)$, where x is a $N(0, 1)$, show that*

$$N(-r) = 1 - N(r)$$

Sol. Since normal curve is symmetrical about $x = 0$,

$$P(0 \leq x \leq r) = P(-r \leq x \leq 0) \qquad \ldots(1)$$



Now $N(-r)=P(x\leqslant -r)$

$$=0\cdot5-P(-r\leqslant x\leqslant 0)$$
$$=0\cdot5-P(0\leqslant x\leqslant r)$$
$$=1-\{0\cdot5+P(0\leqslant x\leqslant r)\}$$
$$=1-P(x\leqslant r)$$
$$=1-N(r)$$

10.4. Geometric Distribution

The prob. dist.

$$P(x)=q^x p,\quad x=0,\ 1,\ 2,\ \ldots\ldots,\ q=1-p$$

is called geometric distribution.

$$\bar{x}=E(x)=\sum_{x=0}^{\infty} xq^x p=p\{q+2q^2+3q^3+\ldots\ldots\}$$
$$=pq\{1+2q+3q^2+\ldots\ldots\}=pq(1-q)^{-2}$$
$$=\frac{q}{p}$$

$$\mu_2'(0)=E(x^2)\sum_{x=0}^{\infty} x^2q^x p=p\sum_{x=0}^{\infty}\{x(x-1)+x\}q^x$$
$$=p\sum_{x=0}^{\infty} x(x-1)q^x+p\sum_{x=0}^{\infty} xq^x$$
$$=p\{2\cdot1q^2+3\cdot2q^3+4\cdot3q^4+\ldots\ldots\}+\frac{q}{p}$$
$$=2q^2p\left\{1+3q+\frac{4\cdot3}{2\,!}q^2+\frac{5\cdot4}{2\,!}q^3\ldots+\right\}+\frac{q}{p}$$
$$=2q^2p(1-q)^{-3}+\frac{q}{p}$$
$$=\frac{2q^2}{p^2}+\frac{q}{p}$$

$$\therefore\quad \mu_2=\mu_2'(0)-\bar{x}^2=\frac{q^2}{p^2}+\frac{q}{p}=\frac{q}{p}\left(\frac{q+p}{p}\right)=\frac{q}{p^2}.$$

Ex. 10-68. *Find M.G.F. of the geometric dist.*

Sol. $M_0(t)=E(e^{tx})=\sum_{x=0}^{\infty} e^{tx}q^x p$

$$=\sum_{x=0}^{\infty}(qe^t)^x p$$
$$=p\cdot\frac{1}{1-qe^t}.$$

Ex. 10·69. *A population is subjected to recurring attacks of a disease and each attack affects a proportion p of the population. Assuming that r attacks are fatal to the individual, find the proportion of dying during the nth exposure.*

Sol. Prob of an attack affecting an individual$=p$

$\therefore$ Prob of an attack not affecting an individual$=1-p$

The individuals dying during the nth exposure will be those who have had the disease $(r-1)$ times in the first $(n-1)$ exposures and catch it again.

Now prob of having disease $(r-1)$ times in the first $(n-1)$ exposures

$$={}^{n-1}c_{r-1}p^{r-1}q^{n-r}$$

where $\qquad q=1-p$

$\therefore$ Prob. of dying during nth exposure

$$=({}^{n-1}c_{r-1}p^{r-1}q^{n-r})(p)={}^{n-1}c_{r-1}q^{n-r}p^r$$

which also gives the proportion of dying during the nth exposure.

10.5. Negative Binomial Distribution

In the last question, proportion of individuals dying during the nth exposure

$$={}^{n-1}c_{r-1}q^{n-r}p^r$$

Since death does not commence until the rth exposure, the proportions of death at the rth, $(r+1)$th, ... exposures are

$$p^r, rqp^r, \frac{r(r+1)}{2\,!}q^2p^r, \ldots\ldots$$

which are the successive terms in the binomial expansion, with negative index, of $p^r(1-q)^{-r}$

$$\therefore \qquad \sum_{n=r}^{\infty} {}^{n-1}c_{r-1}q^{n-r}p^r=p^r(1-q)^{-r}=1$$

The dist

$$P(x)={}^{x+r-1}c_{r-1}q^xp^r,\ x=0, 1, 2, \ldots\ldots\ldots$$

is called *Negative Binomial Distribution.*

10.5.1. Mean and Variance

$$\bar{x}=\sum_{x=0}^{\infty} x.{}^{x+r-1}c_{r-1}q^xp^r$$

$$=p^r\left\{rq+2\,\frac{(r+1)r}{2\,!}q^2+\frac{3(r+2)(r+1)r}{3\,!}q^3+\ldots\ldots\right\}$$

$$=rqp^r\{1+(r+1)q+\frac{(r+2)(r+1)}{2\,!}q^2+\ldots\ldots\}$$

$$=rqp^r(1-q)^{-r-1}=\frac{rq}{p}$$

$$\mu_2'(0)=\sum_{x=0}^{\infty} x^2\, {}^{x+r-1}c_{r-1}q^x p^r$$

$$=\sum_{x=0}^{\infty} \{x(x-1)+x\}\, {}^{x+r-1}c_{r-1}q^x p^r$$

$$=p^r\left\{2.1\frac{(r+1)r}{2\,!}q^2+3.2\frac{(r+2)(r+1)r}{3\,!}q^3+\ldots\ldots\right\}+\frac{rq}{p}$$

$$=p^r(r+1)rq^2(1-q)^{-r-2}+\frac{rq}{p}=\frac{r(r+1)q^2}{p^2}+\frac{rq}{p}$$

$\therefore\quad \mu_2=\mu_2'(0)-\bar{x}^2$

$$=\frac{r(r+1)q^2}{p^2}+\frac{rq}{p}-\frac{r^2q^2}{p^2}$$

$$=\frac{rq}{p^2}(q+p)=\frac{rq}{p^2}$$

Since $\frac{1}{p}>1$, $\mu_2>\bar{x}$

10.5.2. Moment Generating Function and Cumulants

$$M_0(t)=\sum_{x=0}^{\infty} e^{tx}.\,{}^{x+r-1}c_{r-1}q^x p^r$$

$$=p^r\sum_{x=0}^{\infty} {}^{x+r-1}c_{r-1}(qe^t)^x=p^r(1-qe^t)^{-r}$$

$\therefore$ Cumulative f^n is given by

$$K_0(t)=\log M_0(t)=r\log p-r\log(1-qe^t)$$

$$=-r\log\left\{\frac{1}{p}-\frac{q}{p}e^t\right\}=-r\log\left\{1-\frac{q}{p}(e^t-1)\right\}$$

$$=-r\log\left\{1-\frac{q}{p}\left(t+\frac{t^2}{2\,!}+\frac{t^3}{3\,!}+\frac{t^4}{4\,!}+\ldots\ldots\right)\right\}$$

$$=r\left[\frac{q}{p}\left(t+\frac{t^2}{2\,!}+\frac{t^3}{3\,!}+\frac{t^4}{4\,!}+\cdots\right)+\frac{1}{2}\frac{q^2}{p^2}\left(t+\frac{t^2}{2\,!}+\frac{t^3}{3\,!}\cdots\right)^2\right.$$

$$\left.+\frac{1}{3}\frac{q^3}{p^3}\left(t+\frac{t^2}{2\,!}\cdots\cdots\right)^3+\frac{1}{4}\frac{q^4}{p^4}(t+\ldots\ldots)^4+\ldots\right]$$

$$\therefore \quad k_1=\frac{rq}{p}, \quad k_2=\frac{rq}{p^2}, \quad k_3=r\left\{\frac{q}{p}+\frac{3q^2}{p^2}+\frac{2q^3}{p^3}\right\}$$

$$=\frac{rq}{p^3}(1+q)$$

and $$k_4=r\left\{\frac{q}{p}+\frac{7q^2}{p^2}+12\,\frac{q^3}{p^3}+6\,\frac{q^4}{p^4}\right\}$$

$$=\frac{rq}{p^4}\{1+4q+q^2\}.$$

Ex. 10-70. *Find the limit of negative binomial distribution when $r \to \infty$ and $q \to 0$ but $\frac{rq}{p}=m$ (a finite constant).*

Sol. For negative binomial dist.

$$P(x)={}^{x+r-1}c_{r-1}q^x p^r$$

$$=\frac{(x+r-1)\,!}{(r-1)\,!\,x\,!}\,q^x p^r$$

$$=\frac{(x+r-1)(x+r-2)\ldots\ldots r}{r^x\,.\,x\,!}\left(\frac{rq}{p}\right)^x(1-q)^{r+x}$$

$$=\frac{1}{x\,!}\left\{\left(1+\frac{x-1}{r}\right)\left(1+\frac{x-2}{r}\right)\ldots\ldots\left(\frac{r}{r}\right)\right\}\left(\frac{rq}{p}\right)^x$$

$$(1-q)^x.\left(1-\frac{mp}{r}\right)^r$$

$$=\frac{1}{x\,!}\left\{\left(1+\frac{x-1}{r}\right)\left(1+\frac{x-2}{r}\right)\ldots\ldots\frac{r}{r}\right\}\left(\frac{rq}{p}\right)^x(1-q)^x$$

$$\left\{\left(1-\frac{mp}{r}\right)^{-\frac{r}{mp}}\right\}^{-mp}$$

Let $r \to \infty$, $q \to 0$ so that $\frac{rq}{p}=m$. Then $p \to 1$.

$$\therefore \qquad P(x) \to \frac{m^x e^{-m}}{x\,!}$$

which is the probability function for Poisson dist.

10-6. Hypergeometric Distribution

Ex. 10-71. *Define hypergeometric distribution and find its mean and variance.*

Sol. Suppose an urn contains Np white and Nq blue balls $(p+q=1)$ and r balls are to be drawn one at a time without replacement. Let $P(x)$ be the prob. that out of r balls drawn x are white. Then

$$P(x)=\frac{Np_{c_x}\,Nq_{c_{r-x}}}{N_{c_r}}$$

$$={}^rc_x\frac{(Np)^{(x)}(Nq)^{(r-x)}}{N^{(r)}}$$

where $x^{(r)}=x(x-1)......(x-r+1)$

Consider

$$(1+y)^{Np}(1+y)^{Nq}=\left(\sum_{s=0}^{Np}Np_{c_s}y^s\right)\left(\sum_{t=0}^{Nq}Np_{c_t}y^t\right)$$

and
$$(1+y)^{N}=\sum_{r=0}^{N}N_{c_r}y^r$$

Since $(1+y)^{Np}(1+y)^{Np}=(1+y)^{N}$, equating co-efficients of y^r.

$$N_{c_r}=\sum_{x=0}^{r}Np_{c_x}.\,Nq_{c_{r-x}}$$

$$\therefore \qquad \sum_{x=0}^{r}P(x)=1$$

$\therefore$ $P(x)$ can be taken to be a probability density function. The distribution

$$P(x)={}^rc_x\,\frac{(Np)^{(x)}(Nq)^{(r-x)}}{N^{(r)}},\ x=0,\ 1,\ 2,\ \ldots r$$

is called **Hypergeometric Distribution.**

10-6.1. Mean and variance of Hypergeometric Distribution.

$$\bar{x}=\sum_{x=0}^{r}x.\ \frac{Np_{c_x}.Nq_{c_{r-x}}}{N_{c_r}}$$

$$=\sum_{x=1}^{r}x.\ \frac{Np_{c_x}\ Nq_{c_{r-x}}}{N_{c_r}}=Np\sum_{x=1}^{r}\frac{Np-1_{c_{x-1}}\,Nq_{c_{r-x}}}{N_{c_r}}$$

$$=\frac{Np}{N_{c_r}}\ Np+Nq-1_{c_{r-1}}=\frac{Np}{N_{c_r}}\ N-1_{c_{r-1}}=rp$$

$$\mu_2'(x) = \sum_{x=0}^{r} x^2 P(x) = \sum_{x=0}^{r} x(x-1)\,P(x) + \sum_{x=0}^{r} xP(x)$$

$$= \sum_{x=0}^{r} x(x-1) \frac{{}^{Np}c_x \, {}^{Nq}c_{r-x}}{{}^{N}c_r} + rp$$

$$= \frac{(Np)(Np-1)}{{}^{N}c_r} \sum_{x=2}^{r} {}^{Np-2}c_{x-2} \, {}^{Nq}c_{r-x} + rp$$

$$= \frac{(Np)(Np-1)}{{}^{N}c_r} \cdot {}^{Np+Nq-2}c_{r-2} + rp$$

$$= \frac{(Np)(Np-1)}{{}^{N}c_r} \; {}^{N-2}c_{r-2} + rp$$

$$\therefore \quad \mu_2'(0) - rp = (Np).(Np-1) \cdot \frac{{}^{N-2}c_{r-2}}{{}^{N}c_r} = (Np)(Np-1)\frac{r(r-1)}{N(N-1)}$$

$$\therefore \quad \mu_2'(0) = \frac{rp(Np-1)(r-1)}{N-1} + rp$$

$$\therefore \quad \mu_2 = \mu_2'(0) - \bar{x}^2 = \frac{rp(Np-1)\,(r-1)}{N-1} + rp - r^2p^2$$

$$= \frac{rp}{N-1}\left\{ Npr - Np - r + 1 + N - 1 - rpN + rp \right\}$$

$$= \frac{rpq}{N-1}\,(N-r)$$

Ex. 10·72. *Find differential equation satisfied by $M_0(t)$ of hypergeometric dist. and deduce the values of moments about mean.*

Sol. $$M_0(t) = \sum_{x=0}^{r} e^{tx} . {}^{r}c_x \, \frac{(Np)^{(x)}(Nq)^{(r-x)}}{N^{(r)}}$$

$$= \sum_{x=0}^{r} e^{tx} . {}^{r}c_x \, \frac{(Np)^{(x)}(Nq)^{(r)}}{N^{(r)}(Nq-r+x)^{(x)}}$$

$$= \frac{(Nq)^{(r)}}{N^{(r)}} \sum_{x=0}^{r} \frac{(Np)^{(x)} . r^{(x)} . e^{tx}}{(Nq-r+x)^{(x)}\; x\,!}$$

Let $F(\alpha, \beta ; \gamma, y)=1+\frac{\alpha\beta}{\gamma}\frac{y}{1!}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{y^2}{2!}+\ldots$

Then

$$M_0(t)=\frac{(Nq)^{(r)}}{N^{(r)}}F(-r, -Np ; Nq-r+1, e^t)$$

$F(\alpha, \beta ; \gamma, y)$ satisfies the differential equation

$$y(1-y)\frac{d^2F}{dy^2}+\{r-(\alpha+\beta+1)y\}\frac{dF}{dy}-\alpha\beta F=0$$

Put $\qquad y=e^t$

Then diff. eq. reduces to

$$(1-e^t)\frac{d^2F}{dt^2}+\frac{dF}{dt}\{\gamma-(\alpha+\beta)e^t-1\}-\alpha\beta e^tF=0$$

$\therefore \quad M_0(t)$ satisfies the equation

$$(1-e^t)\frac{d^2M_0(t)}{dt^2}+\frac{dM_0(t)}{dt}\{Nq-r+1-(-r-Np)e^t-1\}$$
$$-rNpe^tM_0(t)=0$$

or $$(1-e^t)\left\{\frac{d^2M_0(t)}{dt^2}-(r+Np)\frac{dM_0(t)}{dt}+rNpM_0(t)\right\}$$
$$+N\frac{dM_0(t)}{dt}-rNpM_0(t)=0$$

To find mean put $M_0(t)=\sum\limits_{s=0} \mu'_s\frac{t^s}{s!}$

Then

$$(1-e^t)\left\{\sum_{s=2}\mu'_s\frac{t^{s-2}}{(s-2)!}-(r+Np)\sum_{s=1}\mu'_s\frac{t^{s-1}}{(s-1)!}+rNp\sum_{s=0}\mu'_s\frac{t^s}{s!}\right.$$
$$+N\sum_{s=1}\mu'_s\frac{t^{s-1}}{(s-1)!}-rNp\sum_{s=0}\mu'_s\frac{t^s}{s!}=0$$

Put $\qquad t=0$

$\therefore \qquad N\mu_1'-rNp=0 \quad$ or $\quad \mu_1'=rp$

Now $\qquad M_{\bar{x}}(t)=e^{-rpt}M_0(t)$

$\therefore \qquad M_0(t)=e^{rpt}M_{\bar{x}}(t)$

Substituting in the diff. eq.

$$(1-e^t)\left[\frac{d^2M_{\bar{x}}(t)}{dt^2}+\{r(p-q)-Np\}\frac{dM_{\bar{x}}(t)}{dt}+(N-r)pqrM_{\bar{x}}(t)\right]$$

$$+N\frac{dM_{\bar{x}}(t)}{dt}=0$$

To find moments about mean put

$$M_{\bar{x}}(t)=\sum_{s=0}\mu_s\frac{t^s}{s\,!}\text{ and }e^t=1+t+\frac{t^2}{2\,!}+$$

$$-\sum_{i=1}^{\infty}\frac{t^i}{i\,!}\left[\sum_{s=2}\mu_s\frac{t^{s-2}}{(s-2)!}+\{r(p-q)-Np\}\sum_{s=1}\mu_s\frac{t^{s-1}}{(s-1)!}\right.$$

$$\left.+(N-r)pqr\sum_{s=0}\mu_s\frac{t^s}{s\,!}\right]+N\sum_{s=0}\mu_s\frac{t^{s-1}}{(s-1)!}=0$$

Equating co-efficients of t, t^2,......

$$\mu_2=\frac{rpq(N-r)}{N-1}$$

$$\mu_3=\frac{rpq(q-p)\,(N-r)\,(N-2r)}{(N-1)\,(N-2)}$$

Ex. 10·73. *Show that when $N\to\infty$ hypergeometric dist. tends to binomial dist.*

Sol. For hypergeometric dist.

$$P(x)={}^rc_x\frac{\{(Np)(Np-1)\ldots(Np-x+1)\}\{(Nq)(Nq-1)\ldots(Nq-r+x+1)\}}{N(N-1)\ldots\ldots(N-r+1)}$$

$$={}^rc_x\frac{\left\{P\left(p-\frac{1}{N}\right)\ldots\left(p-\frac{x-1}{N}\right)\right\}\cdot\left\{q\left(q-\frac{1}{N}\right)\ldots\left(q-\frac{r-x-1}{N}\right)\right\}}{\frac{N}{N}\left(1-\frac{1}{N}\right)\ldots\left(1-\frac{r-1}{N}\right)}$$

Let $\quad N\to\infty$

Then $\quad P(x)\to{}^rc_xp^xq^{r-x}$

which is the probability function for binomial dist.

Ex. 10 74. *Deduce the moments (about mean) of binomial dist. from those of hypergeometric dist.*

Sol. For hypergeometric dist.

$$\mu_2 = \frac{rpq(N-r)}{N-1} = \frac{rpq\left(1-\frac{r}{N}\right)}{1-\frac{1}{N}}$$

and
$$\mu_3 = \frac{rpq\,(q-p)(N-r)(N-2r)}{(N-1)(N-2)}$$

$$= \frac{rpq(q-p)\left(1-\frac{r}{N}\right)\left(1-\frac{2r}{N}\right)}{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)}$$

Let $N \to \infty$

Then μ_2 for B.D. $= rpq$

and μ_3 for B.D. $= rpq(q-p)$

10.7. Multinomial Distribution

Let there be a series of n independent trials where each trial may result in one of the several outcomes say $E_1, E_2, \ldots\ldots E_k$ with respective probabilities $p_1, p_2, \ldots p_k$ in each trial where

$$p_1 + p_2 + \ldots\ldots + p_k = 1$$

Suppose the event E_i occurs x_i times. $(i=1, 2, \ldots k)$

Then $x_1 + x_2 + \ldots\ldots + x_n = n$

By the theorem of compound prob., prob. of E_1 occurring x_1 times, E_2 occurring x_2 times and so on in any fixed definite order $= p_1^{x_1} p_2^{x_2} \ldots\ldots p_k^{x_k}$

Now out of n trials x_1 trails can be had in ${}^nc_{x_1}$ ways and out of remaining $(n-x_1)$ trials, x_2 trials can be had in ${}^{n-x_1}c_{x_2}$ ways and so on.

$\therefore$ The total number of ways of getting E_1-x_1 times, E_2-x_2 times, ... E_k-x_k times.

$$= {}^nc_{x_1} \cdot {}^{n-x_1}c_{x_2} \ldots\ldots {}^{n-x_1-x_2\ldots\ldots-x_{k-1}}c_{x_k}$$

$$= \frac{n!}{x_1!\,(n-x_1)!} \cdot \frac{(n-x_1)!}{(n-x_1-x_2)!} \ldots\ldots \frac{(n-x_1-x_2\ldots\ldots-x_{k-1})!}{x_k!\,(n-x_1\ldots\ldots-x_k)!}$$

$$= \frac{n!}{x_1!\,x_2!\ldots\ldots x_k!} \qquad (\because\ (n-x_1\ldots\ldots-x_k)! = 0! = 1!)$$

$\therefore$ By total prob. theorem, prob. of getting E_1-x_1 times, E_2-x_2 times,...... E_k-x_k times

$$=\frac{n!}{x_1!\,x_2!\,......\,x_k!}\,p_1^{x_1}\;p_2^{x_2}\;......\;p_k^{x_k}$$

$$\text{Total prob.}=\sum_{x_1=0}^{n}\sum_{x_2=0}^{n-x_1}......\sum_{x_k=0}^{(n-x_1...-x_{k-1})}\frac{n!}{x_1!\,x_2!\,......\,x_k!}\,p_1^{x_1}\;p_2^{x_2}\;......\;p_k^{x_k}$$

Consider $(p_1+p_2......+p_k)^n=\{p_1+(p_2+p_3......+p_k)\}^n$

$$=\sum_{x_1=0}^{n}{}^{n}c_{x_1}\;p_1^{x_1}\;(p_2+p_3......p_k)^{n-x_1}$$

$$=\sum_{x_1=0}^{n}{}^{n}c_{x_1}\;p_1^{x_1}\left\{\sum_{x_2=0}^{n-x_1}{}^{n-x_1}c_{x_2}\;p_2^{x_2}\,(p_3+\,...\,+p_k)^{n-x_1-x_2}\right\}$$

$$=\sum_{x_1=0}^{n}\sum_{x_2=0}^{n-x_1}{}^{n}c_{x_1}\;{}^{n-x_1}c_{x_2}\;p_1^{x_1}\;p_2^{x_2}(p_3+...+p_k)^{n-x_1-x_2}$$

....

$$=\sum_{x_1=0}^{n}\sum_{x_2=0}^{n-x_1}......\sum_{x_k=0}^{n-x_1...-x_{k-1}}\left\{{}^{n}c_{x_1}\;{}^{n-x_1}c_{x_2}\;...\;{}^{n-x_1...-x_{k-1}}c_{x_k}\;.\,p_1^{x_1}\;p_2^{x_2}\;...\;p_k^{x_k}\right\}$$

$$=\sum_{x_1=0}^{n}\sum_{x_2=0}^{n-x_1}......\sum_{x_k=0}^{n-x_1...-x_{k-1}}\frac{n!}{x_1!\,x_2!\,......\,x_k!}\,p_1^{x_1}\;p_2^{x_2}\;...p_k^{x_k}$$

$\therefore$ Total prob.$=(p_1+p_2+...+p_k)^n=1$

Hence the function

$$P(x_1,x_2,\,...\,x_k)=\frac{n!}{x_1!\,x_2!\,......\,x_k!}\,p_1^{x_1}\;p_2^{x_2}\;...\;p_k^{x_k}$$

can be taken to be probability function. The dist formed by $P(x_1, x_2, \ldots x_n)$ together with the values of $x_1, x_2, \ldots x_k$ is called *Multinomial Dist.*

10.7.1. Moment Generating Function, Moments, Covariance etc., for Multinomial Distribution.

Now $M_0(t_1, t_2 \ldots, t_k) = E\left\{ e^{t_1x_1+t_2x_2+\ldots+t_kx_k} \right\}$

$$=\sum_{x_1=0}^{n} \sum_{x_2=0}^{n-x_1} \cdots\cdots \sum_{x_k=0}^{n-x_1\ldots-x_{k-1}} \frac{n\,!}{x_1\,!\,x_2\,!\ldots\ldots x_k\,!}\, p_1^{x_1}\, p_2^{x_2} \ldots p_k^{x_k} \;.e^{t_1x_1+\ldots+t_kx_k}$$

$$=\sum_{x_1} \sum_{x_2} \cdots\cdots \sum_{x_k} \frac{n\,!}{x_1\,!\,x_2\,!\ldots x_k\,!} \left(p_1e^{t_1} \right)^{x_1} \cdots \left(p_ke^{t_k} \right)^{x_k}$$

$$=\left(p_1e^{t_1} + p_2e^{t_2} \ldots\ldots + p_ke^{t_k} \right)^n$$

$$\therefore \quad E(x_i) = \left\{ \frac{\partial M_0}{\partial t_i} \right\}_{t_j=0,\; j=1,\, 2\,\ldots\, k}$$

$$= \left\{ np_ie^{t_i} \left(p_1e^{t_1} + p_2e^{t_2} \ldots + p_ke^{t_k} \right)^{n-1} \right\}_{\substack{t_j=0 \\ j=1,\, 2\, \ldots k}}$$

$$= np_i$$

$$E(x_i^2) = \left\{ \frac{\partial^2 M_0}{\partial t_i^2} \right\}_{t_j=0,\; j=1,\, 2,\, \ldots\, k}$$

$$= \left\{ np_ie^{t_i} \left(p_1e^{t_1} + \ldots + p_ke^{t_k} \right)^{n-1} + n(n-1)p_i^2e^{2t_i} \left(p_1e^{t_1} + \ldots\ldots + p_ke^{t_k} \right)^{n-2} \right\}_{\substack{t_j=0 \\ j=1,\, 2,\, \ldots,\, k}}$$

$$= np_i + n(n-1)p_i^2$$

$$\therefore \quad \text{Var}(x_i) = np_i + n(n-1)p_i^2 - n^2p_i^2 = np_i(1-p_i)$$

$$E(x_iy_j) = \left\{ \frac{\partial^2 M_0}{\partial t_j \partial t_i} \right\}_{t_1=0,\, \ldots\ldots t_k=0}$$

$$= \left\{ n(n-1)p_ip_je^{t_i}\,.e^{t_j} \left(p_1e^{t_1} + \ldots\ldots + p_ke^{t_k} \right)^{n-2} \right\}_{t_1=0,\, t_2=0,\, \ldots\ldots t_k=0}$$

$$= n(n-1)p_ip_j$$

$$\therefore \quad \text{Cov}\,(x_i, x_j) = E(x_ix_j) - E(x_i)E(x_j)$$

$$= n(n-1)p_ip_j - n^2p_ip_j = -np_ip_j$$

Aliter

$$E(x_i)=\sum x_i \frac{n!}{x_1!\,x_2!\ldots x_k!}\,p_1^{x_1}\,p_2^{x_2}\ldots\ldots p_k^{x_k}$$

$$=p_i\frac{\partial}{\partial p_i}\sum \frac{n!}{x_1!\,x_2!\ldots x_k!}\,p_1^{x_1}\ldots\ldots p_k^{x_k}$$

$$=p_i\frac{\partial}{\partial p_i}(p_1+p_2\ldots\ldots+p_k)^n$$

$$=np_i(p_1+p_2\ldots\ldots+p_k)^{n-1}=np_i$$

$$E(x_i^2)=\sum x_i^2\frac{n!}{x_1!\,x_2!\ldots x_k!}\,p_1^{x_1}\,p_2^{x_2}\ldots\ldots p_k^{x_k}$$

$$=p_i\frac{\partial}{\partial p_i}\sum x_i\frac{n!}{x_1!\,x_2!\ldots x_k!}\,p_1^{x_1}\,p_2^{x_2}\ldots\ldots p_k^{x_k}$$

$$=p_i\frac{\partial}{\partial p_i}\left\{p_i\frac{\partial}{\partial p_i}\sum\frac{n!}{x_1!\,x_2!\ldots x_k!}\,p_1^{x_1}\,p_2^{x_2}\ldots\ldots p_k^{x_k}\right\}$$

$$=p_i\frac{\partial}{\partial p_i}\left\{p_i\frac{\partial}{\partial p_i}(p_1+p_2+\ldots\ldots+p_k)^n\right\}$$

$$=p_i\frac{\partial}{\partial p_i}\{np_i(p_1+p_2\ldots\ldots+p_k)^{n-1}\}$$

$$=p_i\{n(p_1+p_2\ldots\ldots+p_k)^{n-1}+n(n-1)p_i(p_1+\ldots\ldots+p_k)^{n-2}\}$$

$$=p_i\{n+n(n-1)p_i\}=np_i+n(n-1)p_i^2.$$

$$E(x_ix_j)=\sum x_ix_j\frac{n!}{x_1!\,x_2!\ldots x_k!}\,p_1^{x_1}\,p_2^{x_2}\ldots\ldots p_k^{x_k}$$

$$=p_i\frac{\partial}{\partial p_i}\left\{\sum x_j\frac{n!}{x_1!\,x_2!\ldots x_k!}\,p_1^{x_1}\,p_2^{x_2}\ldots\ldots p_k^{x_k}\right\}$$

$$=p_i\frac{\partial}{\partial p_i}\left\{p_j\frac{\partial}{\partial p_j}(p_1+p_2\ldots\ldots p_k)^n\right\}$$

$$=n(n-1)p_ip_j.$$

EXERCISES

1. The incidence of occupational disease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of six workers 4 or more will catch the disease ?

$$\left[\textbf{Ans. } \frac{53}{3125}\right]$$

2. If 5 coins are tossed, what is the probability that there shall be at least 4 heads ?

$$\left[\textbf{Ans. } \frac{3}{16}\right]$$

3. Out of 4000 families with 4 children each, how many would you expect to have at least 1 boy ? Assume that the probability of a male birth is $\frac{1}{2}$. [Ans. 3750]

4. Find the probability of guessing correctly at least 6 of the 10 answers on a true-false examination.

$$\left[\text{Ans. } \frac{193}{512}\right]$$

5. If we take 100 sets of 10 tosses of a perfect coin, in how many cases should we expect to get 7 heads and 3 tails ? [Ans. 12]

6. In the above example in how many cases should we expect to get 7 heads at least ? [Ans. 17]

7. An ordinary six-sided die is thrown 4 times. What are the probabilities of obtaining 4, 3, 2, 0 aces ?

$$\left[\text{Ans. } \frac{625}{1296}, \frac{25}{216}, \frac{5}{324}, \frac{1}{1296}\right]$$

8. An experiment succeeds twice as often as it fails. Find the chance that in the next six trials there will be at least 4 successes. $\left[\text{Ans. } \frac{496}{729}\right]$

9. A teacher claims that he could often tell while his students were still in their first year whether they will obtain I, II, III divisions or fail in their final examinations. To demonstrate his claim he forecasts the fates of 8 students. Find the probability of his being correct in 4 cases. $\left[\text{Ans. } \frac{2835}{32768}\right]$

10. In litters of 4 mice the number of litters which contained 0, 1, 2, 3, 4 females were noted. The figures are given in the table below :

No. of female mice	0	1	2	3	4	Total
No. of litters	9	30	35	24	5	103

If the chance of obtaining a female in a single trial is assumed constant, estimate this constant of unknown probability. Find also expected frequencies.

[Ans. 0·466, expected frequencies are the respective terms in the binomial expansion of $103(0{\cdot}534+0{\cdot}466)^4$].

11. Ten coins are tossed 1024 times and the following frequencies observed. Compare these frequencies with the expected frequencies obtained by fitting binomial dist. to the data.

No. of heads (x)	0	1	2	3	4	5	6
Frequencies (f)	3	8	39	106	188	257	226

No. of heads (x)	7	8	9	10
Frequencies (f)	128	59	7	3

[Ans.

x:	0	1	2	3	4	5	6	7	8	9	10
f:	1	10	45	120	210	252	210	120	45	10	1

]

12. Find the mean and standard deviation for the table of deaths of women over 85 years old recorded in a three years period :

No. of death recorded in a day.	0	1	2	3	4	5	6	7
No. of days.	364	376	218	89	33	13	2	1

Find the expected number of days with one death recorded for the Poisson series fitted to the data. [Ans. 1·18, 1·17, 397]

13. Red blood cell deficiency may be determined by examining a specimen of the blood under a microscope. Suppose a certain small fixed volume contains on the average 20 red cells for normal persons. Using Poisson distribution, obtain the probability that a specimen from a normal person will contain less than 15 red cells.

$$\left[\text{Ans. } e^{-20}\sum_{x=0}^{14}\frac{(20)^x}{x\,!}\right]$$

14. A large number of observations on a given solution, which contained bacteria, were made taking samples of 1 c.c. each and noting down the number of bacteria present in each sample. Assuming the Poisson distribution and given that 10% samples contained no bacteria, find the average number of bacteria per c.c. [Ans. 2·3026]

15. In 1000 extensive sets of trials for an event of small probability the frequencies 'f' of the number x of successes are found to be

x :	0	1	2	3	4	5	6	7
f :	305	365	210	80	28	9	2	1

Fitting Poisson distribution to the above data calculate theoretical frequencies.

[Ans. 301, 361, 217, 87, 26, 6, 1 and 2]

16. The following data gives the frequency distribution of the number of men killed by the kick of a horse in 10 Prussian Army Corps per army corps per annum over 20 years.

No. of deaths.	0	1	2	3	4	Total
Frequency.	109	65	22	3	1	200

Show that the distribution is roughly Poissonian and calculate the theoretical frequencies.

$(e^{-0.61}=0.5434)$

[**Ans.** 109, 66, 20, 4 and 1 (4 and over)]

17. A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more that 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality? $(e^{-5}=0.006738)$

$$\left[\textbf{Ans.}\quad 1-(0.0067)\sum_{x=0}^{10}\frac{5^x}{x\,!}\right]$$

18. Letters were received in an office on each of 100 days. Assuming the following data to form a random sample from a Poisson distribution, find the expected frequencies, correct to the nearest unit. $(e^{-4}=0.0183)$

No. of letters.	0	1	2	3	4	5	6	7	8	9	10
Frequency.	1	4	15	22	21	20	8	6	2	0	1

[**Ans.** 1·8, 7·3, 14·6, 19·5, 19·5, 15·6, 10·4, 5·9, 3·0, 1·3, 0·5 or 2, 7, 15, 20, 20, 16, 10, 6, 3, 1, 1].

19. Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting six heads x times.

$$\left[\textbf{Ans.}\quad e^{-100}\,\frac{(100)^x}{x\,!}\right]$$

20. An area of 144 square kilometres was selected for which the mean density of bombs appeared constant. To test the hypothesis that the bombs fell in clusters, the area was divided into 576 squares of $\frac{1}{4}$ kilometre each and a count made of the numbers of squares containing 0, 1, 2 etc., bombs, of which there were 537 altogether. The data is given below :

No. of flying bombs per square.	0	1	2	3	4	5 and over
Actual no. of square.	229	211	93	35	7	1

Calculate the theoretical Poisson frequencies.

[**Ans.** 227, 211, 98, 31, 7 and 2]

21. Express as an integral the probability that a normal variate with mean 5 and s.d. 2 would be observed between 2 and 3.

22. The following table gives frequencies of occurrence of a variable x between certain limits :—

Variable x	Frequency
Less than 40	30
40 or more but less than 50	33
50 and more	37

The distribution is exactly normal. Find the distribution and also obtain the frequencies between $x=50$ and $x=60$.
[Ans. 11·68, 46·125, 25]

23. In a certain examination the percentages of passes and distinction were 45 and 10 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively. (Assume the distribution of marks to the normal). [Ans. 36·1]

24. The marks obtained in a certain paper are found to be normally distributed. If 12·5% of the candidates obtain 60% or more marks, 39% obtain less than 30 marks, find the mean number of marks obtained by the candidates. Given

$\frac{x}{\sigma}$	0·27	0·28	0·29	1·14	1·15	1·6
A	0·6064	0·6102	0·6141	0·8727	0·8749	0·8770

[Ans. 36]

25. The height measurements of 600 adult males are arranged in ascending order and it is observed that the 180th and 450th measurements are 64·2″ and 67·8″ respectively. Assuming that the sample of heights is drawn from a normal population, estimate the mean and the s.d. of the population.

26. Steel rods are manufactured to be 3 inches in diameter but they are acceptable if they are inside the limits 2·99 inches and 3·01 inches. It is observed that 5% are rejected under size. Assuming that the diameters are normally distributed, find the s.d. of the distribution. Hence calculate what proportion of rejects would be if the permissible limits were widened to 2·985 inches and 3·015 inches. [Ans. 1·36%]

27. In an examination it is laid down that a student passes if he secures 30% or more marks. He is placed in the first, second or third division according as he secures 60% or more marks, marks between 45% and 60% and marks between 30% and 45% respectively. He gets a distinction in case he secures 80% or more marks. It is noticed from the results that 10% of the

students failed in the examination, whereas 5% of them obtained distinction. Calculate the percentage of students placed in the second division. (Assume marks to be distributed normally).

[**Ans.** 34%]

28. If x is a normal variate with mean 50 and s.d. 10, find $P(y < 3137)$ where $y = x^2 + 1$.

Hint.
$$\begin{aligned} P(y < 3137) &= P(x^2+1 < 3137) \\ &= P(x^2 < 3136) \\ &= P(|x| < 56) \end{aligned}$$

Put $z = \dfrac{x-50}{10} \Rightarrow x = 50 + 10z$

$$\begin{aligned} \therefore \quad P(y < 3137) &= P(-56 < x < 56) \\ &= P\{-56 < 50 + 10z < 56\} \\ &= P\{-10\cdot6 < z < 0\cdot6\} \\ &= P\{-10\cdot6 < z < 0\} + P\{0 < z < 0\cdot6\} \\ &= 0\cdot5 + 0\cdot2258 \\ &= 0\cdot7258. \end{aligned}$$

11

Bivariate Distribution

11.1. Discrete Bivariate Distributions

In the case of discrete bivariate distributions there are two discrete variates x, y and the pairs of values of x and y are considered. The frequencies or probabilities are for the pairs.

Let $x_1, x_2 \dots x_m$ and $y_1, x_2 \dots y_n$ be the values of x and y respectively and the frequency for the pair (x_i, y_j) be denoted by f_{ij}.

Then $$\sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij} = N \quad \text{(say)}$$

is the total frequency.

If x and y are random variates, and probability for the pair (x_i, y_j) is donated by p_{ij} then the function p s.t.

$$p(x_i, y_j) = p_{ij}$$

is called the **joint probability function** of x and y

Definitions

(1) Let $p_x(x_i) = p_{i1} + p_{i2} + \dots\dots + p_{in}$

and $p_y(y_j) = p_{1j} + p_{2j} + \dots\dots + p_{mj}$

p_x (or p_y) is called **marginal probability function of** x (or y)

(2) $$p(x_i/y_j) = \frac{p_{ij}}{p_y(y_j)}$$

is called **conditional probability function of x given y=y$_j$**

Similarly $$p(y_j/x_i) = \frac{p_{ij}}{p_x(x_i)}$$

is called conditional probability function of y given $x = x_i$.

(3) x and y are said to be independent if

$$p_{ij} = p_x(x_i) \cdot p_y(y_j)$$

Otherwise they are said to be dependent.

Ex. 11-1. *x and y are two random variables having the joint density function*

$$f(x, y)=\frac{1}{27}(x+2y),$$

where x and y can assume only the integer values 0, 1, 2. Find the conditional distribution of y for given x.

Sol. By given

$f(x, y)=\frac{1}{27}(x+2y)$, $x=0, 1, 2$; $y=0, 1, 2$. The table below gives various values of f

$x \downarrow$ \ $y \rightarrow$	0	1	2	f_x
0	0	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{6}{27}$
1	$\frac{1}{27}$	$\frac{3}{27}$	$\frac{5}{27}$	$\frac{9}{27}$
2	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{6}{27}$	$\frac{12}{27}$

The last column headed f_x gives the marginal probability function of x.

The table giving the values of conditional probability function of y for given x.

$x \downarrow$ \ $y \rightarrow$	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$
2	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{6}{12}$

This is obtained by dividing each row by corresponding entry under f_x.

Ex. 11-2. *Two unbiased dice are tossed simultaneously. If x and y be the numbers on two dice respectively, find*

(i) $P\{x+y=6 \mid y=2\}$. (ii) $P(x-y=2)$

Sol. Both x and y can take values 0, 1, 2, 3, 4, 5, 6 each with probability $\frac{1}{6}$.

The joint probability function of x and y is given by

$$p(x, y)=\frac{1}{6} \cdot \frac{1}{6}=\frac{1}{36} \quad \forall\ x \text{ and } \forall\ y.$$

$\therefore$ The table listing the values of p is

$y\rightarrow$ $x\downarrow$	1	2	3	4	5	6
1	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
p_y	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The last row gives the probability function of y

(*i*) $\therefore$ $P\{x+y=6 \mid y=2\}=P\{x=4 \mid y=2\}$

$$=\frac{P\{x=4,\ y=2\}}{p_y(y=2)}$$

$$=\frac{1}{36}\Big/\frac{1}{6}=\frac{1}{6}\ .$$

(*ii*) $P(x-y=2)=P(x=3,\ y=1)+P(x=4,\ y=2)+P(x=5,\ y=3)$
$+P(x=6,\ y=4)$

$$=\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}=\frac{1}{9}$$

Ex. 11-3. *The joint probability distribution of a pair (x, y) of random variables is given by the following table*

$x\rightarrow$ $y\downarrow$	*1*	*2*	*3*
1	*0·1*	*0·1*	*0·2*
2	*0·2*	*0·3*	*0·1*

Find (*i*) *the marginal distributions*

(*ii*) *the conditional distribution of x given $y=2$*

(*iii*) $P\{x+y<3\}$.

Sol. The given table is

$x\rightarrow$ / $y\downarrow$	1	2	3	p_y
1	0·1	0·1	0·2	0·4
2	0·2	0·3	0·1	0·6
p_x	0·3	0·4	0·3	

(*i*) The marginal distribution of y is given by column headed p_y and the marginal distribution of x is given by row headed p_x.

(*ii*) The conditional distribution of x for $y=1$ is given by dividing first row by corresponding entry in column headed p_y.

$\therefore$ Conditional distribution of x for $y=1$ is

$$p(x \mid y=1) : \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{2}$$

(*iii*) $P(x+y \leqslant 3)=P(x+y=2)+P(x+y=3)$

$$=P(x=1, y=1)+P(x=1, y=2)+P(x=2, y=1)$$
$$=0{\cdot}1+0{\cdot}2+0{\cdot}1$$
$$=0{\cdot}4.$$

Ex. 11·4. *Two discrete random variables x and y have*

$$p(0, 0)=\frac{2}{9}, \quad p(0, 1)=\frac{1}{9}, \quad p(1, 0)=\frac{1}{9}, \quad p(1, 1)=\frac{5}{9}.$$

Test whether x and y are independent.

Sol. The given data can be put in the form of table below

$x\rightarrow$ / $y\downarrow$	0	1	p_y
0	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{3}{9}$
1	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{6}{9}$
p_x	$\frac{3}{9}$	$\frac{6}{9}$	

Since $p(x, y) \neq p_x(x) \cdot p_y(y)$, x and y are not independent.

11-2. Continuous Bivariate Distributions

In the case of continuous bivariate distributions, variates x and y are continuous. Here various terms are defined as below :

(4) Probability Density function. A continuous function $f(x, y)$ s.t. the probability of the value of the variate to lie in infinitesimal intervals $\left[x-\frac{dx}{2}, y+\frac{dy}{2}\right]$ and $\left[y-\frac{dy}{2}, y+\frac{dy}{2}\right]$ can be expressed in the form $f(x, y)\, dx\, dy$, is called probability density function or simply the density function.

The density function f has the following properties :

(*i*) $f(x, y) \geq 0, \forall x$ and $\forall y$.

(*ii*) $\iint f(x, y)\, dxdy = 1$

where the integral is extended over the entire range of (x, y).

(5) Probability Differential

$f(x, y)\, dx\, dy$ is called probability differential. Moreover

$$P\left[a \leq x \leq b,\ c \leq y \leq d\right] = \int_{x=a}^{b} \int_{y=c}^{d} f(x, y) dx\, dy$$

(6) Marginal Distributions and Marginal Density Functions

Let
$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

and
$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

Then, $g(x)$ [or $h(y)$] is called marginal density function of x (or y).

$g(x)dx$ is called marginal distribution of x and $h(y)dy$ is called marginal distribution of y.

(7) Conditional Density Function

$$f(x/y) = \frac{f(x, y)}{h(y)}$$

is called conditional density function of x given y.

Similarly, $f(y/x)=\dfrac{f(x, y)}{g(x)}$

is called conditional density function of y given x.

(8) Two variates x and y are said to be independent (or stochastically independent) if

$$f(x, y)=g(x).h(y).$$

Ex. 11-5. *If x and y are two random variables having joint density function*

$$f(x, y)=\frac{1}{8}(6-x-y),\ 0<x<2,\ 2<y<4$$

Find (*i*) $P\{x\leqslant 1, y<3\}$

(*ii*) $P\{x+y<3\}$

(*iii*) $P\{x<1/y<3\}$

Solution.

(*i*) $$P\{x\leqslant 1, y<3\}=\frac{1}{8}\int_{x=0}^{1}\int_{y=2}^{3}(6-x-y)dy\,dx$$

$$=\frac{1}{8}\int_{0}^{1}dx\int_{2}^{3}(6-x-y)dy$$

$$=\frac{1}{8}\int_{0}^{1}dx\left\{6-x-\frac{5}{2}\right\}$$

$$=\frac{1}{8}\left\{6-\frac{1}{2}-\frac{5}{2}\right\}=\frac{3}{8}$$

(*ii*) $$P(x+y<3)=\frac{1}{8}\int_{x=0}^{1}\int_{y=2}^{3-x}(6-x-y)dx\,dy$$

$$=\frac{1}{8}\int_{0}^{1}dx\left\{6(1-x)-x(1-x)-\frac{1}{2}(5+x^2-6x)\right\}$$

$$=\frac{1}{8}\int_0^1 dx\left\{\frac{x^2}{2}-4x+\frac{7}{2}\right\}$$

$$=\frac{1}{8}\left\{\frac{1}{6}-2+\frac{7}{2}\right\}=\frac{5}{24}$$

(*iii*) $$P\{x<1/y<3\}=\frac{P(x\leq 1,\ y\leq 3)}{P\{y\leq 3\}}$$

$$P(y\leq 3)=\int_{x=0}^{2}\int_{2}^{3}\frac{1}{8}(6-x-y)dx\,dy$$

$$=\frac{1}{8}\int_0^2 dx\left\{6-x-\frac{5}{2}\right\}$$

$$=\frac{1}{8}\left(12-2-5\right)$$

$$=\frac{5}{8}$$

$\therefore$ $$P\{x<1/y<3\}=\frac{3/8}{5/8}=3/5$$

Ex. 11-6. *Let x and y have the joint density function*

$$f(x,y)=\frac{1}{2},\ 0\leq y\leq x\leq 2$$

Find the marginal and conditional probability density functions. Are x and y independent ?

Solution

$$\int_{x=0}^{2}\int_{y=0}^{x} f(x,\ y)dx\,dy=\frac{1}{2}\int_{x=0}^{2} dx\int_{y=0}^{x} dy$$

$$=\frac{1}{2}\int_0^2 x\,dx$$

$$=1$$

(*i*) $$g(x)=\int_{y=0}^{x} f(x,y)\,dy$$

$$=\frac{1}{2}\int_0^x dy=\frac{1}{2}x$$

$$h(y)=\int_y^2 f(x, y)\,dx$$

$$=\frac{1}{2}\int_y^2 dx=\frac{1}{2}(2-y)$$

(ii) $$f(x/y)=\frac{f(x, y)}{h(y)}=\frac{1}{2-y}$$

$$f(y/x)=\frac{f(x, y)}{g(x)}=\frac{1}{x}$$

(iii) Now, $$g(x)\,h(y)=\frac{1}{2}x.\ \frac{1}{2}(2-y)$$

$$=\frac{1}{4}x(2-y)\neq f(x, y)$$

$\therefore$ x and y are not independent.

Ex. 11-7. *The joint density function of a bivariate distribution is given as below :*

$$f(x, y)=\frac{1}{3}(x+y),\ 0<x<2,\ 0<y<1$$
$$=0 \quad \text{elsewhere}$$

Determine marginal distributions and show that x and y are stochastically dependent.

Solution.

$$\int_{x=0}^{2}\int_{y=0}^{1} f(x, y)dx\ dy=\frac{1}{3}\int_0^2 dx\int_0^1 (x+y)dy$$

$$=\frac{1}{3}\int_0^2 dx\left\{x+\frac{1}{2}\right\}$$

$$=\frac{1}{3}\left\{\frac{x^2}{2}+\frac{1}{2}x\right\}_0^2=1$$

(*i*)

$$g(x)=\frac{1}{3}\int_0^1 (x+y)dy$$

$$=\frac{1}{3}\left(x+\frac{1}{2}\right)$$

$$h(y)=\frac{1}{3}\int_0^2 (x+y)dx=\frac{2}{3}\left(1+y\right)$$

(*ii*) Now $\quad g(x)\,h(y)=\frac{2}{9}\left(x+\frac{1}{2}\right)(1+y)\neq f(x, y)$

$\therefore$ x and y are stochastically dependent.

Ex. 11-8. *The joint density function of a bivariate distribution is given by*

$$f(x, y)=4xy\, e^{-(x^2+y^2)}\ ,\ x\geqslant 0,\ y\geqslant 0.$$

Find the marginal and conditional probability density functions. Are x and y independent ?

Solution.

$$\int_0^\infty\int_0^\infty f(x, y)dx\ dy=4\int_{x=0}^\infty\int_0^\infty xye^{-(x^2+y^2)}dx\ dy$$

$$=4\int_0^\infty dx.xe^{-x^2}\int_0^\infty ye^{-y^2}dy$$

$$=\left|-e^{-x^2}\right|_0^\infty\left|-e^{-y^2}\right|_0^\infty$$

$$=1$$

(*i*)

$$g(x)=\int_{y=0}^\infty 4xye^{-(x^2+y^2)}dy$$

$$=4x\,e^{-x^2}\int_0^\infty ye^{-y^2}dy$$

$$=2xe^{-x^2}\Big|-e^{-y^2}\Big|_0^{\infty}=2xe^{-x^2}$$

$$h(y)=\int_{x=0}^{\infty}4xye^{-(x^2+y^2)}dx\,dy$$

$$=2ye^{-y^2}$$

(ii) $$f(x/y)=\frac{f(x,y)}{h(y)}=\frac{4xye^{-(x^2+y^2)}}{2ye^{-y^2}}$$

$$=2xe^{-x^2}$$

$$f(y/x)=\frac{4xye^{-(x^2+y^2)}}{2xe^{-x^2}}=2ye^{-y^2}$$

(iii) $f(x,y)=4\,xy\;e^{-(x^2+y^2)}$
$\qquad\qquad=g(x)\,h(y)$

$\therefore$ x and y are independent.

Ex. 11-9. *The joint density function of a bivariate distribution is given by*

$$f(x,y)=c\sin\frac{\pi}{2}(x+y),\ 0<x<1,\ 0<y<1$$
$$=0\ \textit{elsewhere.}$$

Find c and marginal ; conditional probability density functions Are x and y independent ?

Sol. c is given by

$$1=\int_{x=0}^{1}\int_{0}^{1}f(x,y)\,dx\,dy$$

$$=c\int_{x=0}^{1}dx\int_{0}^{1}\sin\frac{\pi}{2}(x+y)\,dy$$

$$=c\int_{x=0}^{1}dx\left\{-\frac{2}{\pi}\cos\frac{\pi}{2}(x+y)\right\}_0^1$$

$$=\frac{2c}{\pi}\int_0^1\left\{\cos\frac{\pi}{2}x-\cos\frac{\pi}{2}(1+x)\right\}dx$$

$$=\frac{4}{\pi^2}c\left|\sin\frac{\pi}{2}x-\sin\frac{\pi}{2}(1+x)\right|_0^1$$

$$=\frac{8}{\pi^2}c.$$

$$\therefore\quad c=\frac{\pi^2}{8}$$

$$(i)\ g(x)=\int_0^1 f(x,y)\,dy$$

$$=c\int_0^1\sin\frac{\pi}{2}(x+y)\,dy$$

$$=\frac{\pi^2}{8}\left|-\cos\left(\frac{\pi}{2}\right)(x+y)\right|_0^1$$

$$=\frac{\pi^2}{8}\left\{\cos\frac{\pi}{2}x-\cos\frac{\pi}{2}(1+x)\right\}$$

$$=\frac{\pi^2}{8}\left\{\cos\frac{\pi}{2}x+\sin\frac{\pi}{2}x\right\}$$

$$h(y)=\int_0^1 f(x,y)\,dx$$

$$=\frac{\pi^2}{8}\int_0^1\sin\frac{\pi}{2}(x+y)\,dx$$

$$=\frac{\pi^2}{8}\left\{\cos\frac{\pi}{2}y-\cos\frac{\pi}{2}(1+y)\right\}$$

$$=\frac{\pi^2}{8}\left\{\cos\frac{\pi}{2}y+\sin\frac{\pi}{2}y\right\}$$

(*ii*) $$f(y/x)=\frac{f(x,y)}{g(x)}=\frac{c.\sin\frac{\pi}{2}(x+y)}{\frac{\pi^2}{8}\left\{\cos\frac{\pi}{2}x+\sin\frac{\pi}{2}x\right\}}$$

$$=\frac{\sin\frac{\pi}{2}(x+y)}{\cos\frac{\pi}{2}x+\sin\frac{\pi}{2}x}$$

Similarly

$$f(x/y)=\frac{\sin\frac{\pi}{2}(x+y)}{\cos\frac{\pi}{2}y+\sin\frac{\pi}{2}y}.$$

(*iii*) Now $f(x,y)\neq g(x)\,h(y)$

$\therefore$ x and y are not independent.

Ex. 11-10. *Show that the conditions for the function*

$$f(x,y)=k\exp\{ax^2+2hxy+by^2\},\ -\infty<x,\ y<\infty$$

to be a density function are

(*i*) $a<0$ (*ii*) $b<0$ (*iii*) $ab-h^2>0$.

Assuming these conditions to be satisfied, find k.

Sol. Let $f(x,y)$ be a density function Then

$$\int_{x=-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy=1$$

i.e., $$k\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\exp\left\{ax^2+2hxy+by^2\right\}dx\,dy=1\qquad\ldots(1)$$

Now $ax^2+2hxy+by^2$

$$=a\left\{\left(x+\frac{hy}{a}\right)^2+\left(\frac{ab-h^2}{a^2}\right)y^2\right\},\text{ if }a\neq0\text{ and}$$

$$=b\left\{\left(y+\frac{hx}{b}\right)^2+\left(\frac{ab-h^2}{b^2}\right)x^2\right\},\text{ if }b\neq0$$

$\therefore$ The integral in (1) converges if

$$a\leqslant0,\ b\leqslant0,\ ab-h^2\geqslant0$$

Assume $a<0,\ b<0,\ ab-h^2>0$

Let $a=-\lambda,\ b=-\mu,\ h=\eta.$

Then $\lambda>0,\ \mu>0$ and $ab-h^2=\lambda\mu-\eta^2>0$

$$\therefore \quad ax^2+2hxy+by^2=-\lambda\left\{\left(x-\frac{\eta}{\lambda}y\right)^2+\frac{\lambda\mu-\eta^2}{\lambda^2}y^2\right\}$$

$$=-\frac{(x\lambda-\eta y)^2}{\lambda}-\frac{\lambda\mu-\eta^2}{\lambda}y^2.$$

$\therefore$ (1) $\Rightarrow$

$$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\exp\left[-\frac{(x\lambda-\eta y)^2}{\lambda}-\frac{\lambda\mu-\eta^2}{\lambda}y^2\right]dx\,dy=1$$

i.e., $$1=k\int_{y=-\infty}^{\infty}\left[\exp\left\{-\frac{\lambda\mu-\eta^2}{\lambda}y^2\right\}\int_{x=-\infty}^{\infty}\exp\left\{-\frac{(x\lambda-\eta y)^2}{\lambda}\right\}dx\right]dy$$

Now $$\int_{x=-\infty}^{\infty}\exp\left\{-\frac{(x\lambda-\eta y)^2}{\lambda}\right\}dx=\frac{1}{\lambda}\int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{\lambda}\right)du$$

where $u=x\lambda-\eta y$

$$=\frac{1}{\sqrt{\lambda}}\int_0^{\infty}t^{\frac{1}{2}-1}\,e^{-t}\,dt$$

where $u^2=\lambda t$

$$=\frac{\sqrt{\pi}}{\sqrt{\lambda}}$$

$$\therefore \quad 1=k\,.\frac{\sqrt{\pi}}{\sqrt{\lambda}}\int_{y=-\infty}^{\infty}\exp\left\{-\frac{\lambda\mu-\eta^2}{\lambda}y^2\right\}dy$$

$$=k\,.\frac{\sqrt{\pi}}{\sqrt{\lambda}}\,.\sqrt{\pi}\,\frac{\sqrt{\lambda}}{\sqrt{\lambda\mu-\eta^2}}$$

$$\therefore \quad k=\frac{1}{\pi}\sqrt{\lambda\mu-\eta^2}$$

$$=\frac{1}{\pi}\sqrt{ab-h^2}$$

11.3. Bivariate Transformation

Let x and y be two random variates having joint probability density function $f(x, y)$. Let x and y be transformed to variates U and V by the transformation

$$x=x(u, v)\,;\quad y=y(u, v)$$

where u, v are continuously differentiable function.

Let $J=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0.$

Then the joint probability density function of u and v is

$$|J| \quad f\{x(u, v), y(u, v)\}.$$

Ex. 11-15. *If the probability density function of two variates x and y is given by*

$$f(x, y)=\begin{cases} 4xy\, e^{-(x^2+y^2)}; & x>0,\ y>0 \\ 0 & \text{else where.} \end{cases}$$

Find the density function of $\sqrt{x^2+y^2}$.

Sol. Put $\qquad u=\sqrt{x^2+y^2}, \qquad v=x$

Then $\qquad v>0, \quad u>v.$

$\therefore \qquad u>0, \quad 0<v<u.$

Now
$$\frac{1}{J}=\frac{\partial(u, v)}{\partial(x, y)}$$

$$=\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$=\begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ 1 & 0 \end{vmatrix}=\frac{-y}{\sqrt{x^2+y^2}}$$

$\therefore$ The probability density function of u and v is given by

$F(u, v)=f(x, y) \quad |J|$

$$=\begin{cases} 4xy\, e^{-(x^2+y^2)} \Big/ \dfrac{y}{\sqrt{x^2+y^2}}, & x>0,\ y>0 \\ 0 \ \text{elsewhere} \end{cases}$$

$$=\begin{cases} 4\, uv\, e^{-u^2} & u>0,\ 0<v<u. \\ 0\ , & \text{elsewhere} \end{cases}$$

$\therefore$ Marginal density function of u is

$$4\int_{v=0}^{u} uv\, e^{-u^2}\, dv=2u^3\, e^{-u^2}, \quad u>0$$

$\therefore$ Density function of u is

$$\begin{cases} 2u^3 e^{-u^2}, & u>0 \\ 0, & \text{elsewhere} \end{cases}$$

Ex. 11-16. *The joint density function of x and y is*

$$f(x, y)=e^{-(x+y)}, \; x>0, \; y>0.$$

Find the prob. density function of $\dfrac{x+y}{2}$.

Sol. Put $u=\dfrac{x+y}{2}$, $v=x$.

Then $u \geqslant 0$, $0 \leqslant v \leqslant 2u$

$$\frac{1}{J}=\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$=\begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ 1 & 0 \end{vmatrix}=-\frac{1}{2}$$

$\therefore$ The prob. density function of u and v is

$$e^{-(x+y)} \cdot |J|$$

$$=2e^{-2u}$$

$\therefore$ Density function of u is

$$2\int_0^{2u} e^{-2u}\, dv$$

$$=4u\, e^{-2u}.$$

Ex. 11-17. *The joint density function of x and y is*

$$f(x, y)=2x\, e^{-y}, \; 0<x<1, \; y>0$$
$$=0 \text{ elsewhere.}$$

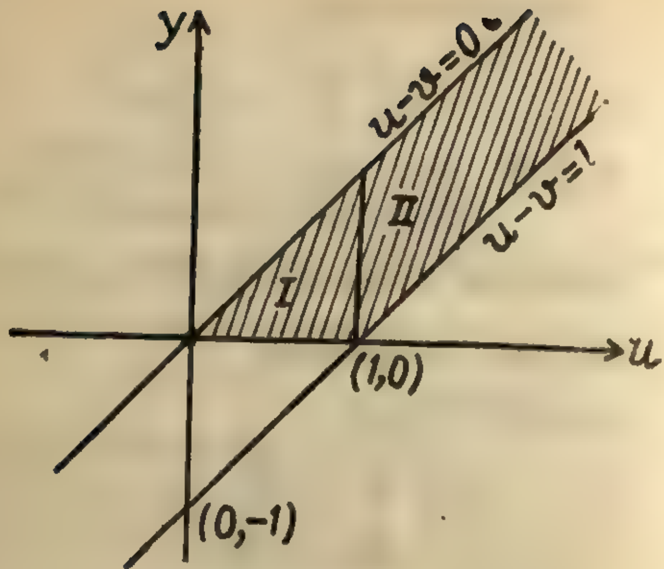
Find the distribution of $x+y$.

Sol. Put $u=x+y$, $v=y$

$\therefore$ $x=u-v$, $y=v$

$$y>0 \Rightarrow v>0$$

$$0<x<1 \Rightarrow 0<u-v<1$$



$\therefore$ u and v vary in shaded portion.

Now $$\frac{1}{J}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}=1$$

$\therefore$ $$J=1$$

$\therefore$ The joint density function of u and v is given by

$$F(u, v)=2(u-v)e^{-v}$$

To find the density function of u divide the shaded portion in two parts marked I and II by the line $x=1$.

In part I, $0<u\leqslant 1$ and for given u, v varies from 0 to u and in part II, $u>1$ and for given u, v varies from $u-1$ to u.

$\therefore$ Density function of u is given by

$$g(u)=\int_0^u 2(u-v)\,e^{-v}\,dv$$

$$=2\left[-(u-v)\,e^{-v}+e^{-v}\right]_{v=0}^{u}$$

$$=2(e^{-u}+u-1) \quad \text{for} \quad 0<u\leq 1$$

and for $u\geq 1$,

$$g(u)=2\int_{u-1}^{u}(u-v)\,e^{-v}\,dv$$

$$=2\left[-(u-v)\,e^{-v}+e^{-v}\right]_{u-1}^{u}$$

$$=2\{e^{-u}+e^{-(u-1)}-e^{-(u-1)}\}$$

$$=2\,e^{-u}.$$

EXERCISES

1. The joint density function of a bivariate distribution is given by

$$f(x, y)=\begin{cases} c\,xy, & 0<y<x<1 \\ 0 & \text{otherwise} \end{cases}$$

(*i*) Find marginal and conditional distributions.

(*ii*) Are x and y independent ? Why ?

2. Obtain the marginal and conditional probability functions for the following distribution

$$f(x, y)=c(2-x-y), \quad 0<y<x<1$$
$$=0 \qquad \text{elsewhere.}$$

3. If the joint density function of a bivariate distribution is given by

$$f(x, y)=c\,.\,e^{-(x+y)}, \quad x>0, \quad y>0$$
$$=0 \qquad \text{elsewhere}$$

Find (*i*) $P(x>y)$

(*ii*) $P(x+y<1)$

4. Given the following bivariate probability distribution.

$x\rightarrow$ / $y\downarrow$	-1	0	1
0	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{1}{15}$

Find (*i*) marginal distributions of x and y

(*ii*) the conditional distribution of x given $y=1$.

5. If x and y are two independent random variables with density functions

$$f(x)=e^{-x},\ x\geq 0$$

and $$g(y)=2\,e^{-2y},\quad y\geq 0.$$

Find the probability distribution of $u=\dfrac{x}{y}$.

$$\left\{\textbf{Ans.}\quad \frac{1}{(u+2)^2}\right\}$$

6. If the joint density function of two random variates x and y is $f(x, y)$, show that the density function of $u=x+y$ is

$$\int_{-\infty}^{\infty} f(v, u-v)\,dv$$

12

Special Continuous Distributions

12.1. Gamma Distribution

Let x be a continuous random variate with probability density function defined as below :

$$f(x)=\frac{1}{\Gamma(\lambda)}\, e^{-x}\, x^{\lambda-1},\ \lambda>0,\ 0<x<\infty$$

x is called **Gamma variate** with parameter λ and is referred to as a $\gamma(\lambda)$-variate. The distribution of x is called a **Gamma distribution.**

12.1.1. Moments about mean and β, γ coefficients

$$\mu'_r(0)=E(x^r)$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^\infty x^r\, e^{-x}\, x^{\lambda-1}\, dx$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-x}\, x^{(\lambda+r)-1}\, dx$$

$$=\frac{1}{\Gamma(\lambda)}\,\Gamma(\lambda+r)$$

$$=(\lambda+r-1)(\lambda+r-2)\ldots\ldots(\lambda)$$

$$\text{mean}=\mu_1'(0)=\lambda$$

$$\mu_2'(0)=\lambda(\lambda+1)$$

$$\mu_3'(0)=\lambda(\lambda+1)(\lambda+2)$$

$$\mu_4'(0)=\lambda(\lambda+1)(\lambda+2)(\lambda+3)$$

$$\therefore\quad \mu_2=\mu_2'(0)-\{\mu_1'(0)\}^2$$

$$=\lambda(\lambda+1)-\lambda^2$$

$$=\lambda$$

$$\mu_3=\mu_3'(0)-3\mu_2'(0)\mu_1'(0)+2\{\mu_1'(0)\}^3$$
$$=\lambda(\lambda+1)(\lambda+2)-3\lambda^2(\lambda+1)+2\lambda^3$$
$$=\lambda^3+3\lambda^2+2\lambda-3\lambda^3-3\lambda^2+2\lambda^3$$
$$=2\lambda$$

$$\mu_4=\mu_4'(0)-4\mu_3'(0)\mu_1'(0)+6\mu_2'(0)\{\mu_1'(0)\}^2-3\{\mu_1'(0)\}^4$$
$$=\lambda(\lambda+1)(\lambda+2)(\lambda+3)-4\lambda^2(\lambda+1)(\lambda+2)+6\lambda^3(\lambda+1)-3\lambda^4$$
$$=\lambda\{\lambda^3+6\lambda^2+11\lambda+6\}-4\lambda^2(\lambda^2+3\lambda+2)+6\lambda^4+6\lambda^3-3\lambda^4$$
$$=3\lambda^2+6\lambda$$

$$\beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{4\lambda^2}{\lambda^3}=\frac{4}{\lambda}$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=3+\frac{6}{\lambda}$$

$$\gamma_1=\sqrt{\beta_1}=\frac{2}{\sqrt{\lambda}}$$

$$\gamma_2=\beta_2-3=\frac{6}{\lambda}$$

12-2. Mode

Density function is

$$f(x)=\frac{1}{\Gamma(\lambda)}\,e^{-x}x^{\lambda-1}$$

$$\therefore \quad f'(x)=\frac{1}{\Gamma(\lambda)}\,e^{-x}\,x^{\lambda-2}\,\{(\lambda-1)-x\}$$

$$\therefore \quad f'(x)=0 \;\Rightarrow\; x=\lambda-1,\ 0$$

for $x=0$, $f'(x)=0$ which is minimum value of $f(x)$.

$\therefore$ for $x=\lambda-1$, $f(x)$ is maximum.

$\therefore$ Mode$=\lambda-1$.

12.1.3. M.G.F. of Gamma Distribution

$$M_0(t)=E\{e^{tx}\}$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{tx}\,e^{-x}\,x^{\lambda-1}\,dx$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-x(1-t)}\,x^{\lambda-1}\,dx$$

$$=\frac{1}{\Gamma(\lambda)}\cdot\frac{1}{(1-t)^{\lambda}}\int_0^{\infty} e^{-y}\ y^{\lambda-1}\,dy,$$

where $y=(1-t)x,\ |t|<1$

$$=\frac{1}{\Gamma(\lambda)}\cdot\frac{1}{(1-t)^{\lambda}}\,\Gamma(\lambda)=\frac{1}{(1-t)^{\lambda}}$$

$$M_{\bar{x}}(t)=E\{e^{(x-\lambda)t}\}$$

$$=e^{-\lambda t}\,M_0(t)$$

$$=e^{-\lambda t}\,(1-t)^{-\lambda},\quad |t|<1.$$

12.1.4. Cumulative Generating Function and Cumulants

$$K_0(t)=\log M_0\,(t)$$

$$=-\lambda\log(1-t),\ |t|<1$$

$$=\lambda\left\{t+\frac{t^2}{2}+\frac{t^3}{3}+\ldots\ldots\right\}$$

$\therefore\qquad k_1(0)=\lambda,\ k_2=\lambda$ etc.

In general,

$$\frac{k_r}{r\,!}=\text{coeff. of } t^r$$

$$=\frac{\lambda}{r}$$

$$\Rightarrow\qquad k_r=\lambda(r-1)\,!$$

12.1.5. Additive Property of Gamma Variates

Theorem. The sum of any finite number of independent Gamma variates is a Gamma variate.

Proof. Let $x_1, x_2,\ldots\ldots x_n$ be independent Gamma variates with parameters $\lambda_1, \lambda_2\ldots\ldots\lambda_n$ respectively.

Then $\quad M_0(t)$ of $x_i=(1-t)^{-\lambda_i}\qquad (i=1, 2\ \ldots\ n)$

Let $\quad X=x_1+x_2\ldots\ldots+x_n$

Then $\quad M_0(t)$ of $X=E\{e^{t(x_1+\ldots\ldots+x_n)}\}$

$$=E\{e^{tx_1}\}\,.\,E\{e^{tx_2}\}\ldots\ldots E\{e^{tx_n}\}$$

$$=(1-t)^{-\lambda_1}\,.\,(1-t)^{-\lambda_2}\ldots.(1-t)^{-\lambda_n}$$

$$=(1-t)^{-(\lambda_1+\ldots+\lambda_n)}$$

which is the m.g.f. of a $\gamma(\lambda_1+\ldots+\lambda_n)$

$\therefore$ X is a $\gamma(\lambda_1+\ldots\ldots+\lambda_n)$.

12.1.6. Limiting Form of Gamma Distribution

Theorem. Show that as $\lambda\to\infty$, Gamma Distribution tends to normal distribution.

Proof. Let x be a $\gamma(\lambda)$. Then $\bar{x}=\lambda$, var$(x)=\lambda$.

Let $$z=\frac{x-\lambda}{\sqrt{\lambda}}$$

$$\therefore\ M_0(t) \text{ of } z=E\left\{e^{t\left(\frac{x-\lambda}{\sqrt{\lambda}}\right)}\right\}$$

$$=e^{-\sqrt{\lambda}\cdot t}E\left\{e^{\left(\frac{t}{\sqrt{\lambda}}\right)x}\right\}$$

$$=e^{-\sqrt{\lambda}\cdot t}\left(1-\frac{t}{\sqrt{\lambda}}\right)^{-\lambda}$$

$$\therefore\ \log\{M_0(t) \text{ of } z\}=-\sqrt{\lambda}\ t-\lambda\log\left(1-\frac{t}{\sqrt{\lambda}}\right)$$

$$=-\sqrt{\lambda}\ t+\lambda\left\{\frac{t}{\sqrt{\lambda}}+\frac{t^2}{2\lambda}+\frac{t^3}{3\lambda\sqrt{\lambda}}+\ldots\right\}$$

$$=\frac{1}{2}t^2+\text{terms containing } \lambda \text{ in the denominator}$$

$$\to\frac{1}{2}t^2 \text{ as } \lambda\to\infty$$

$$\therefore\quad M_0(t) \text{ of } z\to e^{\frac{1}{2}t^2} \text{ as } \lambda\to\infty$$

which is the m.g.f. of a standard normal variate.

$\therefore$ In the limiting case, z and hence x is a normal variate.

12.2. Beta Distribution First Kind

Let x be a continuous random variate with probability density function defined as below :

$$f(x)=\frac{1}{\beta(m,n)}x^{m-1}(1-x)^{n-1},\quad m,n>0\ ,\ 0<x<1.$$

x is known as a Beta variate of first kind with parameters m and n. It is referred to as $\beta_1(m,n)$ variate.

The distribution of x is called Beta distribution of first kind.

12-2.1. Moments and Harmonic Mean

$$\therefore \quad \mu_r'(0)=E(x^r)$$

$$=\frac{1}{\beta(m, n)}\int_0^1 x^r \cdot x^{m-1}(1-x)^{n-1}\, dx$$

$$=\frac{1}{\beta(m, n)}\int_0^1 x^{m+r-1}(1-x)^{n-1}\, dx$$

$$=\frac{\beta(m+r, n)}{\beta(m, n)}$$

$$=\frac{\Gamma(m+r)\,\Gamma(m+n)}{\Gamma(m+n+r)\,\Gamma(m)}$$

$$=\frac{(m+r-1)(m+r-2)\,\ldots\ldots(m)}{(m+n+r-1)\ldots\ldots\ldots(m+n)}$$

$$\therefore \quad \text{Mean}=\mu_1'(0)=\frac{m}{m+n}$$

$$\mu_2'(0)=\frac{m(m+1)}{(m+n)(m+n+1)}$$

$$\therefore \quad \mu_2=\mu_2'(0)-\{\mu_1'(0)\}^2$$

$$=\frac{m(m+1)}{(m+n)(m+n+1)}-\left\{\frac{m}{m+n}\right\}^2$$

$$=\frac{mn}{(m+n)^2(m+n+1)}$$

The harmonic mean H is given by

$$\frac{1}{H}=E\left(\frac{1}{x}\right)$$

$$=\frac{1}{\beta(m, n)}\int_0^1 \frac{1}{x}\cdot x^{m-1}(1-x)^{n-1}\, dx$$

$$=\frac{1}{\beta(m, n)}\int_0^1 x^{m-2}(1-x)^{n-1}\, dx$$

$$=\frac{\beta(m-1, n)}{\beta(m, n)}$$

$$=\frac{\Gamma(m-1)\,\Gamma(m+n)}{\Gamma(m)\,\Gamma(m+n-1)}$$

$$=\frac{m+n-1}{m-1}$$

$$\therefore \qquad H=\frac{m-1}{m+n-1}.$$

12-3. Beta Distribution of Second Kind

Let x be a continuous random variate with probability density function defined as below :

$$f(x)=\frac{1}{\beta(m, n)}\,\frac{x^{m-1}}{(1+x)^{m+n}},\ m,\ n>0,\ 0<x<\infty$$

x is known as Beta Variate of second kind with parameters, m and n. It is referred to as $\beta_2(m, n)$ variate. The distribution of x is called Beta distribution of second kind.

Remarks : If x is a $\beta_2(m, n)$ variate, then

$$y=\frac{1}{1+x}$$

is a $\beta_1(m, n)$ variate.

12.3-1. Moments and Harmonic Mean

$$\mu_r'(0)=E(x^r)$$

$$=\frac{1}{\beta(m, n)}\int_0^\infty x^r.\ \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$=\frac{1}{\beta(m, n)}\int_0^\infty \frac{x^{m+r-1}}{(1+x)^{m+n}}\,dx$$

$$=\frac{1}{\beta(m, n)}\int_0^\infty \frac{x^{m+r-1}}{(1+x)^{(m+r)+(n-r)}}\,dx$$

$$=\frac{1}{\beta(m, n)}\,\beta(m+r, n-r)$$

$$=\frac{\Gamma(m+r)\,\Gamma(n-r)}{\Gamma(m)\,\Gamma(n)}$$

$$=\frac{(m+r-1)\,(m+r-2)\ldots(m)}{(n-1)\,(n-2)\ldots(n-r)}$$

$\therefore$ $$\text{Mean}=\mu_1'(0)=\frac{m}{n-1}$$

$$\mu'_2(0)=\frac{m\,(m+1)}{(n-1)(n-2)}$$

$\therefore$ $$\mu_2=\mu'_2(0)-\{\mu'_1(0)\}^2$$

$$=\frac{m(m+1)}{(n-1)(n-2)}-\left(\frac{m}{n-1}\right)^2$$

$$=\frac{m(m+n-1)}{(n-1)^2(n-2)}$$

The harmonic mean H is given by

$$\frac{1}{H}=E\left(\frac{1}{x}\right)$$

$$=\frac{1}{\beta(m,n)}\int_0^\infty \frac{1}{x}\;\frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$=\frac{1}{\beta(m,n)}\int_0^\infty \frac{x^{(m-1)-1}}{(1+x)^{(m-1)+(n+1)}}\,dx$$

$$=\frac{1}{\beta(m,n)}\,\beta(m-1,n+1)$$

$$=\frac{\Gamma(m-1)\,\Gamma(n+1)}{\Gamma(m)\,\Gamma(n)}$$

$$=\frac{n}{m-1}$$

$\therefore$ $$H=\frac{m-1}{n}$$

Ex. 12-1. *If x is a $N\,(m,\sigma)$, then*

$$z=\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2$$

is a $\gamma\left(\frac{1}{2}\right)$.

Sol. Distribution of x is

$$dP=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx \qquad -\infty<x<\infty$$

Put $$z=\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2$$

$\Rightarrow$ $$x=m+\sigma\sqrt{2z}$$

$\therefore$ $$dx=\sqrt{2}\,\sigma\frac{dz}{\sqrt{z}}$$

$\therefore$ Distribution of z is

$\therefore$ $dP=c\,.\,e^{-z}\,.\,z^{-1/2}\,dz,\quad 0<z<\infty$

where c is constant to be obtained s.t

$$\int_0^\infty dP=1$$

$\therefore$ c is given by

$$c\int_0^\infty e^{-z}\,z^{-1/2}\,dz=1$$

i.e., $$c\,\Gamma\left(\frac{1}{2}\right)=1$$

i.e., $$c=\frac{1}{\Gamma\left(\frac{1}{2}\right)}$$

$\therefore$ Distribution of z is

$$dP=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\,e^{-z}\,z^{-1/2}\,dz,\quad 0<z<\infty$$

which implies that z is a $\gamma\left(\frac{1}{2}\right)$.

Ex. 12·2. *If x is a $\gamma(\lambda)$, find $E(\sqrt{x})$. Deduce mean deviation about mean for a normal variate.*

Sol. Distribution of x is

$$dP=\frac{1}{\Gamma(\lambda)}\,e^{-x}\,.\,x^{\lambda-1}\,dx,\qquad 0<x<\infty.$$

$$E(\sqrt{x})=\frac{1}{\Gamma(\lambda)}\int_0^\infty\sqrt{x}\,.\,e^{-x}\,.\,x^{\lambda-1}\,dx$$

$$=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-x}\,.\,x^{(\lambda+\frac{1}{2})-1}\,dx$$

$$=\frac{1}{\Gamma(\lambda)}\,\Gamma\left(\lambda+\frac{1}{2}\right)$$

Deduction. Let x be a $N(m, \sigma)$.

Then $$z=\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2 \text{ is a } \gamma\left(\frac{1}{2}\right)$$

Now $|x-m|=\sigma\sqrt{2z}$

$\therefore$ Mean deviation about mean

$$=\sigma E\,|x-m|$$

$$=\sigma\sqrt{2}\,E(\sqrt{z})$$

$$=\sigma\sqrt{2}.\frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} \qquad \left(\because\ \lambda=\frac{1}{2}\right)$$

$$=\sigma\sqrt{\frac{2}{\pi}}.$$

Ex. 12·3. *If x and y are independent gamma variates, find the distribution of*

(*i*) $x+y$ (*ii*) $\dfrac{x}{x+y}$ (*iii*) $\dfrac{x}{y}$.

Sol. Let x and y be gamma variates with parameters λ and μ respectively.

Then distributions of x and y are

$$dP=\frac{1}{\Gamma(\lambda)}\,e^{-x}\,.\,x^{\lambda-1}\,dx,\quad 0<x<1$$

$$dP=\frac{1}{\Gamma(\mu)}\,e^{-y}\,.\,y^{\mu-1}\,dy,\quad 0<y<1$$

Since x and y are independent, joint distribution of x and y is

$$dP=\frac{1}{\Gamma(\lambda)\,\Gamma(\mu)}\,e^{-(x+y)}\,.\,x^{\lambda-1}\,y^{\mu-1}\quad x\,dy$$

Put $u=x+y,\quad v=\dfrac{x}{x+y}$

$$\frac{\partial(u, v)}{\partial(x, y)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix}$$

$$=-\frac{(x+y)}{(x+y)^2}=-\frac{1}{x+y}=-\frac{1}{u}$$

Now $\qquad x=uv, \quad y=u(1-v)$

$\therefore$ Joint distribution of u and v is

$$dP=\frac{1}{\Gamma(\lambda)\,\Gamma(\mu)}\; e^{-u}\;(uv)^{\lambda-1}\;\{u(1-v)\}^{\mu-1}\;u.du\,dv$$

$$=\frac{1}{\Gamma(\lambda)\,\Gamma(\mu)}\; e^{-u}\,.\,u^{\lambda+\mu-1}\,.\,v^{\lambda-1}\;(1-v)^{\mu-1}\,du\,dv$$

$$=\left\{\frac{1}{\Gamma(\lambda+\mu)}\;e^{-u}\,.\,u^{\lambda+\mu-1}\,du\right\}\left\{\frac{1}{\beta(\lambda,\mu)}\;v^{\lambda-1}(1-v)^{\mu-1}dv\right\}$$

$\Rightarrow$ u and v are independent variates. u is a $\gamma(\lambda+\mu)$ variate and v is $\beta_1(\lambda,\mu)$ variate.

To find dist. of $\frac{x}{y}$ proceed as below :

Put $\quad u=x+y, \; v=\frac{x}{y}$

$$\therefore \qquad \frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix}$$

$$=-\frac{x+y}{y^2}$$

Now $\qquad x=\frac{uv}{1+v}, \quad y=\frac{u}{1+v}$

$$\therefore \qquad \frac{\partial(u,v)}{\partial(x,y)}=-\frac{(1+v)^2}{u}$$

$\therefore$ Joint distribution of u and v is

$$dP=\frac{1}{\Gamma(\lambda)\Gamma(\mu)}\; e^{-u}\,.\left(\frac{uv}{1+v}\right)^{\lambda-1}\left(\frac{u}{1+v}\right)^{\mu-1}du\,dv\;\frac{u}{(1+v)^2}$$

$$=\frac{1}{\Gamma(\lambda)\Gamma(\mu)}\cdot e^{-u}\cdot u^{\lambda+\mu-1}\cdot\frac{v^{\lambda-1}}{(1+v)^{\lambda+\mu}}du\,dv$$

$$=\left\{\frac{1}{\Gamma(\lambda+\mu)}\; e^{-u}\cdot u^{\lambda+\mu-1}du\right\}\left\{\frac{1}{\beta(\lambda,\mu)}\;\frac{v^{\lambda-1}}{(1+v)^{\lambda+\mu}}\;dv\right\}$$

$\Rightarrow$ u and v are independent variates.

u is a $\gamma(\lambda+\mu)$ variate

and v is $\beta_2(\lambda,\mu)$ variate.

Ex. 12·4. *If x and y are two independent standard normal variates, find the distributions of (i) x^2 (ii) x^2+y^2 (iii) $\frac{x^2}{y^2}$.*

Sol. (*i*) Put $u=x^2$

Then $\frac{1}{2}u$ is a $\gamma\left(\frac{1}{2}\right)$ variate.

$\therefore$ Dist. of $\frac{1}{2}u$ is

$$dP=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\,e^{-\frac{u}{2}}\left(\frac{u}{2}\right)^{\frac{1}{2}-1}d\left(\frac{u}{2}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{u}{2}}\,u^{-\frac{1}{2}}\,du$$

which gives the distribution of u.

(*ii*) $\frac{1}{2}x^2$ is a $\gamma\left(\frac{1}{2}\right)$ variate

$\frac{1}{2}y^2$ is a $\gamma\left(\frac{1}{2}\right)$ variate

$\therefore$ $\frac{1}{2}(x^2+y^2)$ is a $\gamma\left(\frac{1}{2}+\frac{1}{2}=1\right)$ variate.

$\therefore$ Dist. of $\frac{x^2+y^2}{2}$ is

$$dP=\frac{1}{\Gamma(1)}\;e^{-\frac{1}{2}(x^2+y^2)}\left(\frac{x^2+y^2}{2}\right)^{1-1}\;\left(\frac{x^2+y^2}{2}\right)$$

$$=\frac{1}{2}\ e^{-\frac{1}{2}\psi^2}\ d\psi^2$$

where $\psi^2=x^2+y^2$

(*iii*) $\frac{1}{2}x^2$ is a $\gamma\left(\frac{1}{2}\right)$ variate.

$\frac{1}{2}y^2$ is a $\gamma\left(\frac{1}{2}\right)$ variate.

$$\therefore\quad u=\frac{\frac{1}{2}x^2}{\frac{1}{2}y^2}=\frac{x^2}{y^2}\ \text{is a}\ \beta_2\left(\frac{1}{2},\ \frac{1}{2}\right).$$

Ex. 12·5. *If x and y are gamma variates with parameters λ and μ. Find the distributions of $x+y$ and $\frac{x-y}{x+y}$.*

Sol. Let $u=x+y,\ v=\frac{x-y}{x+y}$

$$\therefore\quad x=\frac{u(1+v)}{2},\ y=\frac{u(1-v)}{2}$$

$$\therefore\quad \frac{\partial(x, y)}{\partial(u, v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{1+v}{2} & \frac{u}{2}\\ \frac{1-v}{2} & -\frac{u}{2}\end{vmatrix}=-\frac{u}{2}$$

Now Dists. of x and y are

$$dP=\frac{1}{\Gamma(\lambda)}\ e^{-x}\ .\ x^{\lambda-1}\ dx$$

$$dP=\frac{1}{\Gamma(\mu)}\ e^{-y}\ .\ y^{\mu-1}\ dy$$

The joint distribution of x and y is

$$dP=\frac{1}{\Gamma(\lambda)\ .\ \Gamma(\mu)}\ e^{-(x+y)}\ x^{\lambda-1}\ y^{\mu-1}\ dx\ dy$$

The joint distribution of u and v is

$$dP=\frac{1}{\Gamma(\lambda)\Gamma(\mu)}\ e^{-u}\cdot\left\{\frac{u(1+v)}{2}\right\}^{\lambda-1}\left\{\frac{u(1-v)}{2}\right\}^{\mu-1}du\,dv\,.\,\frac{u}{2}$$

$$=\left\{\frac{1}{\Gamma(\lambda+\mu)}e^{-u}.u^{\lambda+\mu-1}\,du\right\}\cdot\left\{\frac{1}{2\beta(\lambda,\mu)}\,(1+v)^{\lambda-1}\,(1-v)^{\mu-1}dv\right\}$$

$\therefore$ u and v are independent and their distributions are

$$dP=\frac{1}{\Gamma(\lambda+\mu)}\ e^{-\mu}\ u^{\lambda+\mu-1}\,du$$

and
$$dP=\frac{1}{2\beta(\lambda,\mu)}\ (1+v)^{\lambda-1}\ (1-v)^{\mu-1}dv \text{ respectively.}$$

Ex. 12-6. *Let x be a $\beta(\lambda, \mu)$, find $E(\sqrt{x})$.*

Sol. Dist. of x is

$$dP=\frac{1}{\beta(\lambda,\mu)}\ x^{\lambda-1}\,(1-x)^{\mu-1}\,dx,\quad 0<x<1$$

$$\therefore\quad E(\sqrt{x})=\frac{1}{\beta(\lambda,\mu)}\int_0^1 x^{\frac{1}{2}}\,.\,x^{\lambda-1}\,(1-x)^{\mu-1}\,dx$$

$$=\frac{1}{\beta(\lambda,\mu)}\int_0^1 x^{\lambda+\frac{1}{2}-1}\,(1-x)^{\mu-1}\,dx$$

$$=\frac{1}{\beta(\lambda,\mu)}\,\beta\left(\lambda+\frac{1}{2},\ \mu\right)$$

$$=\frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(\lambda+\mu)}{\Gamma(\lambda)\ \Gamma\left(\lambda+\mu+\frac{1}{2}\right)}$$

EXERCISES

1. If $x_1, x_2, \ldots, x_n$ are gamma variates each with parameter λ, find the distribution of $\bar{x}=\dfrac{x_1+x_2+\ldots+x_n}{n}$.

2. For a Beta distribution of first kind, show that

   (*i*) A.M > H.M.

   (*ii*) $\log G=\dfrac{1}{\beta(\lambda,\mu)}\ \dfrac{\partial}{\partial\lambda}\,\beta(\lambda,\ \mu)$

13

Correlation Co-efficient and Linear Regression

13.1. Introduction

In bivariate distributions there are two variates x and y. If the change in one affects the change in other, the variables are said to be correlated. Otherwise they are said to be uncorrelated. If the increase (or decrease) in one results the increase (or decrease) in other, the correlation is said to be positive otherwise negative.

In bivariate distribution the data is of the form (x_1, y_1), (x_2, y_2) ... (x_n, y_n). Each of these value pairs can be represented by a point in xy-plane. The resulting set of points is called a scatter diagram. From the scatter diagram, one can get a fairly good idea, though vague, about the correlation between the variables. If the points are very dense (*i.e.*, very close to each other) a fairly good amount of correlation is expected and if the points are widely scattered correlation is poor.

As a measure of degree of linear relationship between the variates, coefficient of correlation is defined. This formula is referred to as product-moment formula for linear correlation. It, being due to Karl Pearson, is sometimes called Karl Pearson's correlation coefficient.

13.2. Covariance. *Covariance between two variates x and y is defined to be*

$$E(x-\bar{x})(y-\bar{y}) \qquad \textit{(for prob. dist.)}$$

or

$$\frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})(y_i-\bar{y}) \qquad \textit{(for freq. dist.)}$$

where $\bar{x}$ and $\bar{y}$ are respective means and is denoted by 'cov(x, y).'

13.2

Correlation Co-efficient. *Correlation co-efficient between two variates x and y is defined to be*

$$\frac{cov\,(x, y)}{(s\,d \text{ of } x)(s.d \text{ of } y)}$$

and is denoted by r_{xy}.

For freq. dist. $r_{xy}=\dfrac{\Sigma f(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma f(x-\bar{x})^2.\Sigma f(y-\bar{y})^2}}$

$$=\frac{\Sigma fxy-\dfrac{1}{N}(\Sigma fx)(\Sigma fy)}{\sqrt{\left\{\Sigma fx^2-\dfrac{1}{N}(\Sigma fx)^2\right\}\left\{\Sigma fy^2-\dfrac{1}{N}(\Sigma fy)^2\right\}}}$$

and for prob. dist. r_{xy}

$$=\frac{E(x-\bar{x})(y-\bar{y})}{\sqrt{E(x-\bar{x})^2E(y-\bar{y})^2}}=\frac{E(xy)-E(x)E(y)}{\sqrt{[E(x)^2-\{E(x)\}^2][E(y^2)-\{E(y)\}^2]}}$$

13.3. Properties of Correlation Coefficient

(*i*) *The correlation co-efficient is numerically independent of origin and scale.*

Sol. Let x and y be two variates with expected values $\bar{x}$ and $\bar{y}$ respectively and r the correlation co-efficient between them.

The transformations corresponding to change of origin and scale are

$$u=\frac{x-a}{h} \text{ and } v=\frac{y-b}{k}$$

where u and v are the variates to which x and y transform, a and b are constants corresponding to the change of origin and h and k are constants corresponding to change of scale.

Now $\quad x=a+uh$ and $y=b+kv$

$\therefore \quad \bar{x}=a+\bar{u}\,h$ and $\bar{y}=b+k\bar{v}$

where $\bar{u}$, $\bar{v}$ are expected values of u, v respectively

Now $\quad r_{xy}=\dfrac{E\{(x-\bar{x})(y-\bar{y})\}}{\sqrt{E(x-\bar{x})^2E(y-\bar{y})^2}}=\dfrac{hkE\{(u-\bar{u})(v-\bar{v})\}}{\sqrt{h^2k^2E(u-\bar{u})^2.E(v-\bar{v})^2}}$

$$=\frac{hk}{|hk|}\,r_{uv}=\pm r_{uv}$$

according as h and k are of same or opposite signs.

$\therefore \quad |r_{xy}|=|r_{uv}|$.

(ii) For two independent variates correlation coefficient is zero.

Sol. Let x and y be two independent variates with expected values $\bar{x}$ and $\bar{y}$ respectively.

Now $\quad \text{cov}(x, y)=E\{(x-\bar{x})(y-\bar{y})\}$

$$=E\{xy-\bar{x}y-\bar{y}x+\bar{x}\bar{y}\}$$
$$=E(xy)-\bar{x}E(y)-\bar{y}E(x)+\bar{x}\bar{y}$$
$$=E(xy)-\bar{x}\bar{y}-\bar{x}\bar{y}+\bar{x}\bar{y}=E(xy)-\bar{x}\bar{y}$$

Since x and y are independent,

$$E(xy)=E(x)E(y)=\bar{x}\bar{y}$$

$\therefore \quad \text{cov}(x, y)=\bar{x}\bar{y}-\bar{x}\bar{y}=0$

$\therefore \quad r_{xy}=0.$

Converse. It is not necessary that, if $r=0$, the variates are independent. To observe this consider the following example :

x :	-3	-2	-1	0	1	2	3
$y=x^2$:	9	4	1	0	1	4	9
xy :	-27	-8	-1	0	1	8	27

Here $\quad \Sigma x=0=\Sigma xy$

$$\therefore \quad r_{xy}=\frac{\Sigma xy-\frac{1}{N}(\Sigma x)(\Sigma y)}{\sqrt{(\Sigma x^2)-\frac{1}{N}(\Sigma x^2)}.\sqrt{\Sigma y^2-\frac{1}{N}(\Sigma y)^2}}=0$$

(iii) The correlation coefficient between linearly related variables is '+1' or '−1'.

Sol. Let x and y be the variates related by the equation $y=mx+c$ and $\bar{x}, \bar{y}$ be their expected values.

Then $\quad \bar{y}=m\bar{x}+c$

Now $\quad r_{xy}=\dfrac{E(x-\bar{x})(y-\bar{y})}{\sqrt{E(x-\bar{x})^2.E(y-\bar{y})^2}}=\dfrac{mE(x-\bar{x})^2}{\sqrt{m^2.E(x-\bar{x})^2}}$

$$=\frac{m}{|m|}=\pm 1$$

according as m is positive or negative.

(iv) The correlation coefficient cannot numerically exceed unity.

Sol. Let x and y be the variates with expected values $\bar{x}$ and $\bar{y}$ respectively.

Let $\quad x'=x-\bar{x}$ and $y'=y-\bar{y}$

Now for any real constant 'a', $(ax'-y')^2 \geq 0$

Since probabilities are non-negative and the sum of the non-negative quantities is non-negative,

$$E(ax'-y')^2 = \sum_i p_i(ax_i'-y_i')^2 > 0$$

$\therefore \quad a^2E(x'^2)+E(y'^2)-2aE(x'y') \geqslant 0$

Put
$$a = \frac{E(x'y')}{E(x'^2)}$$

Then
$$E(y'^2) \geqslant \frac{\{E(x'y')\}^2}{E(x'^2)}$$

i.e., $1 \geqslant \left\{\dfrac{E(x'y')}{\sqrt{E(x'^2)E(y')^2}}\right\}^2 = \left\{\dfrac{E(x-\bar{x})(y-\bar{y})}{\sqrt{E(x-\bar{x})^2E(y-\bar{y})^2}}\right\}^2 = r_{xy}^2$

$\therefore \quad |r_{xy}| \leqslant 1$ or $-1 \leqslant r_{xy} \leqslant 1$.

Ex. 13·1. *Show that, if x', y' are the deviations of the variables x, y from their means,*

$$r = 1 - \frac{1}{2N}\sum_i f_i\left(\frac{x_i'}{\sigma_x} - \frac{y_i'}{\sigma_y}\right)^2 \quad \textit{(Symbols have their usual meanings)}$$

and $$r = -1 + \frac{1}{2N}\sum_i f_i\left(\frac{x_i'}{\sigma_x} + \frac{y_i'}{\sigma_y}\right)^2$$

Deduce that $-1 \leqslant r \leqslant 1$.

Sol. Consider $1 - \dfrac{1}{2N}\displaystyle\sum_i f_i\left(\frac{x_i'}{\sigma_x} - \frac{y_i'}{\sigma_y}\right)^2$

$$= 1 - \frac{1}{2N}\sum_i f_i\left\{\frac{x_i'^2}{\sigma_x^2} + \frac{y_i'^2}{\sigma_y^2} - \frac{2x_i'y_i'}{\sigma_x\sigma_y}\right\}$$

$$= 1 - \frac{1}{2\sigma_x^2}\left(\frac{1}{N}\sum_i f_ix_i'^2\right) - \frac{1}{2\sigma_y^2}\left(\frac{1}{N}\sum_i f_iy_i'^2\right) + \frac{1}{\sigma_x\sigma_y}\left(\frac{1}{N}\sum_i f_ix_i'y_i'\right)$$

$$= 1 - \frac{1}{2\sigma_x^2}\cdot\sigma_x^2 - \frac{1}{2\sigma_y^2}\cdot\sigma_y^2 + \frac{\text{cov}(x, y)}{\sigma_x\sigma_y}$$

$$= r$$

Similarly second result can be proved.

Now $$1-r=\frac{1}{2N}\sum_i f_i\left(\frac{x_i'}{\sigma_x}-\frac{y_i'}{\sigma_y}\right)^2 \geqslant 0 \quad i.e., \quad 1 \geqslant r$$

and $$1+r=\frac{1}{2N}\sum_i f_i\left(\frac{x_i'}{\sigma_x}+\frac{y_i'}{\sigma_y}\right)^2 \geqslant 0 \quad i.e., \quad r \geqslant -1$$

$$\therefore \qquad -1 \leqslant r \leqslant 1.$$

Ex. 13-2. *Show that the co-efficient of correlation between two variates x and y may be expressed in the form*

$$\frac{1}{\sigma_x\sigma_y}\left(\frac{1}{n}\sum xy-\bar{x}\bar{y}\right)$$

where $\bar{x}, \bar{y}$ are A.Ms. and σ_x, σ_y are s.ds.

Sol. By def.

$$r_{xy}=\frac{\text{cov}(x, y)}{(s.d.\text{ of } x)(s.d.\text{ of } y)}=\frac{\frac{1}{n}\sum(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y}$$

$$=\frac{1}{n\sigma_x\sigma_y}\left\{\sum(xy-\bar{x}y-\bar{y}x+\bar{x}\bar{y})\right\}$$

$$=\frac{1}{n\sigma_x\sigma_y}\left\{\sum xy-n\bar{x}\bar{y}-n\bar{y}\bar{x}+n\bar{x}\bar{y}\right\}$$

$$=\frac{1}{\sigma_x\sigma_y}\left\{\frac{1}{n}\sum xy-\bar{x}\bar{y}\right\}$$

Note. *For the numerical data r_{xy} is calculated by changing the variates x and y to the new variates u and v defined by*

$$u=\frac{x-a}{h} \quad \text{and} \quad v=\frac{y-b}{k}$$

where a, b, h and k are constants to be chosen suitably so as to simplify the calculations and using the fact $r_{uv}=\pm r_{xy}$.

Ex. 13-3. *The ages (x) and systolic blood pressures (y) of 12 women are given below :*

Ages in years (x)	*Blood pressure (y)*
56	147
42	125
72	160
36	118
63	149
47	128
55	150
49	145
38	115
42	140
68	152
60	155

Calculate the correlation co-efficient between x and y.

Sol. Define the new variates

$$u=x-52, \quad \text{and} \quad v=y-140.$$

Then we have the table of values as below :

x	y	u	v	u^2	v^2	uv
56	147	4	7	16	49	28
42	125	-10	-15	100	225	150
72	160	20	20	400	400	400
36	118	-16	-22	256	484	352
63	149	11	9	121	81	99
47	128	-5	-12	25	144	60
55	150	3	10	9	100	30
49	145	-3	5	9	25	-15
38	115	-14	-25	196	625	350
42	140	-10	0	100	0	0
68	152	16	12	256	144	192
60	155	8	15	64	225	120
		4	4	1552	2502	1766

$$r=\frac{1766-\frac{1}{12}(4)(4)}{\sqrt{1552-\frac{1}{12}(4)^2}\sqrt{2502-\frac{1}{12}(4)^2}}=0{\cdot}896$$

Ex. 13·4. *Find the co-efficient of correlation for the following table :*

$x\downarrow$ \ $y\rightarrow$	67	72	77	82	87	92	97
92	—	—	—	1	2	3	1
87	—	—	1	3	8	1	5
82	4	4	6	4	9	1	—
77	3	3	7	6	4	—	—
72	2	3	5	6	1	1	—
67	3	2	—	—	—	—	—
62	1	—	—	—	—	—	—

Let $u=\dfrac{x-77}{5}$

and $v=\dfrac{y-82}{5}$

Sol. Calculation of Co-eff. of Correlation.

$v\rightarrow$ $u\downarrow$	-3	-2	-1	0	1	2	3	Total f	fu	fu^2	fuv
3	—	—	—	0 1	6 2	18 3	9 1	7	21	63	33
2	—	—	−2 1	0 3	16 8	4 1	30 5	18	36	72	48
1	−12 4	−8 4	−6 6	0 4	9 9	2 1	—	28	28	28	−15
0	0 3	0 3	0 7	0 6	0 4	—	—	23	0	0	0
−1	6 2	6 3	5 5	0 6	−1 1	−2 1	—	18	−18	18	14
−2	18 3	8 2	—	—	—	—	—	5	−10	20	26
−3	9 1	—	—	—	—	—	—	1	−3	9	9
Total f	13	12	19	20	24	6	6	100	54	210	115
fv	−39	−24	−19	0	24	12	18	−28			
fv^2	117	48	19	0	24	24	54	286			
fuv	21	6	−3	0	30	22	39	115			

$$r=\frac{115-\dfrac{1}{100}(54)(-28)}{\sqrt{210-\dfrac{1}{100}(54)^2}\sqrt{286-\dfrac{1}{100}(-28)^2}}=0.58$$

Ex. 13·5. *A computer while calculating r_{xy} from 25 pairs of observations obtained the following constants :*

$$n=25,\ \Sigma x=125,\ \Sigma x^2=650,\ \Sigma y=100,\ \Sigma y^2=460,\ \Sigma xy=508.$$

A recheck showed that he had copied down two pairs (6, 14) (8, 6) while the correct values were (8, 12), (6, 8). Obtained the correct value of the correlation co-efficient.

Sol. $n=25$

Correct value of $\Sigma x=125-6-8+8+6=125$

Correct value of $\Sigma x^2=650-36-64+64+36=650$

Correct value of $\Sigma y=100-14-6+12+8=100$

Correct value of $\Sigma y^2=460-196-36+144+64=436$

Correct value of $\Sigma xy=508-84-48+96+48=520$

$\therefore$ Correct value of co-efficient of correlation

$$=\frac{n(\Sigma xy)-(\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2-(\Sigma x)^2}\sqrt{n\Sigma y^2-(\Sigma y)^2}}$$

$$=\frac{(25)(520)-(125)(100)}{\sqrt{(25)(650)-(125)^2}\sqrt{(25)(436)-(100)^2}}$$

$$=\frac{520-500}{\sqrt{650-625}\sqrt{436-400}}=\frac{20}{5.6}=\frac{2}{3}$$

Ex. 13·6. *The following table gives the number of students having different hts. and wts. :*

wts. in lbs.

hts. in inches	80—90	90—100	100—110	110—120	120—130	Total
50—55	1	3	7	5	2	18
55—60	2	4	10	7	4	27
60—65	1	5	12	10	7	35
65—70	0	3	8	6	3	20
Total	4	15	37	28	16	100

Calculate co-efficient of correlation between hts. and wts.

Sol. Let x and y be the variates for the wts. and hts. respectively.

Calculation of Co-eff. of Correlation.

$x \rightarrow$ $y \downarrow$			80—90	90—100	100—110	110—120	120—130	Total f	fv	fv^2	fuv
	Mid points		85	95	105	115	125				
		$u \rightarrow$ $v \downarrow$	-2	-1	0	1	2				
50—55	52·5	-3	1^{6}	3^{9}	7^{0}	5^{-15}	2^{-12}	18	-54	162	-12
55—60	57·5	-1	2^{4}	4^{4}	10^{0}	7^{-7}	4^{-8}	27	-27	27	-7
60—65	62·5	1	1^{-2}	5^{-5}	12^{0}	10^{10}	7^{14}	35	35	35	17
65—70	67·5	3	—	3^{-9}	8^{0}	6^{18}	3^{18}	20	60	180	27
Total f			4	15	37	28	16	100	14	404	25
fu			-8	-15	0	28	32	37			
fu^2			16	15	0	28	64	123			
fuv			8	-1	0	6	12	25			

$u = \dfrac{(x-105)}{10}$

$v = \dfrac{(y-60)}{2 \cdot 5}$

Ex. 13-5. *A computer while calculating r_{xy} from 25 pairs of observations obtained the following constants :*

$$n=25,\ \Sigma x=125,\ \Sigma x^2=650,\ \Sigma y=100,\ \Sigma y^2=460,\ \Sigma xy=508.$$

A recheck showed that he had copied down two pairs (6, 14) (8, 6) while the correct values were (8, 12), (6, 8). Obtained the correct value of the correlation co-efficient.

Sol. $n=25$

Correct value of $\Sigma x=125-6-8+8+6=125$

Correct value of $\Sigma x^2=650-36-64+64+36=650$

Correct value of $\Sigma y=100-14-6+12+8=100$

Correct value of $\Sigma y^2=460-196-36+144+64=436$

Correct value of $\Sigma xy=508-84-48+96+48=520$

$\therefore$ Correct value of co-efficient of correlation

$$=\frac{n(\Sigma xy)-(\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2-(\Sigma x)^2}\sqrt{n\Sigma y^2-(\Sigma y)^2}}$$

$$=\frac{(25)(520)-(125)(100)}{\sqrt{(25)(650)-(125)^2}\sqrt{(25)(436)-(100)^2}}$$

$$=\frac{520-500}{\sqrt{650-625}\sqrt{436-400}}=\frac{20}{5.6}=\frac{2}{3}$$

Ex. 13-6. *The following table gives the number of students having different hts. and wts. :*

wts. in lbs.

hts. in inches	80—90	90—100	100—110	110—120	120—130	Total
50—55	1	3	7	5	2	18
55—60	2	4	10	7	4	27
60—65	1	5	12	10	7	35
65—70	0	3	8	6	3	20
Total	4	15	37	28	16	100

Calculate co-efficient of correlation between hts. and wts.

Sol. Let x and y be the variates for the wts. and hts. respectively.

Calculation of Co-eff. of Correlation.

$x \rightarrow$ / $y \downarrow$			80—90	90—100	100—110	110—120	120—130	Total f	fv	fv^2	fuv
	Mid points		85	95	105	115	125				
		$u \rightarrow$ / $v \downarrow$	−2	−1	0	1	2				
50—55	52·5	−3	1 (6)	3 (9)	7 (0)	5 (−15)	2 (−12)	18	−54	162	−12
55—60	57·5	−1	2 (4)	4 (4)	10 (0)	7 (−7)	4 (−8)	27	−27	27	−7
60—65	62·5	1	1 (−2)	5 (−5)	12 (0)	10 (10)	7 (14)	35	35	35	17
65—70	67·5	3	—	3 (−9)	8 (0)	6 (18)	3 (18)	20	60	180	27
Total f			4	15	37	28	16	100	14	404	25
fu			−8	−15	0	28	32	37			
fu^2			16	15	0	28	64	123			
fuv			8	−1	0	6	12	25			

$$u = \frac{(x-105)}{10}$$

$$v = \frac{(y-60)}{2 \cdot 5}$$

$$r=\frac{25-\frac{1}{100}(37)(14)}{\sqrt{123-\frac{1}{100}(37)^2}\sqrt{404-\frac{1}{100}(14)^2}}=0{\cdot}09$$

Ex. 13·7. *Find the variance of the variate*

$$u=a_1x_1+a_2x_2+\ldots\ldots+a_nx_n \qquad \textit{(where a's are constants)}$$

in terms of variances of x_1, x_2 *etc.*

Sol. Let $\bar{u}$, $\bar{x}_1$, $\bar{x}_2$, etc., be the expected values of the variates.

Then $\quad \bar{u}=E(u)=E(a_1x_1+a_2x_2+\ldots\ldots+a_nx_n)$

i.e., $\quad \bar{u}=a_1\bar{x}_1+a_2\bar{x}_2+\ldots\ldots+a_n\bar{x}_n$

$$\therefore \quad \text{Var}(u)=E(u-\bar{u})^2=E\{a_1(x_1-\bar{x}_1)+a_2(x_2-\bar{x}_2)+\ldots\ldots+a_n(x_n-\bar{x}_n)\}^2$$

$$=E\{a_1^2(x_1-\bar{x}_1)^2+a_2^2(x_2-\bar{x}_2)^2+\ldots\ldots+a_n^2(x_n-\bar{x}_n)^2+2a_1a_2(x_1-\bar{x}_1)(x_2-\bar{x}_2)+\ldots\ldots\}$$

$$=a_1^2E(x_1-\bar{x}_1)^2+a_2^2E(x_2-\bar{x}_2)^2+\ldots\ldots+a^2_nE(x_n-\bar{x}_n)^2+2a_1a_2E(x_1-\bar{x}_1)(x_2-\bar{x}_2)+\ldots\ldots$$

$$=a_1^2\text{ var }(x_1)+a_2^2\text{ var }(x_2)+\ldots\ldots+a_n^2\text{ var }(x_n)+2a_1a_2\text{ Cov }(x_1, x_2)+\ldots\ldots$$

Ex. 13·8. *If* σ_x^2 σ_y^2 *and* σ_{x-y}^2 *be the variances of* x, y *and* $x-y$ *respectively, show that*

$$r_{xy}=\frac{\sigma_x^2+\sigma_y^2-\sigma_{x-y}^2}{2\sigma_x\sigma_y}.$$

Sol. $\sigma_{x-y}^2=\text{var }(x-y)=\text{var }(x)+\text{var }(y)-2\text{ cov }(x, y)$

$$=\sigma_x^2+\sigma_y^2-2r_{xy}\sigma_x\sigma_y$$

$$\therefore \quad r_{xy}=\frac{\sigma_x^2+\sigma_y^2-\sigma_{x-y}^2}{2\sigma_x\sigma_y}.$$

Ex. 13·9. *Find the correlation co-efficient between* x *and* $a-x$.

Sol. Let $\quad u=a-x$

Then $\quad \bar{u}=a-\bar{x}$

where $\bar{u}$, $\bar{x}$ are expected values

$$\therefore \quad \text{var }(u)=E(u-\bar{u})^2=E(x-\bar{x})^2=\sigma^2 \text{ (say)}$$

$$\text{cov }(x, u)=E\{(x-\bar{x})(u-\bar{u})\}=-E(x-\bar{x})^2=-\sigma^2$$

$$\therefore \quad r_{xy}=\frac{-\sigma^2}{\sigma\cdot\sigma}=-1.$$

Ex. 13·10 *Find the correlation co-efficient between* $x+y$ *and* $x-y$ *(it is given that* x *and* y *are uncorrelated).*

Sol. Let $$u=x+y,\ v=x-y$$

Then $$\bar{u}=\bar{x}+\bar{y}, \quad \bar{v}=\bar{x}-\bar{y}$$

where $\bar{u}$, $\bar{v}$ etc., are A.Ms.

$$\therefore \quad \text{cov}\,(u, v)=E\{(u-\bar{u})(v-\bar{v})\}$$
$$=E\{((x-\bar{x})+(y-\bar{y}))((x-\bar{x})-(y-\bar{y}))\}$$
$$=E\{(x-\bar{x})^2-(y-\bar{y})^2\}=\sigma_x^2-\sigma_y^2$$

where σ_x, σ_y etc., are s.ds.

Also var (u)=var (x)+var (y)+2 cov (x, y)

i.e., $$\sigma_u^2=\sigma_x^2+\sigma_y^2$$

and $$\sigma_v^2=\sigma_x^2+\sigma_y^2$$

$\therefore$ If r be the correlation co-efficient between u and v,

$$r=\frac{\sigma_x^2-\sigma_y^2}{\sigma_x^2+\sigma_y^2}$$

Ex. 13-11. *If x and y are two correlated variables with the same s.d. and the correlation co-efficient r, show that the correlation co-efficient between x and x+y is* $\sqrt{\dfrac{1+r}{2}}$.

Sol. Let $u=x+y$ and σ the s.d. of x or y.

Then $$\bar{u}=\bar{x}+\bar{y}$$

where $\bar{u}$, $\bar{x}$, $\bar{y}$ are A.Ms.

Now var (u)=var (x)+var (y)+2 cov $(x, y)=2\sigma^2+2r\sigma^2$
$$=2\sigma^2(1+r)$$

$$\text{cov}\,(u, x)=E[(u-\bar{u})(x-\bar{x})]=E[\{(x-\bar{x})+(y-\bar{y})\}(x-\bar{x})]$$
$$=E(x-\bar{x})^2+E\{(x-\bar{x})(y-\bar{y})\}=\sigma^2+\text{cov}\,(x, y)$$
$$=\sigma^2(1+r)$$

$\therefore$ Correlation co-efficient between u and x is given by

$$r_{ux}=\frac{\sigma^2(1+r)}{\sigma^2\sqrt{2(1+r)}}=\sqrt{\frac{1+r}{2}}.$$

Ex. 13-12. *If x_1, x_2 and x_3 be uncorrelated variables each having the same s.d., obtain the correlation co-efficient between $u=x_1+x_2$ and $v=x_2+x_3$.*

Sol. Let $\bar{u}$ and $\bar{v}$ be the A.Ms of u and v respectively.

Then $$\bar{u}=\bar{x}_1+\bar{x}_2 \text{ and } \bar{v}=\bar{x}_2+\bar{x}_3$$

where $\bar{x}_1$, $\bar{x}_2$ etc., are A.Ms of x_1 and x_2 etc., respectively.

Now var (u)=var (x_1)+var $(x_2)=2\sigma^2$

where var $(x_1)=\sigma^2$ etc.

Similarly $\text{var}(v)=2\sigma^2$

$$\text{cov}(u, v)=E\{(u-\bar{u})(v-\bar{v})\}=E[\{(x_1-\bar{x}_1)+(x_2-\bar{x}_2)\}\{(x_2-\bar{x}_2)+(x_3-\bar{x}_3)\}]$$

$$=E(x_2-\bar{x}_2)^2 \qquad (\because \text{cov}(x_1, x_3)=0 \text{ etc., as } x_1, x_2, x_3 \text{ are uncorrelated})$$

$$=\sigma^2$$

$\therefore$ Correlation co-efficient between u and v is given by

$$r=\frac{\sigma^2}{2\sigma^2}=\frac{1}{2}$$

Ex. 13.13. *Two variates x and y have zero means, the same variance σ^2 and zero correlation. Show that*

$$U=x\cos\alpha+y\sin\alpha \text{ and } V=x\sin\alpha-y\cos\alpha$$

have the same variance σ^2 and zero correlation.

Sol. Since x and y have zero correlation, $\text{cov}(x, y)=0$ *i.e.,* $E(xy)=0$.

Now $\text{var}(U)=\cos^2\alpha\ \text{var}(x)+\sin^2\alpha\ \text{var}(y)=\sigma^2$

$\text{var}(V)=\sin^2\alpha\ \text{var}(x)+\cos^2\alpha\ \text{var}(y)=\sigma^2$

$\text{cov}(U, V)=E[(U-\bar{U})(V-\bar{V})]$

Now $\bar{U}=\cos\alpha\,\bar{x}+\sin\alpha\,\bar{y}=0$

and $\bar{V}=\bar{x}\sin\alpha-\bar{y}\cos\alpha=0$

$$\therefore \cos(U, V)=E(UV)=E\{(x\cos\alpha+y\sin\alpha)(x\sin\alpha-y\cos\alpha)\}$$

$$=\cos\alpha\sin\alpha\,E(x^2)-\sin\alpha\cos\alpha\,E(y^2)$$

$$=\cos\alpha\sin\alpha\,\sigma^2-\sin\alpha\cos\alpha\,\sigma^2=0$$

Ex. 13-14. *If $u=ax+by$ and $v=bx-ay$, where x and y represent deviations from the respective means and if the co-efficient of correlation between x and y is r and u and v are uncorrelated, show that*

$$\sigma_u\sigma_v=(a^2+b^2)\sigma_x\sigma_y\sqrt{1-r^2}$$

where σ_u, σ_v etc., are s.d.s of u, v etc.

Sol. Now $\sigma_u^2=\text{var}(u)=a^2\sigma_x^2+b^2\sigma_y^2+2ab\ \text{cov}(x, y)$

$\sigma_v^2=\text{var}(v)=b^2\sigma_x^2+a^2\sigma_y^2-2ab\ \text{cov}(x, y)$

$\text{cov}(u, v)=E\{(ax+by)(bx-ay)\}$

$=ab(\sigma_x^2-\sigma_y^2)+(b^2-a^2)\ \text{cov}(x, y)$

$$\therefore \quad \sigma_u^2\sigma_v^2-\text{cov}^2(u, v)=(a^2+b^2)^2\sigma_x^2\sigma_y^2-\text{cov}^2(x, y)$$

$$=(a^2+b^2)^2\sigma_x^2\sigma_y^2\left\{1-\frac{\text{cov}^2(x, y)}{\sigma_x^2\sigma_y^2}\right\}$$

$$=(a^2+b^2)^2\sigma_x^2\sigma_y^2(1-r^2)$$

But $\text{cov}(u, v)=0$

$$\therefore \qquad \sigma_u\sigma_v=(a^2+b^2)\,\sigma_x\sigma_y\sqrt{1-r^2}.$$

Ex. 13-15. *x_1, x_2 are two variates with variances σ_1^2 and σ_2^2 respectively and ρ is the correlation coefficient between them. Determine the values of the constants a and b which are independent of ρ such that x_1+ax_2 and x_1+bx_2 are uncorrelated.*

Sol. Let $u=x_1+ax_2$ and $=x_1+bx_2$.

$$\therefore \qquad \bar{u}=\bar{x}_1+a\bar{x}_2 \text{ and } \bar{v}=\bar{x}_1+b\bar{x}_2$$

$$\therefore \text{cov}(u, v)=E\{(u-\bar{u})(v-\bar{v})\}$$

$$=E\{(x_1-\bar{x}_1)+a(x_2-\bar{x}_2)\}\{(x_1-\bar{x}_1)+b(x_2-\bar{x}_2)\}$$

$$=E(x_1-\bar{x}_1)^2+(a+b)E\{(x_1-\bar{x}_1)(x_2-\bar{x}_2)\}+abE(x_2-\bar{x}_2)^2$$

$$=\sigma_1^2+(a+b)\,\text{cov}(x_1, x_2)+ab\sigma_2^2$$

$$=\sigma_1^2+ab\sigma_2^2+(a+b)\rho\sigma_1\sigma_2$$

Now $\qquad \text{cov}(u, v)=0$

$$\therefore \quad (\sigma_1^2+ab\sigma_2^2)+(a+b)\rho\sigma_1\sigma_2=0$$

Since a and b are independent of ρ, a and b are given by

$$\sigma_1^2+ab\sigma_2^2=0$$

and

$$a+b=0$$

$$\therefore \qquad a=-b=\frac{\sigma_1}{\sigma_2}.$$

Ex. 13-16. *If x and y are two variates each with mean zero and variance unity and $r_{xy}=r(\neq -1)$, find 'b' so that 'x+y' and 'x+by' may be uncorrelated.*

Sol. Let $u=x+y$ and $v=x+by$

Then $\qquad \bar{u}=\bar{x}+\bar{y}=0$ and $\bar{v}=\bar{x}+b\bar{y}=0.$

$$\therefore \qquad 0=\text{cov}(u, v)=E(x+y)(x+by)$$

$$=E(x^2)+(1+b)E(xy)+bE(y^2)$$

$$=(1+b)(1+r)$$

$$\therefore \qquad 1+b=0 \quad \text{as } r\neq -1.$$

$$\therefore \qquad b=-1.$$

Ex. 13-17. *If x and y are independent random variates, show that*

$$r(x+y,\ x-y)=r^2(x, x+y)-r^2(y, x+y)$$

where $r(x+y, x-y)$ denotes the coefficient of correlation between $x+y$ and $x-y$.

Sol. Since x and y are independent,

$$\text{cov}(x, y)=0 \qquad \ldots(1)$$

Put $\qquad X=x+y, \quad Y=x-y$

$\therefore \qquad \text{var}(X)=\text{var}(x)+\text{var}(y)+2\,\text{cov}(x, y)$

$$=\sigma_x^2+\sigma_y^2$$

and $\qquad \text{var}(Y)=\text{var}(x)+\text{var}(y)-2\,\text{cov}(x, y)$

$$=\sigma_x^2+\sigma_y^2$$

Now $\qquad \bar{X}=\bar{x}+\bar{y}, \quad \bar{Y}=\bar{x}-\bar{y}$

$$\therefore \quad \text{cov}(X, Y)=E\{X-\bar{X}\}\{Y-\bar{Y}\}$$

$$=E\{(x-\bar{x})+(y-\bar{y})\}\{(x-\bar{x})-(y-\bar{y})\}$$

$$=E\{(x-\bar{x})^2-(y-\bar{y})^2\}$$

$$=\sigma_x^2-\sigma_y^2$$

$$r(X, Y)=\frac{\sigma_x^2-\sigma_y^2}{\sigma_x^2+\sigma_y^2}$$

$$\text{cov}(x, X)=E(x-\bar{x})(X-\bar{X})$$

$$=E(x-\bar{x})\{(x-\bar{x})+(y-\bar{y})\}$$

$$=E(x-\bar{x})^2+E(x-\bar{x})(y-\bar{y})$$

$$=\sigma_x^2+\text{cov}(x, y)$$

$$=\sigma_x^2$$

$$\text{cov}(y, X)=E(y-\bar{y})(X-\bar{X})$$

$$=E(y-\bar{y})\{(x-\bar{x})+(y-\bar{y})\}$$

$$=E(x-\bar{x})(y-\bar{y})+E(y-\bar{y})^2$$

$$=\sigma_y^2$$

$$\therefore \quad r(x, X)=\frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_y^2}\,.\,\sigma_x}=\frac{\sigma_x}{\sqrt{\sigma_x^2+\sigma_y^2}}$$

and $$r(y, X)=\frac{\sigma_y}{\sqrt{\sigma_x^2+\sigma_y^2}}$$

$$\therefore \quad r(X, Y)=r^2(x, X)-r^2(y, X)$$

Ex. 13·18. *x_1, x_2, x_3 are three variables each with variance σ^2 and the correlation coefficient between any two of them is r. If*

$$\bar{x}=\frac{x_1+x_2+x_3}{3},$$

show that

$$var\,(\bar{x})=\frac{\sigma^2}{3}\,(1+2r)$$

Deduce that $\quad r \geq -\frac{1}{2}$

Sol. $$\bar{x}=\frac{x_1+x_2+x_3}{3}$$

$$E(\bar{x})=\frac{\bar{x}_1+\bar{x}_2+\bar{x}_3}{3}$$

$\therefore$
$$\begin{aligned}\text{var}\,(\bar{x})&=E\{\bar{x}-E(\bar{x})\}^2\\&=E\left\{\frac{(x_1-\bar{x}_1)+(x_2-\bar{x}_2)+(x_3-\bar{x}_3)}{3}\right\}^2\\&=\frac{1}{9}E\{(x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2+(x_3-\bar{x}_3)^2\\&\qquad+2(x_1-\bar{x}_1)(x_2-\bar{x}_2)+2(x_2-\bar{x}_2)(x_3-\bar{x}_3)\\&\qquad+2(x_1-\bar{x}_1)(x_3-\bar{x}_3)\}\\&=\frac{1}{9}\{3\sigma^2+2\,\text{cov}\,(x_1,x_2)+2\,\text{cov}\,(x_2,x_3)\\&\qquad+2\,\text{cov}\,(x_1,x_3)\}\\&=\frac{1}{9}\{3\sigma^2+6\sigma^2r\}\quad(\because\ \text{cov}\,(x_1,x_2)=\sigma^2r\ \text{etc.})\\&=\frac{1}{3}\,\sigma^2\,(1+2r)\end{aligned}$$

Since $\quad \text{var}\,(\bar{x})\geq 0,$

$\quad 1+2r\geq 0$

$\Rightarrow \quad r\geq -\frac{1}{2}.$

Ex. 13.19. *x and y are independent random variables each with mean zero and variance 1. Find 'a' so that the correlation coefficient between x+ay and x+y is maximum.*

Sol. By given

$$\bar{x}=0=\bar{y},\ \sigma_x=\sigma_y=1$$
$$\text{cov}\,(x,y)=0$$

Put $\quad X=x+ay,\ Y=x+y$

Then $\quad \bar{X}=0=\bar{Y}$

$\therefore$ Cov $(X, Y)=(E(XY)$
$$\begin{aligned}&=E\{(x+ay)(x+y)\}\\&=E(x^2)+(1+a)\,E(xy)+aE(y^2)\\&=1+a\end{aligned}$$

$$\text{var}(X) = \text{var}(x) + a^2 \text{ var}(y)$$
$$= 1 + a^2$$
$$\text{var}(Y) = \text{var}(x) + \text{var}(y)$$
$$= 2$$

$\therefore$
$$r_{xy} = \frac{1+a}{\sqrt{2}\ \sqrt{1+a^2}}$$

Now maximum value of $r_{xy} = 1$

$\therefore$
$$\frac{1+a}{\sqrt{2(1+a^2)}} = 1$$

$\Rightarrow$
$$1 + a^2 + 2a = 2 + 2a^2$$

i.e.,
$$a^2 - 2a + 1 = 0$$

$\Rightarrow$
$$a = 1$$

Ex. 13.20. *If x and y are uncorrelated random variates with means zero and variances σ_1^2 and σ_2^2 respectively. Show that the correlation coefficient between*

$$u = x \sin \alpha + y \cos \alpha$$

and
$$v = x \cos \alpha - y \sin \alpha$$

is

$$\frac{\sigma_1^2 - \sigma_2^2}{\{(\sigma_1^2 - \sigma_2^2)^2 + 4\sigma_1^2\sigma_2^2 \operatorname{cosec}^2 2\alpha\}^{1/2}}$$

Sol. By given

$$\bar{x} = 0 = \bar{y},$$
$$\text{var}(x) = \sigma_1^2, \ \text{var}(y) = \sigma_2^2$$

and
$$\text{cov}(x, y) = 0$$

Now
$$u = x \sin \alpha + y \cos \alpha$$
$$v = x \cos \alpha - y \sin \alpha$$

$\therefore$
$$\bar{u} = 0 = \bar{v}$$

$$\sigma_u^2 = \text{var}(u) = \sin^2 \alpha \text{ var}(x) + \cos^2 \alpha \text{ var}(y)$$
$$= \sin^2 \alpha\ \sigma_1^2 + \cos^2 \alpha\ \sigma_2^2$$
$$\sigma_v^2 = \text{var}(v) = \cos^2 \alpha \text{ var}(x) + \sin^2 \alpha \text{ var}(y)$$
$$= \cos^2 \alpha\ \sigma_1^2 + \sin^2 \alpha\ \sigma_2^2$$

$\therefore$
$$\sigma_u^2\sigma_v^2 = (\sin^2 \alpha \sigma_1^2 + \cos^2 \alpha \sigma_2^2)(\cos^2 \alpha \sigma_1^2 + \sin^2 \alpha \sigma_2^2)$$
$$= \sin^2 \alpha \cos^2 \alpha \{\sigma_1^4 + \sigma_2^4 - 2\sigma_1^2\sigma_2^2\} + \sigma_1^2\sigma_2^2 \{\sin^4 \alpha + \cos^4 \alpha + 2 \sin^2 \cos^2 \alpha\}$$
$$= \sin^2 \alpha \cos^2 \alpha\ (\sigma_1 - \sigma_2)^2 + \sigma_1^2\sigma_2^2$$

$$\text{Cov}(u, v) = E(uv)$$

$$= E\{(x \sin \alpha + y \cos \alpha)(x \cos \alpha - y \sin \alpha)\}$$

$$= E\{(x^2 - y^2) \sin \alpha \cos \alpha + xy (\cos^2 \alpha - \sin^2 \alpha)\}$$

$$= (\sigma_1^2 - \sigma_2^2) \sin \alpha \cos \alpha + \text{cov}(x, y)\{\cos^2 \alpha - \sin^2 \alpha\}$$

$$= (\sigma_1^2 - \sigma_2^2) \sin \alpha \cos \alpha$$

$$r_{uv} = \frac{(\sigma_1^2 - \sigma_2^2) \cos \alpha \sin \alpha}{\{\sin^2 \alpha \cos^2 \alpha (\sigma_1^2 - \sigma_2^2)^2 + \sigma_1^2 \sigma_2^2\}^{1/2}}$$

$$= \frac{(\sigma_1^2 - \sigma_2^2)}{\{(\sigma_1^2 - \sigma_2^2)^2 + 4\sigma_1^2 \sigma_2^2 \operatorname{cosec}^2 2\alpha\}^{1/2}}$$

Ex. 13.21. *If x and y are random variates with correlation coefficient γ between them, show that*

$$u = x \sin \alpha + y \cos \alpha$$

$$v = y \sin \alpha - x \cos \alpha$$

are uncorrelated if

$$\tan 2\alpha = \frac{2\gamma \sigma_x \sigma_y}{\sigma_y^2 - \sigma_x^2}$$

Sol. By given

$$u = x \sin \alpha + y \cos \alpha$$

$$v = y \sin \alpha - x \cos \alpha$$

$$\bar{u} = \bar{x} \sin \alpha + \bar{y} \cos \alpha$$

$$\bar{v} = \bar{y} \sin \alpha - \bar{x} \cos \alpha$$

$$\text{cov}(u, v) = E\{(u - \bar{u})(v - \bar{v})\}$$

$$= E\{(x - \bar{x}) \sin \alpha + (y - \bar{y}) \cos \alpha\} \{(y - \bar{y}) \sin \alpha - (x - \bar{x}) \cos \alpha\}$$

$$= E\{-\cos \alpha \sin \alpha (x - \bar{x})^2 + \sin \alpha \cos \alpha (y - \bar{y})^2 + (x - \bar{x})(y - \bar{y}) (\sin^2 \alpha - \cos^2 \alpha)\}$$

$$= \sin \alpha \cos \alpha (\sigma_y^2 - \sigma_x^2) - \text{cov}(x, y) \cos 2\alpha$$

$$= \sin \alpha \cos \alpha (\sigma_y^2 - \sigma_x^2) - \gamma \sigma_x \sigma_y \cos 2\alpha$$

Now, u and v are uncorrelated if

$$\text{cov}(u, v) = 0$$

$$\Rightarrow \quad \sin \alpha \cos \alpha (\sigma_y^2 - \sigma_x^2) - \gamma \sigma_x \sigma_y \cos 2\alpha = 0$$

$$\Rightarrow \quad \tan 2\alpha = \frac{2\gamma \sigma_x \sigma_y}{\sigma_y^2 - \sigma_x^2}$$

13.4. Rank Correlation

13.4.1. Non-Repeated Ranks. Let n be the number of individuals which are ranked according to two different characters A and B. Let x and y be the ranks *w.r.t.* A and B respectively. Assuming that ranks are not repeated in either series, both x and y take the same values 1, 2, ...n.

Then $\Sigma x = \Sigma y = 1+2+\ldots\ldots+n = \dfrac{n(n+1)}{2}$

and $\Sigma x^2 = \Sigma y^2 = 1^2+2^2+\ldots\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$\therefore \quad \text{var}(x) = \text{var}(y) = \frac{1}{n}\Sigma x^2 - \left(\frac{\Sigma x}{n}\right)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left\{\frac{n(n+1)}{2n}\right\}^2 = \frac{n^2-1}{12}$$

Let $\qquad d = x \sim y$

$$\therefore \qquad \Sigma d^2 = \Sigma(x-y)^2 = \Sigma\{(x-\bar{x})-(y-\bar{y})\}^2$$

where $\bar{x}$ and $\bar{y}$ are A.Ms.

$$\therefore \qquad \frac{1}{n}\Sigma d^2 = \frac{1}{n}\Sigma(x-\bar{x})^2 + \frac{1}{n}\Sigma(y-\bar{y})^2 - 2\frac{1}{n}\Sigma(x-\bar{x})(y-\bar{y})$$

$$= \text{var}(x) + \text{var}(y) - 2\,\text{cov}(x, y)$$

$$= \frac{n^2-1}{12} + \frac{n^2-1}{12} - 2\,\text{cov}(x, y)$$

$$\therefore \quad \text{cov}(x, y) = \frac{n^2-1}{12} - \frac{1}{2n}\Sigma d^2$$

$\therefore$ Correlation co-efficient between x and y is given by

$$r = \frac{\text{cov}(x, y)}{(s.d \text{ of } x)(s.d \text{ of } y)} = 1 - \frac{6}{n(n^2-1)}\Sigma d^2$$

'r' is called Spearman's rank correlation co-efficient.

Ex. 13-22. *Calculate Spearman's rank correlatinn coefficient from the following data. Two numbers within brackets denote the ranks of the students in papers A and B respectively.*

(1, 1) ; (2, 10) ; (3, 3) ; (4, 4) ; (5, 5) ; (6, 7) ; (7, 2) ; (8, 6) ; (9, 8) ; (10, 11) ; (11, 15) ; (12, 9) ; (13, 14) ; (14, 12) ; (15, 16) ; (16, 13).

Sol. Let R_1 and R_2 be the ranks for A and B respectively.

Calculation of Coeff. of Rank Correlation

R_1	R_2	$d=R_1\sim R_2$	d^2	R_1	R_2	$R_1\sim R_2=d$	d^2
1	1	0	0	9	8	1	1
2	10	8	64	10	11	1	1
3	3	0	0	11	15	4	16
4	4	0	0	12	9	3	9
5	5	0	0	13	14	1	1
6	7	1	1	14	12	2	4
7	2	5	25	15	16	1	1
8	6	2	4	16	13	3	9
							136

$\therefore$ Spearman's Rank Correlation Coefficient is given by

$$r=1-\frac{6\Sigma d^2}{n(n^2-1)}=1-\frac{6.(136)}{16.255}$$
$$=0{\cdot}8.$$

Ex. 13-23. *Ten competitors in a voice test are ranked by three judges in the following data :*

First Judge :	*1*	*6*	*5*	*10*	*3*	*2*	*4*	*9*	*7*	*8*
Second Judge :	*3*	*5*	*8*	*4*	*7*	*10*	*2*	*1*	*6*	*9*
Third Judge :	*6*	*4*	*9*	*8*	*1*	*2*	*3*	*10*	*5*	*7*

Use the method of rank correlation to gauge which pair of judges have the nearest approach to common likings in voice.

Sol. Let R_1, R_2 and R_3 be the ranks due to three judges respectively.

Calculation of Rank Coeff. of Correlation

Comp. No.	R_1	R_2	R_3	R_1-R_2 $=d_{12}$	d_{12}^2	R_2-R_3 $=d_{23}$	d_{23}^2	R_1-R_3 $=d_{13}$	d_{13}^2
1	1	3	6	-2	4	-3	9	-5	25
2	6	5	4	1	1	1	1	2	4
3	5	8	9	-3	9	-1	1	-4	16
4	10	4	8	6	36	-4	16	2	4
5	3	7	1	-4	16	6	36	2	4
6	2	10	2	-8	64	8	64	0	0
7	4	2	3	2	4	-1	1	1	1
8	9	1	10	8	64	-9	81	-1	1
9	7	6	5	1	1	1	1	2	4
10	8	9	7	-1	1	2	4	1	1
					200		214		60

$\therefore$ Rank coeff. of correlation between first and second judge

$$=1-\frac{1200}{10.99}\simeq-0{\cdot}212$$

Rank Coeff. of correlation between first and third judge

$$=1-\frac{(6)(60)}{10.99}\simeq 0{\cdot}636.$$

and rank co-efficient of correlation between second and third judge

$$1=-\frac{6.124}{10.99}\simeq-0{\cdot}297.$$

Since the correlation between first and second judge is $-$ve, opinions regarding voice test are opposite of each other. Similarly opinions of second and third judge are opposite of each other. But the opinions of first judge and third are of similar type as their correlation is positive *i.e.*, their likings and dislikings are very much common.

Hence first and third judges have nearest approach to the common likings.

13.4.2. Repeated Ranks

In this case two or more individuals are bracketed equal in either or both classifications. Here, common ranks are given to the bracketed individuals. This common rank is the average of the ranks which these individuals would have assumed had they been slightly different in ranks from each other.

The rank co-eff. of correlation when $t_1, t_2 \ldots t_p$ figures are given same rank, is given by

$$r=1-\frac{6(\Sigma d^2+T)}{n(n^2-1)}$$

where $T=\sum_{i=1}^{p}\frac{1}{12}(t_i^3-t_i)$

Ex. 13·24. *Find spearman's/rank correlation coefficient for the data given below :*

Students :	1	2	3	4	5	6	7	8	9	10	11	12
Marks in Exam. A :	15	13	17	14	18	12	20	16	18	17	19	21
Marks in Exam. B :	18	16	18	15	19	16	18	15	21	17	18	20

Sol. Calculation of Rank Coeff. Correlation.

S.N.	Ranks in A R_1	Ranks in B R_2	$R_1 \sim R_2$ d	d^2
1	9	5·5	3·5	12·25
2	11	9·5	1·5	2·25
3	6·5	5·5	1·0	1·00
4	10	11·5	1·5	2·25
5	4·5	3	1·5	2·25
6	12	9·5	2·5	6·25
7	2	5·5	3·5	12·25
8	8	11·5	3·5	12·25
9	4·5	1	3·5	12·25
10	6·5	8	1·5	2·25
11	3	5·5	2·5	6·25
12	1	2	1	1·00
				72·50

Here in paper A, two students have got 18 marks each and two 17 marks each. While marking ranks, ranks 1, 2, 3 are given to students getting marks 21, 20, 19, ranks 4th and 5th are to be given to students getting 18 each. As they have got equal marks, their ranks should be same and hence each is given the rank

$$\frac{4+5}{2}=4{\cdot}5.$$

Similarly the ranks of students getting 17 marks each is

$$\frac{6+7}{2}=6{\cdot}5 \text{ each}$$

In paper B, there are four students getting marks 18 each and the ranks to be given to them are 4, 5, 6, 7.

$\therefore$ Rank of each student getting marks 18

$$=\frac{4+5+6+7}{4}=\frac{22}{4}=5{\cdot}5.$$

Similarly rank of each student getting 16 marks in B

$$=\frac{9+10}{2}=9{\cdot}5$$

and rank of each student getting 15 marks

$$=\frac{11+12}{2}=11{\cdot}5$$

As in paper A, two students get the same rank 4·5 and two students get the same rank 6·5, for paper A we have $t_1=2$, $t_2=2$.

In paper B, there are 4 students getting rank 5·5, two students getting rank 9·5 and two students getting rank 11·5.

$\therefore \quad t_3=4,\ t_4=2,\ t_5=2.$

$$\therefore \quad T=\frac{1}{12}\{2^3-2\}+\frac{1}{12}\{2^3-2\}+\frac{1}{12}\{4^3-4\}+\frac{1}{12}\{2^3-2\}$$
$$+\frac{1}{12}\{2^3-2\}$$

$$=\frac{1}{3}\{8-2\}+\frac{1}{12}\{64-4\}$$

$$=2+5=7.$$

$$\therefore \quad r=1-\frac{6(72\cdot5+7)}{(12)(143)}$$

$$=1-\frac{79\cdot5}{286}=\frac{206\cdot5}{286}=0\cdot722.$$

Ex. 13·25. *The coefficients of rank correlation of the marks obtained by 10 students in Physics and Maths was found to be 0·4. It was later on discovered that the difference in ranks for one student was wrongly taken as 2 instead of 3. Find the correct coefficient of rank correlation.*

Sol. Here $r=0\cdot4,\quad n=10$

$$\therefore \qquad 1-\frac{6\Sigma d^2}{n(n^2-1)}=0\cdot4$$

$$\Rightarrow \qquad 0\cdot6=\frac{6\Sigma d^2}{10(99)}$$

$$\Rightarrow \qquad \Sigma d^2=99.$$

Now, corrected value of $\Sigma d^2=99-4+9=104.$

Corrected value of rank correlation coefficient

$$=1-\frac{6(104)}{10.99}=0\cdot37$$

13.4.3. Limits for the Rank Correlation Coefficient

The formula for rank correlation coefficient is

$$r=1-\frac{6.\Sigma d^2}{n(n^2-1)}$$

where r is the rank correlation coefficient. Now, r is maximum when Σd^2 is minimum, which is so only when each d is minimum *i.e.*, zero. This is achieved only when ranks of each individuals are same in either classification.

$\therefore$ Minimum value of $\Sigma d^2=0$

$\therefore$ Maximum value of $r=1$

r is minimum, when Σd^2 is maximum. This is achieved only when the ranks in two classifications are in reverse order *i.e.*, if the rank of an individual in one classification is r, its rank in other classification is $n-r-1=n+1-r$. In this case corresponding value of d is

$$|\, n-(2r-1)\,|$$

$\therefore$ Maximum value of Σd^2

$$= \sum_{r=1}^{n} \{n-(2r-1)\}^2$$

$$= \sum_{r=1}^{n} \{n^2+(4r^2-4r+1)-2n(2r-1)\}$$

$$= \sum_{r=1}^{n} \{(n+1)^2-4(n+1)r+4r^2\}$$

$$= n(n+1)^2-4(n+1)\sum_{r=1}^{n} r+4\sum_{r=1}^{n} r^2$$

$$= n(n+1)^2-4(n+1)\cdot\frac{n(n+1)}{2}+4\,\frac{n(n+1)(2n+1)}{6}$$

$$= -n(n+1)^2+\frac{2}{3}n(n+1)(2n+1)$$

$$= \frac{1}{3}n(n+1)\{4n+2-3(n+1)\}$$

$$= \frac{1}{3}n(n^2-1)$$

Min. value of $r=1-\dfrac{6}{n(n^2-1)}\cdot\dfrac{1}{3}n(n^2-1)$

$$= -1$$

13.5. Regression and Lines of Regression

In case there is some relationship between the variates, the points of the scatter diagram will be more or less concentrated round a curve. This curve is called curve of regression. From this curve, it is possible to estimate one of the variables (the dependent variable) from the other (the independent variable). This process of estimation is often referred to as, regression. If y (or x) is estimated from x (or y), regression curve is of y on x (or x on y).

In case this curve is a straight line, it is called the line of regression and the regression is said to be linear.

Evidently the line of regression is the straight line which gives the 'best fit in the least square sense' to the given data.

In case y is treated as dependent and x as independent variable, the line of regression is called the 'line of regression of y on x' and gives the best estimate of y for any given value of x. In the contrary case it is called the 'line of regression of x on y' and gives the best estimate of x for any given value of y.

13.5.1. Equations of Lines of Regression

Consider the bivariate freq. dist.

$$\begin{array}{c} x\rightarrow \\ y\rightarrow \\ f\rightarrow \end{array}\begin{pmatrix} x_1 & x_2 \ldots\ldots x_n \\ y_1 & y_2 \ldots\ldots y_n \\ f_1 & f_2 \ldots\ldots f_n \end{pmatrix}$$

where $f_1+f_2+\ldots\ldots f_n=N$.

The line of regression is the straight line best fitted in the least square sense to the given distribution.

Let $y=mx+c$ be the equation of line of regression of y on x, where m and c are unknown to be determined by the method of least squares.

Let $Y_i=mx_i+c$

and $$S=\sum_{i=1}^{n} f_i(Y_i-y_i)^2=\sum_{i=1}^{n} f_i(mx_i+c-y_i)^2$$

According to the method of least squares, m and c are to be determined so that S is minimum.

The Normal equations are :

$$0=\frac{\partial S}{\partial c}=\sum_{i=1}^{n} 2f_i(mx_i+c-y_i)$$

i.e., $$m\sum_{i=1}^{n} f_ix_i+c\sum_{i=1}^{n} f_i=\sum_{i=1}^{n} f_iy_i$$

i.e., $$m\bar{x}+c=\bar{y} \qquad \ldots(1)$$

and $$0=\frac{\partial S}{\partial m}=\sum_{i=1}^{n} 2f_i(mx_i+c-y_i)(x_i)$$

i.e., $$m\sum_{i=1}^{n} f_ix_i^2+c\sum_{i=1}^{n} f_ix_i=\sum_{i=1}^{n} f_ix_iy_i \qquad \ldots(2)$$

Now by def,

$$\sigma_x^2=\frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})^2=\frac{1}{N}\sum_{i=1}^{n} f_i\{x_i^2+\bar{x}^2-2x_i\bar{x}\}$$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i x_i^2-\bar{x}^2$$

$$\therefore \qquad \sum_{i=1}^{n} f_i x_i^2=N(\sigma_x^2+\bar{x}^2)$$

and $$\mu_e=\text{cov}\,(x, y)=\frac{1}{N}\sum_{i=1}^{n} f_i(x_i-\bar{x})(y_i-\bar{y})$$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i\{x_i y_i-\bar{x}y_i-x_i\bar{y}+\bar{x}\bar{y}\}$$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i x_i y_i-\bar{x}\bar{y}$$

$$\therefore \qquad \sum_{i=1}^{n} f_i x_i y_i=N(\mu_e+\bar{x}\bar{y})$$

Substituting in (2)

$$m(\sigma_x^2+\bar{x}^2)+\bar{x}c=\mu_e+\bar{x}\bar{y} \qquad \ldots(3)$$

Solving (1) and (3) for m and c

$$m=\frac{\mu_e}{\sigma_x^2} \text{ and } c=\bar{y}-\frac{\mu_e}{\sigma_x^2}\bar{x}$$

$\therefore$ Eq. of line of regression of y on x is

$$y-\bar{y}=\frac{\mu_e}{\sigma_x^2}(x-\bar{x})$$

Similarly the line of regression of x on y can be shown to have its equation

$$x-\bar{x}=\frac{\mu_e}{\sigma_y^2}(y-\bar{y}).$$

Ex. 13-25. *If α is the angle between the two regression lines in the case of two variables x and y show that*

$$\tan \alpha = \frac{1-r^2}{r} \quad \frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}$$

where the symbols have their usual meanings.

Sol. The equations of two lines of regression are

$$y-\bar{y}=\frac{\mu_{s}}{\sigma_x^2}(x-\bar{x}) \qquad \text{...(1)} \ (y \text{ on } x)$$

$$x-\bar{x}=\frac{\mu_{s}}{\sigma_y^2}(y-\bar{y}) \qquad \text{...(2)} \ (x \text{ on } y)$$

Case I. If $\mu_s \neq 0$.

Slope of (1)$=\frac{\mu_s}{\sigma_x^2}=r\ \frac{\sigma_y}{\sigma_x}$

Slope of (2)$=\frac{\sigma_y^2}{\mu_s}=\frac{\sigma_y}{r\sigma_x}$

where r is the correlation co-efficient between x and y.

Now $$\tan \alpha = \frac{\frac{1}{r}\ \frac{\sigma_y}{\sigma_x}-\frac{r\sigma_y}{\sigma_x}}{1+\frac{\sigma_y^2}{\sigma_x^2}}=\frac{1-r^2}{r}\ \frac{\sigma_y\sigma_x}{\sigma_x^2+\sigma_y^2}$$

Case II. If $\mu_s=0$.

Lines (1) and (2) become

$$y=\bar{y} \quad \text{and} \quad x=\bar{x}$$

which are parallel to co-ordinate axes

$\therefore$ $\alpha=90°$.

13-5.2. Regression Co-efficients. *The quantities* $\frac{\mu_s}{\sigma_x^2}$ *and* $\frac{\mu_s}{\sigma_y^2}$ *are called regression co-efficients of 'y on x' and 'x on y' respectively and are denoted by b_{yx} and b_{xy} respectively.*

13.5.3. Properties of Regression Coefficients

(*i*) *The correlation co-efficient is the geometric mean between regression co-efficients.*

Regression co-efficients are given by

$$b_{yx}=\frac{\mu_s}{\sigma_x^2} \text{ and } b_{xy}=\frac{\mu_s}{\sigma_y^2}$$

$$\therefore \qquad b_{yx}.b_{xy}=\left\{\frac{\mu_s}{\sigma_x\sigma_y}\right\}^2=r^2$$

where r is the correlation coefficient.

(ii) The correlation co-efficient cannot numerically exceed the arithmetic mean between regression co-efficients.

Regression co-efficients are given by

$$b_{yx}=\frac{\mu_{11}}{\sigma_x^2}=r\,\frac{\sigma_y}{\sigma_x} \qquad \text{and} \qquad b_{xy}=\frac{\mu_{11}}{\sigma_y^2}=r\,\frac{\sigma_x}{\sigma_y}$$

$$\therefore \qquad \left|\frac{b_{yx}+b_{xy}}{2}\right|=|r|\,\frac{\sigma_y^2+\sigma_x^2}{2\sigma_x\sigma_y}$$

$$\therefore \quad \left|\frac{b_{yx}+b_{xy}}{2}\right|-|r|=|r|\left\{\frac{\sigma_y^2+\sigma_x^2}{2\sigma_x\sigma_y}-1\right\}$$

$$=|r|\,\frac{(\sigma_y-\sigma_x)^2}{2\sigma_y\sigma_x}>0$$

$$\therefore \qquad \left|\frac{b_{yx}+b_{xy}}{2}\right|>|r|$$

Remarks. (1) b_{yx} is the slope of line of regression of y on x and b_{xy} is the reciprocal of the slope of line of regression of x on y.

(2) b_{yx}, b_{xy} and r are of same signs.

Ex. 13-26. *The ages (X) and systolic blood pressures (Y) of 12 women are given below :*

Age in years (X)	*Blood Pressure* (Y)
56	147
42	125
72	160
36	118
63	149
47	128
55	150
49	145
38	115
42	140
68	152
60	155

Determine the least squares regression line of Y on X and find the value of the regression co-efficient of Y on X.

Also estimate the blood pressure of a woman whose age is 45 years.

Sol.

X	Y	x	y	x^2	xy	
56	147	4	7	16	28	
42	125	−10	−15	100	150	
72	160	20	20	400	400	
36	118	−16	−22	256	352	
63	149	11	9	121	99	$x=X-52$
47	128	−5	−12	25	60	$y=Y-140$
55	150	3	10	9	30	
49	145	−3	5	9	−15	
38	115	−14	−25	196	350	
42	140	−10	0	100	0	
68	152	16	12	256	192	
60	155	8	15	64	120	
		4	4	1552	1766	

Let the line of regression of y on x be

$$y=a+bx$$

where the co-efficients 'a' and 'b' are given by the equations

$$\Sigma y=na+b\Sigma x$$

and $$\Sigma xy=a\Sigma x+b\Sigma x^2$$

Substituting the values of Σx etc.,

$$4=12a+4b \quad \text{or} \quad 1=3a+b$$

and $$1766=4a+1552b \quad \text{or} \quad 883=2a+776b$$

Solving

$$a=-0{\cdot}046 \text{ and } b=1{\cdot}138$$

$\therefore$ The equation of line of regression of y on x is

$$y=(-0{\cdot}046)+(1{\cdot}138)x$$

$\therefore$ The equation of line of regression of Y on X is

$$Y-140=(-0{\cdot}046)+(1{\cdot}138)(X-52)$$

or $$Y=(1{\cdot}138)X+(80{\cdot}778)$$

$\therefore$ Regression co-efficient of Y on $X=(1{\cdot}138)$

Now value of x for $X=45$ is $(45-52)=-7$.

$\therefore$ Estimate of $y=-0{\cdot}046-7{\cdot}966$

$$=-8{\cdot}012$$

$\therefore$ Estimate of Y for $X=45$

$$=140-8{\cdot}012$$
$$=131{\cdot}988$$

Ex. 13-27. *For the following table :*

Ages of husbands in years	*Ages of wives in years.* 10—20	20—30	30—40	40—50	50—60	Total
15—25	6	3	—	—	—	9
25—35	3	16	10	—	—	29
35—45	—	10	15	7	—	32
45—55	—	—	7	10	4	21
55—65	—	—	—	4	5	9
Total	9	29	32	21	9	100

Find (i) the co-efficient of correlation

(ii) The two regression lines.

Sol. The calculating table is on page 13.30.

(*i*) $\therefore$ The co-efficient of correlation is given by

$$r=\frac{98-\frac{1}{100}(-8)(-8)}{\sqrt{122-\frac{1}{100}(-8)^2}\sqrt{122-\frac{1}{100}(-8)^2}}$$

$$=\frac{9800-64}{(12200-64)}$$

$$=\frac{9736}{12136}=0{\cdot}802$$

(*ii*) $\bar{u}=\bar{v}=\frac{(-8)}{100}=-0{\cdot}08$

and $$\sigma_u^2=\sigma_v^2=\frac{1}{100}\{122\}-\left(-\frac{8}{100}\right)^2$$

$$=\frac{1}{(100)^2}\{12200-64\}=1{\cdot}2136$$

$\therefore$ $b_{uv}=b_{vu}=r=0{\cdot}802.$

$\therefore$ The equations of lines of regression of v on u and u on v respectively are

$$(v+0{\cdot}08)=0{\cdot}802(u+0{\cdot}08)$$

and $$(u+0{\cdot}08)=(0{\cdot}802)(v+0{\cdot}08)$$

Let x be the variable for the ages of wives and y be the variable for the ages of husbands.

$x \rightarrow$ $y \downarrow$			10—20	20—30	30—40	40—50	50—60	Total f	fv	fv^2	fuv	
	Mid points		15	25	35	45	55					
		$u \rightarrow$ $v \downarrow$	−2	−1	0	1	2					
15—25	20	−2	6^{24}	3^{6}	—	—	—	9	−18	36	30	
25—35	30	−1	3^{6}	16^{16}	10^{0}	—	—	29	−29	29	22	$u = \frac{(x-35)}{10}$
35—45	40	0	—	10^{0}	15^{0}	7^{0}	—	32	0	0	0	$v = \frac{(y-40)}{10}$
45—55	50	1	—	—	7^{0}	10^{10}	4^{8}	21	21	21	18	
55—65	60	2	—	—	—	4^{8}	5^{20}	9	18	36	28	
Total f			9	29	32	21	9	100	−8	122	98	
fu			−18	−29	0	21	18	−8				
fu^2			36	29	0	21	36	122				
fuv			30	22	0	18	28	98				

$\therefore$ The equations of lines of regression of y on x and x on y respectively are

$$\left(\frac{y-40}{10}+0{\cdot}08\right)=0{\cdot}802\left(\frac{x-35}{10}+0{\cdot}08\right)$$

or $$y=0{\cdot}802x+11{\cdot}772$$

and $$\left(\frac{x-35}{10}+0{\cdot}08\right)=(0{\cdot}802)\left(\frac{y-40}{10}+0{\cdot}08\right)$$

or $$x=0{\cdot}802y+2{\cdot}762.$$

Ex. 13·28. *In a partially destroyed laboratory record of the correlation analysis of data, the following results only are legible: var $(x)=9$, regression lines $8x-10y+66=0$ and $40x-18y=214$.*

Supply (i) mean values of x and y.

(ii) the s.d of y.

(iii) the correlation co-efficient between x and y.

Sol. (*i*) The equations of lines of regression of y on x and x on y respectively are

$$y-\bar{y}=b_{yx}(x-\bar{x}) \quad \ldots(1)$$

and $$x-\bar{x}=b_{xy}(y-\bar{y}) \quad \ldots(2)$$

$\therefore$ Thus $\bar{x}$ and $\bar{y}$ are the values of x and y which satisfy both the regression equations simultaneously.

$\therefore$ Solving regression equations

$$\bar{x}=13 \quad \text{and} \quad \bar{y}=17$$

(*ii*) It is not given, of the two regression equations which represents the line of regression of y on x. So we assume.

$$8x-10y+66=0$$

to be the equation representing the line of regression of y on x. Then equation representing the line of regression of x on y must be

$$40x-18y=214$$

$\therefore$ Comparing with (1) and (2)

$$b_{yx}=\frac{4}{5} \quad \text{and} \quad b_{xy}=\frac{9}{20}$$

$\therefore$ The co-eff. of correlation r is given by

$$r^2=b_{yx}.b_{xy}=\frac{4}{5}\cdot\frac{9}{20}=\frac{9}{25}(<1)$$

Since r^2 comes out to be less than unity, our assumption is correct.

Since b_{yx} and b_{xy} are positive, r must be positive and hence

$$r=\frac{3}{5}=0{\cdot}6$$

(*iii*) Now $b_{yx}=r\dfrac{\sigma_y}{\sigma_x}$ where symbols have their usual meanings.

$$\therefore \qquad \frac{4}{5}=\frac{3}{5}\cdot\frac{\sigma_y}{3}$$

or $$\sigma_y=4.$$

Ex. 13·29. *Two random variables have the least squares regression lines $3x+2y-26=0$ and $6x+y-31=0$. Find the mean values and the correlation co-efficient.*

Sol. Solving regression equations

$$\bar{x}=4 \quad \text{and} \quad \bar{y}=7$$

Now slopes of regression lines are

$$-\frac{3}{2} \quad \text{and} \quad -6$$

Since $\qquad r^2<1,$

$$b_{yx}=-\frac{3}{2} \qquad \text{and} \qquad b_{xy}=-\frac{1}{6}$$

$$\therefore \qquad r^2=b_{yx}.b_{xy}=\frac{1}{4}\ (<1)$$

$$\therefore \qquad r=-0\cdot5 \qquad\qquad (\because\ b_{yx}, b_{xy} \text{ are } <0)$$

Ex. 13.30. *For a bivariate distribution, the lines of regression are $3x+12y=19$ and $3y+9x=46$. Find the mean of the distribution and the correlation coefficient.*

Sol. Solving regression equations

$$\bar{x}=5 \text{ and } \bar{y}=\frac{1}{3}$$

Now slopes of regression lines are $-\dfrac{1}{4}$ and -3

$$\therefore \qquad b_{yx}=-\frac{1}{4} \text{ and } b_{xy}=-\frac{1}{3}$$

$$\therefore \qquad r^2=b_{yx}.b_{xy}=\frac{1}{12}$$

$$\therefore \qquad r=-\frac{1}{2\sqrt{3}} \qquad\qquad (\because\ b_{yx}, b_{xy}<0)$$

Ex. 13·31. *Given that the lines of regression of y on x and x on y are respectively $y=x$ and $4x-y-3=0$ and the second moment about the origin for x is 2; calculate* (*i*) *the mean for x* (*ii*) *the mean for y* (*iii*) *variance of x* (*iv*) *variance of y* (*v*) *the correlation coefficient between x and y.*

Sol. Solving regression equations

$\therefore \quad \bar{x}=1=\bar{y}$

Now slopes of regression equations are 1 and 4

$\therefore \quad b_{yx}=1,\ b_{xy}=\dfrac{1}{4}$

$\therefore \quad r^2=\dfrac{1}{4}$ i.e., $r=0{\cdot}5$

Now for x, $\mu_2'(0)=2$

$\therefore \quad \sigma_x^2=\mu_2'(0)-\bar{x}^2=2-1=1.$

Also $\quad 1=b_{yx}=\dfrac{r\sigma_y}{\sigma_x}=\dfrac{1}{2}\sigma_y$

$\therefore \quad \sigma_y=2.$

Ex. 13·32. *Given $x=4y+5$ and $y=kx+4$ are the regression lines of x on y and y on x respectively. Show that $0<4k<1$. If $k=\dfrac{1}{16}$, find the means of the two variables and the co-efficient of correlation between them.*

Sol. Here $b_{yx}=k$ and $b_{xy}=4$

$\therefore \quad r^2=b_{yx}.b_{xy}=4k.$

Now since r^2 is the square of the real quantity, it should be nonnegative.

Since $\quad bxy\neq 0,\ r\neq 0$

Also as two lines of regression are different, $r^2\neq 1$.

$\therefore \quad 0<r^2<1$

$\Rightarrow \quad 0<4k<1$

When $k=\dfrac{1}{16}$, the equations of lines of regression become

$$x=4y+5 \qquad (x \text{ on } y)$$
$$16y=x+64 \qquad (y \text{ on } x)$$

Solving regression equations

$\therefore \quad \bar{x}=28$ and $\bar{y}=5{\cdot}75$

Also $\quad r^2=4k=4.\dfrac{1}{16}=\dfrac{1}{4}$

$\therefore \quad r=\dfrac{1}{2}=0{\cdot}5. \qquad (b_{yx}, b_{xy}>0)$

Ex. 13.33. *For two variables x and y the two regression lines are*

$$x+2y-5=0$$

and $$2x+3y-8=0$$

Also $$var(x)=12.$$

Find $$\bar{x}, \bar{y}, \sigma_y \text{ and } r.$$

Sol. Solving regression equations

$$\bar{x}=1 \text{ and } \bar{y}=2.$$

Slopes of regression lines are $-\frac{1}{2}$ and $-\frac{2}{3}$

$$\therefore \quad b_{yx}=-\frac{1}{2} \text{ and } b_{xy}=-\frac{3}{2}$$

$$\therefore \quad r^2=\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)=\frac{3}{4} \quad (<1)$$

$$\therefore \quad r=-\frac{\sqrt{3}}{2} \qquad (\because b_{yx}, b_{xy} < 0)$$

Also $\quad b_{yx}=r\,\frac{\sigma_y}{\sigma_x}$ and $\sigma_x=2\sqrt{3}$

$$\therefore \quad \left(-\frac{1}{2}\right)=\left(-\frac{\sqrt{3}}{2}\right)\frac{\sigma_y}{2\sqrt{3}}$$

$$\therefore \quad \sigma_y=2.$$

Ex. 13-34. *For 10 observations on price (x) and supply (y) the following data were obtained (in appropriate units) :*

$\Sigma x=130$, $\Sigma y=220$, $\Sigma x^2=2288$, $\Sigma y^2=5506$, $\Sigma xy=3467$.

Obtain the line of regression of y on x and estimate the supply when the price is 16 units.

Sol. Let the equation of the line of regression of y on x be

$$y=a+bx.$$

where the co-efficients 'a' and 'b' are given by the normal equations

$$\Sigma y=na+b\Sigma x$$

and $$\Sigma xy=a\Sigma x+b\Sigma x^2.$$

Substituting the values

$$220=10a+130b \text{ or } 22=a+13b$$

and $$3467=130a+2288b$$

$$\therefore \quad a=8{\cdot}8 \text{ and } b=1{\cdot}015$$

$\therefore$ The equation of line of regression of y on x is

$$y=8{\cdot}8+1{\cdot}015x$$

$\therefore$ Estimate of supply (y) when the price (x) is 16 units

$$=8{\cdot}8+16{\cdot}240=25{\cdot}04.$$

Ex. 13-35. *From the data given below estimate the most likely height of a father whose son's height is 70".*

Fathers : Mean height is 67" with a s.d. of 3·5"

Sons : Mean height is 65" with a s.d. of 2·5".

Co-efficient of correlation between the heights of fathers and sons is +0·8.

Sol. Let y be the variable corresponding to the height of fathers and x be the variable for the son's heights.

Then $\bar{x}=65$, $\bar{y}=67$, $\sigma_x=2{\cdot}5$, $\sigma_y=3{\cdot}5$ and $r_{xy}=0{\cdot}8$.

$$\therefore \qquad b_{yx}=\frac{(0{\cdot}8)(3{\cdot}5)}{(2{\cdot}5)}=1{\cdot}12$$

$\therefore$ The equation of line of regression of y on x is

$$y-\bar{y}=b_{yx}(x-\bar{x})$$

or $$y-67=1{\cdot}12(x-65)$$

or $$y=1{\cdot}12x-5{\cdot}8$$

$\therefore$ Most likely height of a father whose sons height is 70"

$$=\text{Estimate of } y \text{ for } x=70$$

$$=78{\cdot}4-5{\cdot}8=72{\cdot}6.$$

Ex. 13-36. *The following statistical co-efficients were deduced in the course of an examination of the relationship between yield of wheat and the amount of rainfall :*

	Yield in lbs. (per acre)	*Annual Rainfall (in inches)*
Mean	*985·0*	*12·8*
s.d.	*70·1*	*1·6*

r(between yield and rainfall)=+0·52.

From the above data, calculate (i) the most likely yield of wheat per acre when the annual rainfall is 9·2" and (ii) the probable annual rainfall for yield of 1,400 lbs. per acre.

Sol. Let y be the variable for yield and x be the variable for annual rainfall.

Then $\bar{x}=12{\cdot}8$, $\bar{y}=985{\cdot}0$, $\sigma_x=1{\cdot}6$, $\sigma_y=70{\cdot}1$ and $r_{xy}=0{\cdot}52$

$$\therefore \qquad b_{yx}=\frac{(0{\cdot}52)(70{\cdot}1)}{(1{\cdot}6)}=22{\cdot}7825$$

and $$b_{xy}=\frac{(0\cdot52)(1\cdot6)}{70\cdot1}=0\cdot01187$$

$\therefore$ The equations of lines of regression are

$$y-985=22\cdot7825(x-12\cdot8) \qquad (y \text{ on } x)$$

and $$x-12\cdot8=(0\cdot01187)(y-985)$$

$\therefore$ The most likely yield of wheat per acre when the annual rainfall is 9·2".

$$=985+(22\cdot7825)(-3\cdot6)=902\ 983$$
$$\simeq 903$$

and the probable annual rainfall for yield of 1,400 lbs. per acre

$$=12\cdot8+(0\cdot01187)(415)$$
$$=17\cdot72605\simeq 17\cdot7''.$$

Ex. 13-37. *The following data give the correlation coefficient, means and s.d. of rainfall and yield of paddy in a certain tract :*

	Yield per acre (in lbs.)	*Annual Rainfall (in inches)*
Mean	*973·5*	*18·3*
s.d.	*38·4*	*2·0*

Co-efficient of correlation=0·58

Estimate the most likely yield of paddy when the annual rainfall is 22" , other factors being assumed to remain the same.

Sol. Let y be the variable for yield and x be the variable for annual rainfall. Then

$$\bar{x}=18\cdot3,\ \bar{y}=973\cdot5,\ \sigma_x=2\cdot0\ \sigma_y=38\cdot4 \text{ and } r_{xy}=0\cdot58.$$

$$\therefore \quad b_{yx}=\frac{(0\cdot58)(38\cdot4)}{(2\cdot0)}=11\cdot136$$

$\therefore$ The equation of line of regression of y on x is

$$y-973\cdot5=11\cdot136(x-18\cdot3)$$

$\therefore$ Estimate of the most likely yield of paddy when the annual rainfall is 22"

$$=\text{Estimate of } y \text{ for } x=22$$
$$=973\cdot5+(11\cdot136)(3\cdot7)$$
$$=973\cdot5+41\cdot2032=1014\cdot7032$$
$$\simeq 1014\cdot7.$$

Ex. 13-38. *If a number x is chosen at random from among the integers 1, 2, 3, 4 and number y is chosen from among these at least as large as x, prove that*

$$cov(x, y)=5/8$$

Also find the line of regression of x on y.

Sol. Since x is to be selected at random from the integers

$$1, 2, 3, 4$$

prob. of x taking each of these values is $\frac{1}{4}$.

Now, when x takes value 1, y is to be chosen out of 1, 2, 3, 4.

$\therefore$ Conditional prob. of y taking each of these values is $\frac{1}{4}$.

$\therefore$ Prob. of each of the pairs

$$(1, 1), (1, 2), (1, 3), (1,4) \text{ is}$$

$$\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$$

Now, when x takes value 2, y is to be chosen out of 2, 3, 4.

$\therefore$ Conditional prob. of y taking value $1=0$.

and conditional prob. of y taking each of the values 2, 3, $4=\frac{1}{3}$

$\therefore$ Prob. of the pairs

$$(2, 1), (2, 2), (2, 3), (2, 4) \text{ are } 0,$$

$$\frac{1}{4}\cdot\frac{1}{3}=\frac{1}{12}, \frac{1}{12}, \frac{1}{12}$$

When x takes values 3, y is to be chosen out of 3, 4.

$\therefore$ Conditional prob. of y taking each of the values 1, 2 is zero.

and conditional prob. of y taking each of the values 3, 4 is $\frac{1}{2}$.

$\therefore$ Prob. of the pairs

$$(3, 1), (3, 2), (3, 3), (3, 4)$$

are $$0, 0, \frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}, \frac{1}{8}$$

When x takes value 4, y can take only one value 4.

$\therefore$ Conditional prob. of y taking each of the values 1, 2, 3 is zero.

and the conditional prob. of y taking value 4 is 1.

$\therefore$ Prob. of the pairs

$$(4, 1), (4, 2), (4, 3), (4, 4)$$

are $$0, \quad 0, \quad 0, \quad \frac{1}{4}.1=\frac{1}{4}$$

Thus, the bivariate distribution is

$x \downarrow$ \ $y \rightarrow$	1	2	3	4
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
2	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
3	0	0	$\frac{1}{8}$	$\frac{1}{8}$
4	0	0	0	$\frac{1}{4}$

The calculating table is on page 13.39.

(*i*) $\operatorname{cov}(x, y)=\Sigma pxy-(\Sigma px)(\Sigma py)$

$$=\frac{420}{48}-\frac{10}{4}\cdot\frac{156}{48}$$

$$=\frac{5}{8}$$

(*ii*) Let. eq. of line of regression of x on y is

$$x=a+by$$

Normal equations are

$$\Sigma px=a+b\Sigma py$$

and $$\Sigma pxy=a\Sigma py+b\Sigma py^2$$

Substituting values, equations reduce to

$$\frac{10}{4}=a+b\left(\frac{156}{48}\right)$$

and $$\frac{420}{48}=\frac{156}{48}\,a+b.\,\frac{548}{48}$$

i.e., $$120=48\,a+156\,b$$

and $$420=156\,a+548\,b$$

$\therefore$ $$a=0{\cdot}13,\qquad b=0{\cdot}73$$

$\therefore$ Eq. of line of regression of x on y is

$$x=0{\cdot}13+0{\cdot}73\,y.$$

Cal. of Covariance

$x\downarrow$ $y\rightarrow$	1	2	3	4	p	px	px^2	pxy
1	1/16 $^{1/16}$	1/16 $^{2/16}$	1/16 $^{3/16}$	1/16 $^{4/16}$	1/4	1/4	1/4	10/16
2	0	1/12 $^{4/12}$	1/12 $^{6/12}$	1/12 $^{8/12}$	1/4	2/4	4/4	18/12
3	0	0	1/8 $^{9/8}$	1/8 $^{12/8}$	1/4	3/4	9/4	21/8
4	0	0	0	1/4 $^{16/4}$	1/4	4/4	16/4	16/4
p	1/16	7/48	13/48	25/48	1	10/4	30/4	420/48
py	1/16	14/48	39/48	100/48	156/48			
py^2	1/16	28/48	117/48	400/48	548/48			
pxy	1/16	22/48	87/48	308/48	420/48			

13.5.4. Standard Errors of Estimate

Find the standard errors of estimate of y and x respectively.

Sol. The eq. of line of regression of y on x is

$$y-\bar{y}=\frac{\mu_{11}}{\sigma_x^2}(x-\bar{x})$$

Let (x_i, y_i), $i=1, 2 \dots n$ be the variate value pair occurring with frequency f_i.

Let
$$Y_i=\bar{y}+\frac{\mu_{11}}{\sigma_x^2}(x_i-\bar{x})$$

The standard error of estimate of y is given by

$$S_y^2=\frac{1}{N}\sum_{i=1}^{n} f_i\{Y_i-y_i\}^2$$

where $\quad N=\sum_{i=1}^{n} f_i$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i\left\{\frac{\mu_{11}}{\sigma_x^2}(x_i-\bar{x})-(y_i-\bar{y})\right\}^2$$

$$=\frac{1}{N}\sum_{i=1}^{n} f_i\left\{\left(\frac{\mu_{11}}{\sigma_x^2}\right)^2(x_i-\bar{x})^2+(y_i-\bar{y})^2 -2\frac{\mu_{11}}{\sigma_x^2}(x_i-\bar{x})(y_i-\bar{y})\right\}$$

$$=\frac{\mu_{11}^2}{\sigma_x^2}+\sigma_y^2-2\frac{\mu_{11}^2}{\sigma_x^2}$$

$$=\sigma_y^2\left\{1-\left(\frac{\mu_{11}}{\sigma_x\sigma_y}\right)^2\right\}=\sigma_y^2(1-r^2)$$

$\therefore \quad S_y=\sigma_y(1-r^2)^{1/2}$

Similarly standard error of estimate of x is given by

$S_x=\sigma_x(1-r^2)^{1/2}$.

Note. If $r=\pm 1$, $S_x=S_y=0$

$\therefore$ All points lie on both lines of regression and hence two regression lines coincide and thus there is a linear functional relation between the variates x and y.

As $r^2 \to 1$, $S_x^2 \to 0$ and $S_y^2 \to 0$ *i.e.*, as r^2 comes nearer to unity, the points are closer to lines of regression which are nearer to coincidence.

$\therefore$ The departure of r^2 from unity can be taken as a measure of departure of the relationship between the two variates from linearity.

Ex. 13·39. *For a given bivariate dist. find the straight line for which the sum of the squares of the normal deviations is minimum.*

Sol. Coincider the bivariate dist.

$$\begin{array}{c} x \to \\ y \to \\ f \to \end{array} \begin{pmatrix} x_1 & x_2 \ldots\ldots x_n \\ y_1 & y_2 \ldots\ldots y_n \\ f_1 & f_2 \ldots\ldots f_n \end{pmatrix}$$

and let the equation of the straight line be

$$x \cos \alpha + y \sin \alpha - p = 0 \qquad \ldots(1)$$

The normal deviation of an observed value pair (x_i, y_i) from the line is the length of perpendicular from the point (x_i, y_i) upon the line *i.e.*, $x_i \cos \alpha + y_i \sin \alpha - p = 0$.

Let $S = \sum\limits_{i=1}^{n} f_i(x_i \cos \alpha + y_i \sin \alpha - p)^2$

Normal eqs. are

$$0 = \frac{\partial S}{\partial \alpha} = \sum_{i=1}^{n} 2f_i(x_i \cos \alpha + y_i \sin \alpha - p)(-x_i \sin \alpha + y_i \cos \alpha) \qquad \ldots(2)$$

and
$$0 = \frac{\partial S}{\partial p} = \sum_{i=1}^{n} -2f_i(x_i \cos \alpha + y_i \sin \alpha - p) \qquad \ldots(3)$$

Eqs. (3) and (2) are equivalent to eqs.

$$\bar{x} \cos \alpha + \bar{y} \sin \alpha = p \qquad \ldots(4)$$

and
$$\cos \alpha \sin \alpha \{ \sum_{i=1}^{n} f_i y_i^2 - \sum_{i=1}^{n} f_i x_i^2 \} + \cos 2\alpha \sum_{i=1}^{n} f_i x_i y_i$$
$$+ p\{\sin \alpha \sum_{i=1}^{n} f_i x_i - \cos \alpha \sum_{i=1}^{n} f_i y_i\} = 0$$

i.e.,
$$\cos \alpha \sin \alpha \{(\sigma_y^2 + \bar{y}^2) - (\sigma_x^2 + \bar{x}^2)\} + \cos 2\alpha(\mu_{11} + \bar{y}\bar{x})$$
$$+ p\bar{x} \sin \alpha - p\bar{y} \cos \alpha = 0$$

i.e., $\{\cos\alpha\sin\alpha(\sigma_y^2-\sigma_x^2)+\cos 2\alpha\ \mu_{,}\}+\{\cos\alpha\sin\alpha(\bar{y}^2-\bar{x}^2)$
$$+\bar{x}\bar{y}(\cos^2\alpha-\sin^2\alpha)+p\bar{x}\sin\alpha-p\bar{y}\cos\alpha\}=0$$

i.e., $\left\{\frac{\sin 2\alpha}{2}(\sigma_y^2-\sigma_x^2)+\cos 2\alpha.\mu_{,}\right\}+\left\{\bar{y}\cos\alpha(\bar{y}\sin\alpha+\bar{x}\cos\alpha-p)\right.$
$$\left.-\bar{x}\sin\alpha(\bar{x}\cos\alpha+\bar{y}\sin\alpha-p)\right\}=0$$

i.e., $$\frac{\sin 2\alpha}{2}(\sigma_y^2-\sigma_x^2)+\cos 2\alpha.\mu_{,}=0 \qquad \text{[using (4)]}$$

$$\therefore \qquad \tan 2\alpha=\frac{2\mu_{,}}{\sigma_x^2-\sigma_y^2} \qquad ...(5)$$

Eq. (5) gives two values of α. If one is θ, the other is $\frac{\pi}{2}+\theta$. The corresponding values of p are given by (4). With these values of α and p, (1) gives the equation of the required line. Evidently there are two such lines which are perpendicular.

13.6. Correlation Ratio

Def. Consider the case when corresponding to any given value of x (say x_i) there are more than one values of y (say y_{ij}). Let the pair (x_i, y_{ij}) occur with frequency f_{ij}.

Let $$\bar{y}_i=(\sum_j f_{ij}y_{ij})/\sum_j f_{ij}$$

Then correlation ratio of y on x (η_{yx}) is defined by

$$\sum_j\sum_j f_{ij}(y_{ij}-\bar{y}_i)^2=N\sigma_y^2(1-\eta_{yx}^2)$$

where $\sum_i\sum_j f_{ij}=N$

Theorem. Show that :

$$r^2\leqslant\eta^2_{yx}\leqslant 1$$

Proof. Evidently $\eta_{yx}^2\leqslant 1$.

To prove $r^2\leqslant\eta_{yx}^2$ first the equation of line of regression of y on x will be obtained. Let it be $y=a+bx$.

The unknowns a and b are given by

$$\sum_i\sum_j f_{ij}y_{ij}=Na+b\sum_i\sum_j f_{ij}x_i$$

and $$\sum_i\sum_j f_{ij}x_iy_{ij}=a\sum_i\sum_j f_{ij}x_i+b\sum_i\sum_j f_{ij}x_i^2$$

i.e., $$\bar{y}=a+b\bar{x}$$

and $$\mu_{,}+\bar{x}\bar{y}=a\bar{x}+b(\sigma_x^2+\bar{x}^2)$$

$$\therefore \qquad b=\frac{\mu_{xy}}{\sigma_x^2} \text{ and } a=\bar{y}-\frac{\mu_{xy}}{\sigma_x^2}\,\bar{x}$$

$\therefore$ Eq. of line of regression is

$$y-\bar{y}=\frac{\mu_{xy}}{\sigma_x^2}(x-\bar{x})$$

Let $$Y_i=\bar{y}+\frac{\mu_{xy}}{\sigma_x^2}(x_i-\bar{x})$$

$$\therefore \sum_i\sum_j f_{ij}(y_{ij}-Y_i)^2=\sum_i\sum_j f_{ij}\left\{(y_{ij}-\bar{y})-\frac{\mu_{xy}}{\sigma_x^2}(x_i-\bar{x})\right\}^2$$

$$=N\left(\sigma_y^2+\frac{\mu_{xy}^2}{\sigma_x^2}-2\frac{\mu_{xy}^2}{\sigma_x^2}\right)=N\sigma_y^2(1-r^2)$$

Now $$\sum_i\sum_j f_{ij}(y_{ij}-Y_i)^2 > \sum_i\sum_j f_{ij}(y_{ij}-\bar{y}_i)^2$$

i.e., the sum of square of deviations in any array is least when they are measured from the mean of the array

$$\therefore \qquad N\sigma_y^2(1-r^2) > N\sigma_y^2(1-\eta_{yx}^2)$$

which implies $$r^2<\eta_{yx}^2$$

$$\therefore \qquad r^2<\eta_{yx}^2<1.$$

Note. Similarly as above correlation ratio of x on y (η_{xy}) can be defined and it can be shown that

$$1\geqslant\eta_{xy}^2\geqslant r^2$$

Ex. 13-40. *Show that the correlation ratio of y on x is the ratio of the standard deviation of the weighted means of the arrays of y's (weighted by the corresponding array frequencies) to the standard deviation of all y's of the dist.*

Sol. Let y_{ij} $(j=1, 2, \ldots)$ be the values of y corresponding to $x=x_i$ and f_{ij} be the frequency of the pair (x_i, y_{ij}).

Now $$N\sigma_y^2=\sum_i\sum_j f_{ij}(y_{ij}-\bar{y})^2 \qquad \text{where } \bar{y}=\text{A.M. of } y$$

$$=\sum_i\sum_j f_{ij}\{(y_{ij}-\bar{y}_i)+(\bar{y}_i-\bar{y})\}^2$$

where $$\bar{y}_i=\left(\sum_j f_{ij}y_{ij}\right) \Big/ \sum_j f_{ij}$$

$$=\sum_i\sum_i f_{ij}(y_{ij}-\bar{y}_i)^2+\sum_i\sum_j f_{ij}(\bar{y}_i-\bar{y})^2$$

$$+2\sum_i\sum_j f_{ij}(y_{ij}-\bar{y}_i)(\bar{y}_i-\bar{y})$$

$$=N\sigma_y^2(1-\eta_{yx}^2)+\sum_i n_i(\bar{y}_i-\bar{y})^2 \qquad \text{where } n_i=\sum_j f_{ij}$$

$$\left(\because \sum_j f_{ij}(y_{ij}-\bar{y}_i)=n_i\bar{y}_i-n_i\bar{y}_i=0\right)$$

$$\therefore \qquad \eta^2_{yx}=\left\{\frac{1}{N}\sum_i n_i(\bar{y}_i-\bar{y})^2\right\}\Big/\sigma_y^2=\frac{\sigma_{my}^2}{\sigma_y^2}$$

where $\sigma_{my}=\sqrt{\dfrac{1}{N}\sum_i n_i(\bar{y}_i-\bar{y})^2}$ is the *s.d.* of the weighted means of arrays of *y*'s (weighted by the corresponding array frequencies).

Note. For correlation ratio of *x* on *y*, $\eta_{xy}^2=\dfrac{\sigma_{mx}^2}{\sigma_x^2}$.

EXERCISES

1. Calculate correlation co-efficient for the following datas ;

x :	5	15	10	20	25	40
y :	21	14	28	7	35	42

[Ans. 0·49]

2.

x :	18·8	19·1	17·6	16·8	18·2	19·5	20·0	21·8	21·9
y :	7·8	7·6	7·7	7·5	7·8	7·2	8·0	7·9	7·8

[Ans. 0·37]

3.

Husband's age (x)	20	30	40	50	60	70	80
Wife's age (y)	14	5	30	32	40	45	65

[Ans. 0·94]

4.

Husband's age (x)	24	27	28	28	29	30	32	33	35	35	40
Wife's age (y)	18	20	22	25	22	28	28	30	27	30	22

[Ans. 0·5]

5.

x :	20	18	16	15	14	12	12	10	8	5
y :	12	16	10	14	12	10	9	8	7	2

[Ans. 0·87]

6.

x :	28	41	40	38	35	33	40	32	36	33
y :	23	34	33	34	30	26	28	31	36	38

[Ans. 0·44]

7. From the index numbers given below, find Karl Pearson's co-efficient of correlation :

Months : (in 1984)	*May*	*June*	*July*	*Aug.*	*Sep.*	*Oct.*	*Nov.*	*Dec.*	*Jan.*	*Feb.*
Index no. of prices in Calcutta (x)	169	182	182	192	198	211	227	238	350	253
Index no. of prices in Bombay (y)	204	222	225	228	231	233	249	266	255	255

[Ans. 0·74]

8. Obtain the co-efficient of correlation between male and female death rates in Delhi city during the period 1930—37.

Year	1930	1931	1932	1933	1934	1935	1936	1937
Male death rate	33	23	24	28	27	28	22	24
Female death rate	45	31	33	40	35	39	32	34

[Ans. 0·97]

9. Calculate the correlation co-efficient between the marks in two examinations given below :

Marks (in Exam. A) x	15	13	17	14	18	12	20	16	18	17	19	21
Marks (in Exam. B) y	18	16	18	15	19	16	18	15	21	17	18	20

[Ans. 0·703]

10. The table below shows the number of vehicles with licences and the number of motor vehicle accidents in a city. Calculate the co-efficient of correlation :

Year	1975	1976	1977	1978	1979	1980	1981	1982
No. of Vehicles with licences ('000)	2·6	2·8	2·9	3·1	3·2	2·3	2·5	1·8
No. of Motor vehicles accidents ('00)	5·9	6·0	6·2	6·2	7·6	7·0	7·4	5·5

[Ans. 0·366]

11. Calculate the co-efficient of correlation between cotton and woollen cloth manufacturers from the following data :

Months	*July*	*Aug.*	*Sep.*	*Oct.*	*Nov.*	*Dec.*
Index nos. of cotton cloth manufacturers (x)	103	105	108	106	104	102
Index nos. of woollen cloth manufacturers (y)	75	73	78	71	80	76

Jan.	*Feb.*	*March.*	*April.*	*May.*	*June.*
108	115	118	114	116	120
68	65	62	60	58	54

[Ans. 0·909]

12. Calculate the co-efficient of correlation between the production of rice and its price from the following table :

Year	1965	1967	1969	1971	1973	1975	1977	1979	1981	1983
Production	250	270	278	325	260	510	428	320	440	310
Price	84	50	62	75	90	170	136	65	72	58

[Ans. 0·74]

13. Find the correlation co-efficient of datas given below :

$y \downarrow$ \ $x \rightarrow$	30—35	35—40	40—45	45—50	50—55	55—60
80—90	2	3	2	—	—	—
90—100	—	2	5	4	2	—
100—110	—	4	8	5	1	—
110—120	—	—	2	3	1	1
120—130	1	—	—	2	1	1

[Ans. 0·43]

14.

$y \downarrow$ \ $x \rightarrow$	18	19	20	21	22	Total
0—5	—	—	—	3	1	4
5—10	—	—	—	3	2	5
10—15	—	—	7	10	—	17
15—20	—	5	4	—	—	9
20—25	3	2	—	—	—	5
Total	3	7	11	16	3	40

[Ans. 0·837]

15. *Ages of daughters (in years)*

Ages of mothers (in years)	5—10	10—15	15—20	20—25	25—30	Total
15—25	6	3	—	—	—	9
25—35	3	16	10	—	—	29
35—45	—	10	15	7	—	32
45—55	—	—	7	10	4	21
55—65	—	—	—	4	5	9
Total	9	29	32	21	9	100

[Ans. 0·802]

16.

Marks in Sanskrit \ Marks in History	0—20	20—40	40—60	60—80	Total
0—20	32	88	15	—	135
20—40	45	436	200	4	685
40—60	16	500	398	25	939
60—80	—	105	532	40	677
80—100	—	8	40	16	64
Total	93	1137	1185	85	2500

[**Ans.** 0·048]

17. The following table gives the ages of husbands and wives at the time of their marriages. Calculate the correlation co-efficient between the ages of husbands and wives :

Ages of husbands \ Ages of wives	10—20	20—30	30—40	40—50
10—20	20	26	—	—
20—30	8	14	37	—
30—40	—	4	18	6
40—50	—	—	4	3

[**Ans.** 0·69]

18. Construct examples of at least 5 pairs of observations with co-efficients of correlation equal to -1, 0 and $+1$.

19. Two independent variates x and y have means 5 and 10 and variances 4 and 9 respectively. Show that the variates $u=3x+4y$, $v=3x-y$ are uncorrelated.

20. The variables x and y are connected by the equation

$$ax+by+c=0$$

Show that the correlation co-efficient between them is -1 if the signs of 'a' and 'b' are alike and $+1$ if they are different.

21. The independent random variables are defined by

$$\left.\begin{aligned} f(x)&=4ax \quad 0 \leq x \leq r \\ &=0 \quad \text{otherwise} \end{aligned}\right\} \qquad \left.\begin{aligned} f(y)&=4by \quad 0 \leq y < s \\ &=0 \quad \text{otherwise} \end{aligned}\right\}.$$

Find the correlation co-efficient between $x+y$ and $x-y$.

$$\left[\textbf{Ans. } \frac{b-a}{b+a}\right]$$

22. (*a*) Show that

$$\text{var}\,(x\pm y)=\text{var}(x)+\text{var}\,(y)$$

provided x and y are uncorrelated.

(*b*) Show that

$$r_{xy} > \text{ or } <0 \text{ according as } \sigma_{x+y} > \text{ or } < \sigma_{x-y}.$$

23. $\bar{x}$ be the A.M. of n independent variates $x_1, x_2 \ldots\ldots x_n$ each of s.d.σ, show that var $(\bar{x}) = \dfrac{\sigma^2}{n}$.

24. If $u = ax + by$, $v = ax - by$, where x, y represent deviations from the means of two measurements by the same individuals. The co-efficient of correlation between x and y is r. If u and v are uncorrelated, show that

$$\sigma_u \sigma_v = 2ab\sigma_x\sigma_y\sqrt{1-r^2}.$$

25. If x_1, x_2, x_3 are three variables with s.ds $\sigma_1, \sigma_2, \sigma_3$ respectively. If any two of the variables are uncorrelated, obtain the coefficient of correlation between $x_1 + x_2$ and $x_2 + x_3$.

26. $x_1, x_2 \ldots x_n$ are random variates each with mean μ and s.d. σ. The correlation coefficient between any two of them is P. Show that

$$\text{var}(\bar{x}) = \frac{\sigma^2}{n} + \left(1 - \frac{1}{n}\right)\rho\sigma^2$$

where $\bar{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$ Deduce that

$$P > -\frac{1}{n-1}.$$

27. x and y are random variates with zero means and unit variances. If

$$r(ax+by, bx+ay) = \frac{1+2ab}{a^2+b^2}.$$

find $r(x, y)$.

28. The coefficient of rank correlation is 0·8. If the sum of the squares of the difference in ranks is 33, find the number of individuals. (**Ans.** 10)

29. The table below shows the respective heights of 12 fathers and their oldest sons

65	63	67	64	68	62	70	66	68	67	69	71
68	66	68	65	69	66	68	65	71	67	68	70

find coefficient of rank correlation.

30. For the following data find the lines of regression :

x :	5	15	10	20	25	30
y :	21	14	28	7	35	43

[**Ans.** $y = 12{\cdot}6 + 0{\cdot}68x$, $x = 9 + 0{\cdot}347y$]

31. Obtain the lines of regression for the following data :

x :	1	2	3	4	5	6	7	8	9
y :	9	8	10	12	11	13	14	16	15

Deduce the value of correlation co-efficient and also obtain an estimate of y which should correspond on the average to $x=6{\cdot}2$.

[**Ans.** $y=0{\cdot}95x+7{\cdot}25$; $x=0{\cdot}95y-6{\cdot}4$; 0·95 ; 13·14]

32. Determine Karl Pearson's co-efficient of correlation between exports and imports given in the following table :

Exports :	45	46	48	50	52	53	51	49	47
Imports :	94	96	98	100	104	105	102	99	97

Obtain also regression equations and standard errors of estimate of x and y.

[**Ans.** 0·99 ; $y=1{\cdot}333x+34{\cdot}111$; $x=0{\cdot}739y-24{\cdot}511$, 0·31, 0·42]

33. Mean soil temperature and germination interval (time between sowing and appearance above ground) for winter wheat 1981—86 for 12 places are recorded below :

Mean soil temp. :	57	42	38	42	45	42	44	40	46	44	43	40
No. of days :	10	26	41	29	27	27	19	18	19	31	29	33

Obtain the regression equation of germination interval on mean soil temperature. [**Ans.** $y=80{\cdot}752-1{\cdot}262x$]

34. Calculate the coefficient of correlation between the marks secured by 12 students in two tests :

Student :	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
Test I :	50	54	56	59	60	62	61	65	67	71	71	74
Test II :	22	25	34	28	26	30	33	30	28	34	36	40

Also obtain the equations of lines of regression.

[**Ans.** 0·774 ; $y=0{\cdot}538x-3{\cdot}125$, $x=1{\cdot}115y+28{\cdot}493$]

35. The following table gives the number of candidates obtaining marks in two subjects *A* and *B* in an examination :

	Marks in B				
Marks in A	30—39	40—49	50—59	60—69	Total
30—39	3	1	1	—	5
40—49	2	6	1	2	11
50—59	1	2	2	1	6
60—69	—	1	1	1	3
Total	6	10	5	4	25

Calculate the co-efficient of correlation. Obtain also the lines of regression.

[**Ans.** 0·39 ; $y=0{\cdot}43x+26{\cdot}961$; $x=0{\cdot}361y+30{\cdot}225$]

36. The following regression equations are obtained from a correlation table :

$$x = 0{\cdot}8456y + 5{\cdot}45$$
$$y = 0{\cdot}7326x + 35{\cdot}86$$

Find the values of (*i*) the correlation co-efficient (*ii*) the mean of x and y.

37. For two variables x and y with the same mean, the two regression equations are $y = ax + b$ and $x = \alpha y + \beta$. Show that $\frac{b}{\beta} = \frac{1-a}{1-\alpha}$. Find also the common mean.

38. Given the regression lines $2y - x = 50$, $3y - 2x = 10$. Show that the estimate of y for $x = 150$ is 100 and the estimate of x for $y = 100$ is 145. Explain the difference.

39. Criticize the following :

$$b_{yx} = 3{\cdot}2 \quad \text{and} \quad b_{xy} = 0{\cdot}8$$

40. Show that $\sin\theta \leqslant (1 - r^2)$ where θ is acute angle between lines of regression.

Hint. We have $\tan\theta = \frac{1-r^2}{|r|} \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}$

Now $\quad \frac{\sigma_x^2 + \sigma_y^2}{2\sigma_x\sigma_y} - 1 = \frac{(\sigma_x - \sigma_y)^2}{2\sigma_x\sigma_y} > 0$

$\therefore \quad \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2} \leqslant \frac{1}{2}$

$\therefore \quad \tan\theta \leqslant \frac{1-r^2}{2|r|}$

$\therefore \quad \cot\theta > \frac{2|r|}{1-r^2}$

$\therefore \quad 1 + \cot^2\theta > 1 + \frac{4r^2}{(1-r^2)^2} = \frac{(1+r^2)^2}{(1-r^2)^2}$

$\therefore \quad \operatorname{cosec}^2\theta \geqslant \frac{(1+r^2)^2}{(1-r^2)^2}$

$\therefore \quad \sin\theta \leqslant \frac{1-r^2}{1+r^2} \leqslant (1-r^2) \qquad (\because \; 1 + r^2 > 1)$

41. If the lines of regression of y on x and x on y are respectively

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0$$

show that $$a_1b_2 \leqslant a_2b_1.$$

14

Sampling Theory and Large Sample Tests

14.1. Introduction

Very often in practice one is interested in drawing valid conclusions about a large group of individuals or objects. Instead of examining the entire group (which may be difficult or impossible) one may think of examining a small part of it This is done with the aim of inferring certain facts about the large group from the results found for smaller part. This process is called **statistical inference**. Various technical words used during this process are explained as below :

Population. *Any collection of individuals or of attributes or the results of operations which can be specified numerically.*

Finite Population. *Population containing finite number of members. Otherwise the population is called infinite population e.g., the population of boys in a college and the population of pressures at various points in the atmosphere are finite and infinite respectively.*

Existent Population. *Population of concrete objects e.g., the population of a city.*

Hypothetical Population. *Population of non-concrete objects e.g., the population of heads and tails obtained by tossing a coin an infinite number of times.*

Sample. *A part or small section selected from the population is called a* **sample** *and the process of such selection is called* **sampling.**

Random Sampling. *When a sample is taken in such a way that each member of the population has the same chance of being selected, the sample obtained is called* **random sample** *and the technique is called* **random sampling.**

Simple Sampling. *When a random sample is drawn from a population in such a way that the chance of selection of a member at any stage is independent of previous selections, the sample obtained is called* **simple sample** *and the technique is called* **simple sampling.**

Stratified Sampling. *In this process the entire heterogeneous population is divided into a number of homogeneous groups (termed as strata) which differ from one another but each of these is homogeneous within itself. The samples are drawn from each stratum (the sample size in each stratum varying according to the relative importance of the stratum in the population). The aggregate of the samples from each of the stratum is called* **stratified sample** *and the technique is called stratified sampling e.g. To estimate the average income of the inhabitants of a city, it is necessary that all sections of the society must be included in the sample otherwise there is a likelihood that more rich people or poor people may be dominating the sample. For this purpose it is better to divide the city into different strata say, according to the localities; slums, middle-class localities and bungalow areas, business localities etc., and then to draw samples from each of these localities. This would ensure that all sections of the society are represented in the sample.*

Sampling with or without Replacement

Sampling where each member of a population may be chosen more than once is called **sampling with replacement** and if each member cannot be chosen more than once, it is called **sampling without replacement.**

Remark. (*i*) From a finite population, a sample with replacement of any size can be drawn without exhausting the population.

(*ii*) For most practical purposes sampling from a finite population (which is very large) can be considered as sampling from an infinite population.

Parameters

A population is considered to be known if the probability function (or density function) $f(x)$ of the associated variable x is known e g., if x is normally distributed, the population is said to be normal.

Certain quantities may appear in $f(x)$ (*e.g* , m and σ in case of normal distribution). Other quantities such as mean, variance etc., can then be obtained in terms of these. Such quantities are called **population parameters** or simply **parameters.**

Remark. When the population is given, the population parameters are taken to be known.

Statistic. It is a statistical measure computed from sample observations alone.

Remark. Statistic is calculated with the purpose of estimating a population parameter. To each population parameter there is a statistic to be computed from the sample. This statistic may not always give the best estimate. One of the important problems of sampling theory is to decide how to form a proper sample statistic so as to get a best estimate of a given population parameter.

Sampling Distributions

The statistic is itself a random variate. Its probability distribution is often called **sampling distribution.** It can be thought of as below :

All possible samples of given size are taken from the population and for each sample the statistic is calculated. The values of the statistic form its sampling distribuiion.

Standard Errors. *The standard deviatian of a sampling distribution of a statistic is known as standard error and is written as 'S.E.'*

Precision. *The reciprocal of S.E. is called precision.*

Probable Error (P.E.). *It is defined by*

$$P.E.=(0{\cdot}67449)\ S.E.$$

Standard Errors of Various Parameters.

(*i*) *Quartiles* $\qquad 1{\cdot}36263\dfrac{\sigma}{\sqrt{n}}$.

(*ii*) *Median* $\qquad 1{\cdot}25331\dfrac{\sigma}{\sqrt{n}}$.

(*iii*) *S.D.* $\qquad \dfrac{\sigma}{\sqrt{2n}}$.

(*iv*) *Variance* $\qquad \sigma^2\sqrt{\dfrac{2}{n}}$.

(*v*) *Co-efficient of correlation* $\qquad \dfrac{(1-r^2)}{\sqrt{n}}$.

(*vi*) μ_3 $\qquad \sigma^3\sqrt{\dfrac{6}{n}}$.

Unbiased Estimate. *A statistic 't' is said to be an unbiased estimate of a parameter θ if* $E(t)=\theta$.

Asymptotically Unbiased Estimate. *A statistic 't_n' is said to be an asymptotically unbiased estimate of a parameter θ if.*

$$\underset{n\to\infty}{Lt}\ F(t_n)=\theta$$

where n is the size of the sample.

Large and Small Samples. *Samples of size greater than 30 are called large samples and of size less than or equal to 30 are called small samples.*

Hypothesis. *Very often it is required to make decisions about populations on the basis of sample information. Such decisions are called* **Statistical decisions.** *In attempting to reach decisions it is often necessary to make assumptions about the population involved. Such assumptions, whice are not necessarily true, are called statistical hypothesis.*

Null Hypothesis. *The hypothesis tested for possible rejection under the assumption that it is true is usually called null hypothesis.*

Tests of Significance. *Procedures which enable us to decide, on the basis of sample information, whether to accept or reject hypothesis or to determine whether observed sampling results differ significantly from expected results are called tests of significance, rules of decision or tests of hypothesis.*

Level of Significance. *The probability level below which we reject the hypothesis is called the level of significance.*

Confidence Interval

It is the interval in which a population parameter is expected to lie with certain probability (mentioned in percentage).

The end enumbers are called **confidence limits** or **fiducial limits.** The probability is called **confidence level.**

14.2. Sampling of Attributes

In the case of sampling of attributes we are concerned only with the presence or absence of some given attribute. The selection of an individual in sampling may be called a trial and the presence of a specified attribute a success and its absence a failure.

By simple sampling of attributes we mean random sampling in which each event has the same chance of success and in which the chances of success of different events are independent whether the previous trials have been made or not.

Mean and s.d.

Suppose we are to draw a simple sample of n individuals from a population. Let p be the chance of success and q the chance of failure for each trial.

Then $p+q=1$

The drawing of a sample is identical with the problem of a series of n independent trials with constant probability p of success.

$\therefore$ The probabilities of 0, 1, 2,...n successes are the successive terms in the binomial expansion of $(q+p)^n$ (from B.D.).

$\therefore$ The probability of x success is given by

$$P(x)={}^nc_x p^x q^{n-x} \quad x=0, 1, 2, \ldots n.$$

The binomial probability distribution so obtained is called sampling distribution of the number of successes in the sample.

$\therefore$ The expected value or mean value of the number of successes

i.e., $\qquad \therefore \quad E(x)=np$

and the *s.d.* of the number of successes

$$=\sqrt{npq.}$$

Now proportion of successes$=\dfrac{x}{n}$

$$\therefore \qquad E\left(\frac{x}{n}\right)=\frac{1}{n}\, E(x)=p$$

and $$\text{var}\left(\frac{x}{n}\right)=\frac{1}{n^2}\ \text{var}\,(x)=\frac{pq}{n}$$

$\therefore$ Standard deviation of $\dfrac{x}{n}=\sqrt{\dfrac{pq}{n}}.$

14.2.1. To test the significance of single proportion for large samples.

Let us suppose that *w.r.t.* attribute A it is possible to classify individuals of a population into two mutually exclusive and collectively exhaustive sets. Let x be the number of individuals possessing A in a single sample of size n. Then the proportion of individuals possessing A is given by

$$p=\frac{x}{n}.$$

Let P be the probability for an individual to possess A *i.e.*, P is the probability of success.

Then x is a binomial variate with expected value nP and *s.d.* $\sqrt{nPQ}.$ (where $Q=1-P$)

$$\therefore \qquad u=\frac{x-nP}{\sqrt{nPQ}}=\frac{p-P}{\sqrt{\dfrac{PQ}{n}}}$$

is a binomial variate with mean zero and *s.d.* unity. Since sample is large u is nearly a $N(0, 1)$.

$$\therefore \quad P(\mid u \mid > 3)=1-P(-3 < u < 3)=1-2P(0 < u < 3)$$
$$=1-2(0{\cdot}49865)=1-0{\cdot}9973=0{\cdot}0027$$

Similarly $P(\mid u \mid > 1{\cdot}96)=0{\cdot}05$

$$P(\mid u \mid > 2{\cdot}58)=0{\cdot}01.$$

The probability P is obtained by setting null hypothesis. On the basis of probabilities obtained above the rules for taking decisions are :

(*i*) If $\mid u \mid > 3$, the difference between the observed and expected number of successes is highly significant and hence the hypothesis is certainly wrong and is to be rejected.

(*ii*) If $2{\cdot}58 < \mid u \mid < 3$, the difference is significant at 1% level of significance.

(*iii*) If $1{\cdot}96 < \mid u \mid < 2{\cdot}58$, the difference is significant at 5% level of significance.

(*iv*) If $\mid u \mid < 1{\cdot}96$, the difference is not significant and the data is said to be consistent with the hypothesis and hence the hypothesis may be accepted.

Note. (*i*) The above test is valid only for large samples since for small samples binomial distribution may not be nearly normal.

(*ii*) The test may furnish evidence against the hypothesis but it cannot prove the hypothesis to be correct. It can at the most provide no evidence against it.

(*iii*) Since the hypothesis can be rejected but cannot be proved, always null hypothesis is set, *e.g.*, to test whether there is any difference it is assumed that there is no difference ; to test whether there is any relationship it is assumed that there is no relationship etc. The rejection of no difference will mean a difference and the rejection of no relationship a relationship.

Ex. 14·1. *A coin is tossed 400 times and it turns up head 216 times. Discuss whether the coin may be regarded as unbiased one.*

Sol. Let x be the number of heads obtained and P the prob of getting head in a toss.

Set the hypothesis : 'Coin is unbiased'.

Then $\quad P=\frac{1}{2}.$

Here $\quad n=400$, and $x=216$

$$\therefore \quad u=\frac{x-nP}{\sqrt{nPQ}}=\frac{216-400\,.\,\frac{1}{2}}{\sqrt{\frac{1}{2}\,.\,\frac{1}{2}\,.\,400}}=1{\cdot}6<1{\cdot}96$$

$\therefore$ The hypothesis may be correct and hence the coin may be regarded as unbiased.

Ex. 14·2. *In some dice-throwing experiments, Weldon threw dice 49,152 times and of these 25,145 yielded a 4, 5 or 6. Is this consistent with the hypothesis that the dice were unbiased ?*

Sol. Set the hypothesis 'Dice were unbiased'.

Then P=prob of 4, 5 or 6 in a throw$=\frac{3}{6}=\frac{1}{2}$

Here $n=49{,}152$ and $x=25{,}145$

$$\therefore \quad u=\frac{25145-49152\,.\,\frac{1}{2}}{\sqrt{49152\,.\,\frac{1}{2}\,.\,\frac{1}{2}}}\simeq\frac{569}{110{\cdot}85}$$

$$\simeq 5{\cdot}13>3$$

$\therefore$ Hypothesis is wrong and hence the dice could not be regarded as unbiased.

Ex. 14·3. *Certain crosses of the pea gave 5,321 yellow and 1,804 green seeds. The expectation is 25% green seeds on a Mendelian hypothesis. Is the divergence significant or might have occurred as due to fluctuations of simple sampling ?*

Sol. Total number of seeds $(n)=5321+1804=7125$

Here P=expected proportions of green seeds

$$=\frac{25}{100}=\frac{1}{4}$$

$\therefore$ The standard error of green seeds

$$=\sqrt{7125\,.\,\frac{1}{4}\,.\,\frac{3}{4}}\simeq 36{\cdot}6$$

$$\therefore \quad u=\frac{1804-\frac{1}{4}\,.\,7125}{36{\cdot}6}=0{\cdot}6<1{\cdot}96$$

$\therefore$ The data is consistent with the hypothesis and hence the divergence may be regarded as due to fluctuations of simple sampling.

Ex. 14·4. *A die is thrown 9,000 times and a throw of 3 or 4 is reckoned as a success. Suppose that 3,240 throws of a 3 or 4 have been made out. Do the data indicate an unbiased die ? If not, find the probable limits of prob of getting 3 or 4.*

Sol. Here $n=9{,}000$

$x=3{,}240$

Set the hypothesis : 'Die is unbiased'.

Then $\quad P=\frac{2}{6}=\frac{1}{3}$

$$\therefore \quad u=\frac{3240-9000 \cdot \frac{1}{3}}{\sqrt{9000 \cdot \frac{1}{3} \cdot \frac{2}{3}}} \simeq 5{\cdot}4>3.$$

$\therefore$ Difference is highly significant and hence the hypothesis is wrong.

$\therefore$ The die cannot be regarded as unbiased.

$$\therefore \quad P \neq \frac{1}{3}$$

In order to find the limits of P, we estimate the standard error of the proportion of successes from the sample.

Now proportion of successes

$$=\frac{3240}{9000}=0{\cdot}36$$

$\therefore$ Estimate of the standard error of the proportion of successes

$$=\sqrt{\frac{(0{\cdot}36)(1-0{\cdot}36)}{9000}}=\sqrt{\frac{(0{\cdot}36)(0{\cdot}64)}{9000}}$$

$$\simeq 0{\cdot}005$$

$\therefore$ Probable limits of P are given by

$$\left|\frac{\frac{x}{n}-P}{0{\cdot}005}\right|<3 \text{ i.e., } \frac{x}{n}-3(0{\cdot}005)<P<\frac{x}{n}+3(0{\cdot}005)$$

$\therefore$ Probable limits of P are

$0{\cdot}36 \pm 3(0{\cdot}005)$ *i.e.,* $0{\cdot}345$ and $0{\cdot}375$.

Ex. 14·5. *In a locality of 18,000 families a sample of 840 families was selected. Of these 840 families, 206 families were found to have a monthly income of Rs. 50 or less. It is desired to estimate how many out of the 18000 families have a monthly income of Rs. 50 or less. Within what limits would you place your estimate ?*

Sol. Let p be the proportion of families with income Rs. 50 or less in the locality.

Then estimate of p from the sample

$$=\frac{206}{840}=\frac{103}{420}=0{\cdot}245$$

$\therefore$ Estimate of the standard error of the proportion of families with income Rs. 50 or less

$$=\sqrt{\frac{103}{420}\left(1-\frac{103}{420}\right)\cdot\frac{1}{840}}\simeq 0{\cdot}015$$

$\therefore$ Probable limits of p are

$0{\cdot}245 \mp 3(0{\cdot}015)$ *i.e.*, $0{\cdot}20$ and $0{\cdot}29$

$\therefore$ The probable limits of the number of families with income Rs. 50 or less are 3600 and 5220.

Ex. 14·6. *A sample of 900 days is taken from meteorological records of a certain district and 100 of them are found to be foggy. What are the probable limits to the percentage of foggy days in the district ?*

Sol. Proportion of foggy days in the district (estimated from the sample)

$$=\frac{100}{900}=\frac{1}{9}=0{\cdot}1111$$

$\therefore$ Estimate of the standard error of the proportion of foggy days in the district

$$=\sqrt{\frac{1}{900}\cdot\frac{1}{9}\cdot\left(1-\frac{1}{9}\right)}=0{\cdot}0105$$

$\therefore$ Probable limits to the percentage of foggy days are

$100\{0{\cdot}1111 \mp 3(0{\cdot}0105)\}$ *i.e.*, $7{\cdot}96\%$ and $14{\cdot}26\%$.

Ex. 14-7. *A sample of 500 pineapples was taken from a large consignment and 65 were found to be bad. Estimate the proportion of bad pineapples in the consignment, as well as the standard error of the estimate. Deduce that the percentage of bad pineapples in the consignment almost certainly lies between 8·5 and 17·5.*

Sol. Here $n=500$,

$p=$ proportion of bad pineapples in a consignment

$$=\frac{65}{500}=0{\cdot}13.$$

$\therefore$ The standard error of the proportion of bad pineapples

$$=\sqrt{\frac{p(1-p)}{n}}=\sqrt{\frac{(0{\cdot}13)(0{\cdot}87)}{500}}\simeq 0{\cdot}015$$

$\therefore$ Probable limits to the percentage of bad pineapples are

$100\{0{\cdot}13 \mp 3(0{\cdot}015)\}$ *i.e.*, $8{\cdot}5\%$ and $17{\cdot}5\%$.

Ex. 14·8. *A biased coin was thrown 400 times and head resulted 240 times. Find the standard error of the observed proportion of heads and deduce that the probability of getting a head in a throw of the coin lies almost certainly between 0·53 and 0·67.*

Sol. Observed proportion of heads

$$=\frac{240}{400}=0{\cdot}6$$

$\therefore$ S.E. of the observed proportion of heads

$$=\sqrt{\frac{(0{\cdot}6)(0{\cdot}4)}{400}}$$

$$=0{\cdot}0245$$

$\therefore$ The probability of getting a head in a throw of a coin lies in

$$0{\cdot}6\mp3(0{\cdot}0245)$$

i.e., $0{\cdot}5265$ and $0{\cdot}6735$

i.e., $0{\cdot}53$ and $0{\cdot}67$.

Ex. 14·9. *A dealer takes 100 samples from a consignment of 1000 items of a certain goods and finds that there are 50 items of grade I worth Rs. 5 per thousand, 30 items of grade II worth Rs. 4 per thousand and 20 items of grade III worth Rs. 3 per thousand. Within what limits should the value of the consignment be fixed?*

Sol. *Grade I.*

Proportion of items$=\frac{50}{100}=0{\cdot}5$

$\therefore$ S.E. of the proportion of items

$$=\sqrt{\frac{1}{100}(0{\cdot}5)(0{\cdot}5)}=0{\cdot}05$$

$\therefore$ Probable limits to the proportion of items are

$0{\cdot}5\mp3(0{\cdot}05)$ *i.e.,* $0{\cdot}35$ and $0{\cdot}65$

Grade II.

Proportion of items$=\frac{30}{100}=0{\cdot}3$

$\therefore$ S.E. of the proportion

$$=\sqrt{\frac{(0{\cdot}3)(0{\cdot}7)}{100}}=0{\cdot}0458$$

$\therefore$ Probable limits to the proportion are

$0{\cdot}3\mp3(0{\cdot}0458)$ *i.e.,* $0{\cdot}1626$ and $0{\cdot}4374$

Grade III.

Proportion of items $= \frac{20}{100} = 0{\cdot}2$

$\therefore$ S.E. of the proportion

$$= \sqrt{\frac{1}{100}(0{\cdot}2)(0{\cdot}8)} = 0{\cdot}04$$

$\therefore$ Probable limits to the proportion are

$0{\cdot}2 \mp 3(0{\cdot}04)$ *i.e.*, $0{\cdot}08$ and $0{\cdot}32$.

Now the highest value that can be given to the consignment is that value for which grade I is the highest and grade III the lowest so that

Proportion of grade I $= 0{\cdot}65$.

and proportion of grade III $= 0{\cdot}08$

$\therefore$ Proportion of grade II $= 1 - 0{\cdot}65 - 0{\cdot}08 = 0{\cdot}27$

$\therefore$ Highest value of the consignment

$= (0{\cdot}65)(5) + (0{\cdot}27)(4) + (0{\cdot}08)(3) =$ Rs. $4{\cdot}57$.

The lowest value that can be given to the consignment is that value for which grade I is the lowest and grade III the highest so that

proportion of grade I $= 0{\cdot}35$

and proportion of grade III $= 0{\cdot}32$

$\therefore$ Proportion of grade II $= 1 - 0{\cdot}35 - 0{\cdot}32 = 0{\cdot}33$

$\therefore$ The least value of the consignment

$= (0{\cdot}35)(5) + (0{\cdot}33)(4) + (0{\cdot}32)3 =$ Rs. $4{\cdot}03$

$\therefore$ The limits of the value of the consignment are Rs. $4{\cdot}03$ and Rs. $4{\cdot}57$.

EXERCISES

1. A coin is tossed 10,000 times and it turns up head 5195 times. Is it reasonable to think that the coin is unbiased? [Ans. No]
2. In 324 throws of a six-faced die odd points appeared 181 times. Can the die be regarded as unbiased? [Ans. Insignificant at 1% level]
3. In breeding certain stocks, 408 hairy and 126 glabrous plant were obtained. If the expectation is one-fourth glabrous, is the divergence significant or might it have occurred as a fluctuation of sampling? [Ans. Insignificant]
4. Experience has shown that 10% of a manufactured product is of top quality. In one day's production of 400 articles only 50 are of top quality. Does this contradict our hypothesis of 10 per cent? [Ans. No]

5. Balls are drawn from a bag containing equal number of black and white balls with replacement. In 2000 drawings, 1100 black and 900 white balls appear. Is there some bias in the drawer ? [**Ans.** Yes]

6. A personnel manager claims that 80% of all single woman hired for secretarial job get married and quit work within two years after they are hired. Test this hypothesis at 5% level of significance if among 200 such secretaries 112 got married within two years after they were hired and quit their jobs. [**Ans.** Hypothesis is wrong.]

7. 400 apples are taken from a large consignment and 50 are found to be bad. Estimate the percentage of bad apples in the consignment and assign the limits within which the percentage lies. [**Ans.** 7·5% and 17·5%]

8. Given that, on the average 40 out of 1000 insured men of age 60 die within a year and that 60 of a particular group of 1000 such men died within a year, show that this group cannot be regarded as representative sample, seeing that the actual deviation of the proportion of deaths is more than three times the standard error of the proportion for samples of this size.

9. A man buys 100 sacks of tomatoes. He finds that out of 100 tomatoes chosen from the sacks at random, 40 are of type A, worth Rs. 10 a sack, 25 are of type B, worth Rs. 7 a sack, 20 are of type C, worth Rs 5 a sack and 15 are of class D, worth Rs. 4 per sack. What are the upper and lower limits for the value of the tomatoes ? [**Ans.** 634·78 and 835·21]

10. 12 dice were thrown 6500 times 4, 5 or 6 being reckoned as a success. What proportion of success do you expect ? If in actual observation the proportion of success is found to be 0·5016, find the standard deviation of proportion with the given number of throws and state whether you would regard the excess of successes as probably significant bias in the dice.

14.2.2. Comparison of large samples

Let the two populations be tested for the prevalence of a certain attribute A by taking from them large simple samples of sizes n_1 and n_2 respectively. Let x_1 and x_2 be the number of individuals possessing A in the two samples.

Let $$p_1=\frac{x_1}{n_1} \quad \text{and } p_2=\frac{x_2}{n_2}$$

Let P_1 and P_2 be probabilities for an individual to possess A for two populations.

Then $$E(x_1)=n_1P_1 \text{ and hence } E(p_1)=P_1$$

$$\text{var}(x_1)=n_1P_1Q_1 \text{ and hence } \text{var}(p_1)=\frac{P_1Q_1}{n_1},\ Q_1=1-P_1$$

Similarly, $E(p_2)=P_2$ and var $(p_2)=\frac{P_2Q_2}{n_2}$, $Q_2=1-P_2$

Now $\because$ $E(p_1-p_2)=E(p_1)-E(p_2)=P_1-P_2$

and var $(p_1-p_2)=\text{var}(p_1)+\text{var}(p_2)=\frac{P_1Q_1}{n_1}+\frac{P_2Q_2}{n_2}$

$$\therefore \qquad u=\frac{(p_1-p_2)-(P_1-P_2)}{\sqrt{\frac{P_1Q_1}{n_1}+\frac{P_2Q_2}{n_2}}}$$

is approximately a $N(0, 1)$.

(i) The hypothesis to be tested is :

'Is the difference (p_1-p_2) significant of a real difference between the two populations *w.r.t.* A'.

To proceed with we set up the hypothesis that two populations are similar *w.r.t.* A. On the basis of this hypothesis

$$P_1=P_2=P \text{ (say)}$$

$$\therefore \qquad u=\frac{p_1-p_2}{\sqrt{PQ\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$$

where $Q=1-P$

we now test the significance with the aid of the normal curve.

Thus if (i) $|u|<1{\cdot}96$, the hypothesis is acceptable at 5% level of significance.

(ii) $1{\cdot}96<|u|<2{\cdot}58$, the difference is significant at 5% level of significance.

(iii) $2{\cdot}58<|u|<3$, the difference is significant at 1% level of significance.

(iv) $|u|>3$, the hypothesis is not acceptable and hence the difference is highly significant.

Generally P is unknown so we have to estimate it from the sample proportions. An unbiased estimate of P is given by

$$\mathbf{P}=\frac{n_1p_1+n_2p_2}{n_1+n_2}=\frac{x_1+x_2}{n_1+n_2}$$

It is unbiased because

$$E(\mathbf{P})=\frac{E(x_1+x_2)}{n_1+n_2}=\frac{n_1P_1+n_2P_2}{n_1+n_2}$$

$$=P \qquad (\because P_1=P_2=P)$$

(*ii*) The hypothesis to be tested is :

'Is the real difference between the populations likely to be hidden in two samples drawn' *i.e.*, If in populations $P_2 < P_1$, is it likely that p_1 and p_2 will be *s.t.* $p_1 \leq p_2$.

i.e., $$p_1 - p_2 \leq 0$$

i.e., $$(P_1 - P_2) + eu \leq 0$$

i.e., $$u \leq -\frac{(P_1 - P_2)}{e}$$

where $$e = \sqrt{\frac{P_1 Q_1}{n_1} + \frac{P_2 Q_2}{n_2}}$$

Now $$P(u > 1{\cdot}645) = 0{\cdot}5 - \int_0^{1{\cdot}645} dP = 0{\cdot}5 - 0{\cdot}45 = 0{\cdot}05$$

$\therefore$ $$P(u < -1{\cdot}645) = 0{\cdot}05$$

$\therefore$ If $$\frac{P_1 - P_2}{e} > 1{\cdot}645$$

i.e., $$-\frac{(P_1 - P_2)}{e} < -1{\cdot}645$$

$$P\left(u \leq -\frac{(P_1 - P_2)}{e}\right) < 0{\cdot}05$$

which implies that at 5% level it is unlikely that difference will be hidden in simple sampling.

Similarly if $\dfrac{P_1 - P_2}{e} > 2{\cdot}327$, the difference is unlikely to be hidden at 1% level of significance.

Ex. 14·10. *In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations?*

Sol. Here $P_1 = 0{\cdot}3$, $P_2 = 0{\cdot}25$, $n_1 = 1200$ and $n_2 = 900$

$\therefore$ $$e = \sqrt{\frac{(0{\cdot}3)(0{\cdot}7)}{1200} + \frac{(0{\cdot}25)(0{\cdot}75)}{900}} \simeq 0{\cdot}0195$$

$\therefore$ $$\frac{P_1 - P_2}{e} = \frac{0{\cdot}05}{0{\cdot}0195} \simeq 2{\cdot}56 (> 1{\cdot}645)$$

$\therefore$ At 5% level the difference is unlikely to be hidden.

Ex. 14.11 *In a simple sample of 600 men from a certain large city, 400 are found to be smokers. In one of 900 from another large city, 450 are smokers. Do the data indicate that the cities are significantly different with respect to the prevalence of smoking among men?*

Sol. Set the hypothesis : Two cities do not differ significantly *w.r.t.* the prevalence of smoking among men.

$$n_1=600,\ x_1=400$$

$$n_2=900,\ x_2=450$$

$$\therefore \qquad p_1=\frac{2}{3} \text{ and } p_2=\frac{1}{2}$$

$$\therefore \qquad P=\frac{x_1+x_2}{n_1+n_2}=\frac{850}{1500}=\frac{17}{30}$$

$$\therefore \qquad Q=1-P=\frac{13}{30}$$

$$\therefore \qquad e^2=PQ\left(\frac{1}{n_1}+\frac{1}{n_2}\right)=\frac{17}{30}\,\frac{13}{30}\left(\frac{1}{600}+\frac{1}{900}\right)$$

$$=0{\cdot}000682$$

$$\therefore \qquad e=0{\cdot}026$$

$$\therefore \qquad u=\frac{p_1\sim p_2}{e}=\frac{\frac{2}{3}-\frac{1}{2}}{0{\cdot}026}=6{\cdot}4\ (>3)$$

$\therefore$ The difference is highly significant and hence the two cities are significantly different with respect to the prevalence of smoking habit among men.

Ex. 14.12 *A railway company installed two sets of 50 Burmaties each. The two sets were treated with creosate by two different processes After a number of years of service it was found that 22 ties of first set and 18 ties of the second set were still in good condition. Are we justified in claiming that there is no real difference between the preserving properties of the two processes?*

Sol. Set the hypothesis : There is no real difference between the preserving properties of the two processes.

Here $\qquad p_1=0{\cdot}44,\ p_2=0{\cdot}36$

$$P=0{\cdot}4,\ Q=0{\cdot}6 \text{ and } e=0{\cdot}098$$

$$\therefore \qquad u=0{\cdot}8<1{\cdot}96$$

$\therefore$ Data provides no evidence against the hypothesis.

Ex. 14.13. *In a referendum submitted to the student body at a university 850 men and 566 women voted. 530 of the men and 304 of the women voted yes. Does this indicate a significant difference of opinion on the matter, at the 1% level, between men and women students ?*

Sol. Set the hypothesis : There is no significant difference of opinion between men and women on the matter.

Here $p_1=0{\cdot}6235,\ p_2=0{\cdot}5371$

$$P=\frac{834}{1416},\quad Q=\frac{582}{1416}\ \text{and}\ e=0{\cdot}0267$$

$\therefore$ $u=3{\cdot}2\ (>3)$

$\therefore$ Hypothesis is wrong.

Ex. 14-14. *On the basis of their total scores, the 200 candidates at a civil service examination are divided into two groups, the upper 30% and the remaining 70%. Consider the first question of this examination. Among the first group, 40 had the correct answer; whereas among the second group 80 had the correct answer. On the basis of these results, can one conclude that the first question is no good at discriminating ability of the type being examined here ?*

Sol. Set the hypothesis : The first question is no good at discriminating ability of the type being examined.

Here $n_1=\frac{30}{100}\cdot 200=60,\ n_2=140,\ x_1=40$, and $x_2=80$

$\therefore$ $p_1=0{\cdot}6667$ and $p_2=0{\cdot}5714$

$$P=0{\cdot}6,\ Q=0{\cdot}4\ \text{and}\ e=\frac{1}{5\sqrt{7}}$$

$\therefore$ $u\simeq1{\cdot}26\ (<1{\cdot}96)$

$\therefore$ The data is consistent with the hypothesis.

Ex. 14-15. *In a year there are 956 births in a town A of which 52·5% were males, while in towns A and B combined this proportion in a total of 1406 births was 0·496. Is there any significant difference in the proportion of male births in the two towns ?*

Sol. Set the hypothesis. There is no significant difference in the proportion of male births in the two towns.

Here $n_1=956,\quad n_1+n_2=1406\quad \therefore\ n_2=450$

$p_1=0{\cdot}525,\quad P=0{\cdot}496\quad \therefore\ Q=0{\cdot}504$

$\therefore$ $p_2=0{\cdot}434$

$\therefore$ $u\simeq3{\cdot}2\ (>3)$

$\therefore$ Hypothesis is wrong.

Ex. 14-16. *A machine puts out 16 imperfect articles in a sample of 500. After machine is overhauled it puts out 3 imperfect articles in a batch of 100. Has the machine been improved ?*

Sol. Set the hypothesis : Machine has not been improved.

Here $\quad p_1=0{\cdot}032,\ p_2=0{\cdot}03,\ \mathbf{P}=\frac{19}{600},\ \mathbf{Q}=\frac{581}{600}$

$\therefore \quad u \simeq 0{\cdot}1\ (<1{\cdot}96)$

$\therefore$ Hypothesis may be correct.

Ex. 14·17. *In a large city A, 20% of a random sample of 900 school boys had a certain slight physical defect. In another large city B 18·5% of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant ?*

Sol. Here $p_1=0{\cdot}2,\ p_2=0{\cdot}185,\ \mathbf{P}=0{\cdot}19,\ \mathbf{Q}=0{\cdot}81$

$\therefore \quad u \simeq 0{\cdot}92\ (<1{\cdot}96)$

$\therefore$ Difference is not significant.

Ex. 14·18. (*a*) *Two large random samples of sizes n_1 and n_2 are taken from two populations. If p_1 and p_2 be the proportions of members possessing the attribute in two samples, give procedure of testing the significance of the difference between p_1 and*

$$p=\frac{n_1p_1+n_2p_2}{n_1+n_2}$$

(*b*) *In two random samples of 400 and 500 students from two different colleges, 300 students in each were found to be failed in an examination. Find out whether the proportion of failures in first college is significantly greater than the proportion of failures in two colleges taken together.*

Sol. Let $\bar{p}_1$ and $\bar{p}$ be the expected values of p_1 and p. Then

$$\bar{p}=\frac{n_1\bar{p}_1+n_2\bar{p}_2}{n_1+n_2}$$

where $\bar{p}_2=E(p_2)$

Now $\quad \operatorname{cov}(p_1, p)=E(p_1-\bar{p}_1)(p-\bar{p})$

$$=\frac{1}{n_1+n_2}E[(p_1-\bar{p}_1)\{n_1(p_1-\bar{p}_1)+n_2(p_2-\bar{p}_2)\}]$$

$$=\frac{1}{n_1+n_2}\{n_1E(p_1-\bar{p}_1)^2+n_2E(p_1-\bar{p}_1)(p_2-\bar{p}_2)\}$$

$$=\frac{n_1}{n_1+n_2}\operatorname{ver}(p_1)$$

($\because$ cov$(p_1, p_2)=0$ as p_1, p_2 are independent)

Now p gives the estimate of population proportion and hence

$\operatorname{var}(p_1)=\frac{pq}{n_1}$ and $\operatorname{var}(p_2)=\frac{pq}{n_2}$

$$\therefore \quad \operatorname{cov}(p_1, p)=\frac{pq}{n_1+n_2}$$

Now $\therefore$ $\quad \text{var}(p)=\dfrac{1}{(n_1+n_2)^2}\ \text{var}\,(n_1p_1+n_2p_2)$

$$=\frac{1}{(n_1+n_2)^2}\,\{n_1^2\ \text{var}(p_1)+n_2^2\ \text{var}(p_2)\}$$

$$=\frac{pq}{n_1+n_2}$$

$$\therefore \qquad \text{var}(p-p_1)=\text{var}(p)+\text{var}(p_1)-2\ \text{cov}(p_1,\,p)$$

$$=\frac{pq}{n_1+n_2}+\frac{pq}{n_1}-\frac{2pq}{n_1+n_2}$$

$$=\frac{n_2}{n_1}\cdot\frac{pq}{n_1+n_2}$$

Assuming the hypothesis that there is no significant difference between p_1 and p, the test statistic becomes

$$u=\frac{p_1\sim p}{\sqrt{\dfrac{n_2}{n_1}\cdot\dfrac{pq}{n_1+n_2}}}$$

which is a $N(0,\,1)$ as n_1 and n_2 are large.

(*b*) Here $n_1=400$, $n_2=500$, $p_1=\dfrac{3}{4}$ and $p_2=\dfrac{3}{5}$

$$\therefore \qquad p=\frac{300+300}{400+500}=\frac{2}{3}$$

$$\therefore \qquad q=1-\frac{2}{3}=\frac{1}{3}$$

$$\therefore \qquad u=\frac{\dfrac{3}{4}-\dfrac{2}{3}}{\sqrt{\dfrac{5}{4}\cdot\dfrac{\dfrac{2}{3}\cdot\dfrac{1}{3}}{900}}}=4{\cdot}7\ (>3)$$

$\therefore$ p and p_1 are significantly different. Evidently $p_1>p$.

EXERCISE

1. In a random sample of 500 persons from towns A, 200 are found to be smokers. In a sample of 400 from town B, 200 are found to be smokers. Discuss the question whether the data reveal a significant difference between A and B so far as the smoking habit among persons is concerned.

[Ans. Significant]

2. In a large city A, 20% of a random sample of 900 schoolboys had defective eye-sight. In another large city B, 15·5% of a random sample of 1600 schoolboys had the same defect. Is the difference between the two proportion significant ?

[**Ans.** Not Significant]

3. In a random sample of 500 men from a particular district, 300 are found to be smokers. In one of 1000 men from another district, 550 are smokers. Do the data indicate that the two districts are significantly different w.r.t. the prevalence of smoking among men ? [**Ans.** Not-Significant]

4. From each of two consignments of eggs, a sample of size 200 is drawn and the number of rotton eggs counted. Test whether the proportion of rotton eggs in the two consignments are significantly different or not, given that

	Size of sample	*No. of rotton eggs*
Sample from consignment A	200	40
Sample from consignment B	200	30

[**Ans.** Not-Significant]

5. In two large populations there are 35% and 30% of fair haired people. Is the difference likely to be hidden in simple samples of 1500 and 1000. [**Ans.** $e=0{\cdot}019$, unlikely]

Ex. 14·19. *Show that standard error of the number of successes is the square root of the mean number of successes provided the mean proportion of successes is small.*

Sol. Mean proportion of successes$=p$

$\therefore$ If p is small, standard error of the number of successes

$$=\sqrt{npq}=\sqrt{np(1-p)}$$

$$\simeq\sqrt{np}$$

$$=\sqrt{\text{Mean number of successes}}$$

Ex. 14·20. *Show that precision of the proportion of successes varies as the square root of the number of members in the sample.*

Ex. 14·21. *If for one half of n events, the chance of success is p and the chance of failure is q, whilst for the other half the chance of success is q and the chance of failure is p. Show that the standard deviation of the number of successes is the same as if the chance of success were p in all the cases i.e., $\sqrt{npq}$ but that the mean of the number of successes is $\frac{n}{2}$ and not np.*

Sol. Let x_1 and x_2 denote the number of successes in two halves.

Then $E(x_1)=\frac{n}{2}\,p$, $\text{var}(x_1)=\frac{n}{2}\,pq$

and $E(x_2)=\frac{n}{2}\,q$, $\text{var}(x_2)=\frac{n}{2}\,pq$

$$\therefore \quad E(x_1+x_2)=\frac{n}{2}\,(p+q)=\frac{n}{2}$$

$$\text{var}(x_1+x_2)=\text{var}(x_1)+\text{var}(x_2)$$

[$\because$ The halves are independent]

$$=\frac{n}{2}\,pq+\frac{n}{2}\,pq=npq.$$

Ex. 14·22. *The sex ratio at birth is sometimes given by the ratio of male to female births, instead of the proportion of male to total births. If z is the ratio i.e., $z=\frac{p}{q}$, show that the standard error of z is approximately $\frac{1}{1+z}\sqrt{\frac{z}{n}}$, n being large so that deviations are small compared with the mean.*

Sol. Let x be the number of male births. Then $(n-x)$ is the number of female births.

Now $$z=\frac{p}{q}=\frac{p}{1-p}$$

$$\therefore \quad p=\frac{z}{1+z} \text{ and } q=\frac{1}{1+z}$$

Also $$z=\frac{x}{n-x}=\frac{x}{n}\cdot\left\{1-\frac{x}{n}\right\}^{-1}=\frac{x}{n}\left(1+\frac{x}{n}+\ldots\ldots\right)\simeq\frac{x}{n}$$

$$\therefore \quad \text{S.E. of } z\simeq\text{S.E. of }\left(\frac{x}{n}\right)=\sqrt{\frac{pq}{n}}$$

$$\simeq\frac{1}{1+z}\sqrt{\frac{z}{n}}$$

Ex. 14·23. *n individuals fall into one or the other two categories with probabilities p and q(=1−p), the number in two categories being n_1 and n_2. Show that $\text{cov}(n_1, n_2)=-npq$. Hence obtain $\text{var}\left(\frac{n_1}{n}-\frac{n_2}{n}\right)$.*

Sol. Evidently $n_2=n-n_1$

Now $$E(n_1)=np$$

$$\therefore \quad E(n_2)=n-E(n_1)=n-np=nq$$

$$\therefore \quad \text{cov}(n_1, n_2)=E\{(n_1-np)(n_2-nq)\}$$
$$=E\{(n_1-np)(n-n_1-nq)\}$$
$$=-E(n_1-np)^2=-\text{var}(n_1)=-npq$$

$$\text{var}\left(\frac{n_1}{n}-\frac{n_2}{n}\right)=\frac{1}{n^2}\text{var}(n_1-n_2)$$
$$=\frac{1}{n^2}[\text{var}(n_1)+\text{var}(n_2)-2\,\text{cov}(n_1, n_2)]$$
$$=\frac{1}{n^2}[npq+npq+2npq[$$
$$=\frac{4pq}{n}$$

14.3. Sample mean

A sample of size n can be described by the values of the random variables. Let $x_1, x_2, \ldots\ldots x_n$ denote the random variables for a sample of size n. Then the sample mean is a random variable defined by

$$\bar{x}=\frac{x_1+x_2+\ldots\ldots+x_n}{n}$$

The probability distribution of $\bar{x}$ is called sampling distribution for the sample mean $\bar{x}$ or the sampling distribution of means.

Remark. In the case of random sample, $x_1, x_2 \ldots\ldots x_n$ are independent.

14.3.1. Central Limit Theorem

Statement. If $x_1, x_2, \ldots\ldots x_n$ be n independent random variables all with same distribution, mean μ and *s.d.* σ and $\bar{x}=\frac{1}{n}\sum_{i=1}^{n} x_i$, then, if m.g.f. of x_i exist, the variate $z=\frac{(\bar{x}-\mu)}{\sigma}\sqrt{n}$ has a distribution that approaches the standard normal distribution as $n\to\infty$,

Proof. $M_0(t)$ of $z=E\left\{e^{t\left(\frac{\bar{x}-\mu}{\sigma}\right)\sqrt{n}}\right\}$

$$=e^{-t\frac{\mu\sqrt{n}}{\sigma}}E\left\{e^{t\frac{\sqrt{n}}{\sigma}\bar{x}}\right\}$$

$$=e^{-t\frac{\mu\sqrt{n}}{\sigma}}E\left\{e^{\frac{t}{\sigma\sqrt{n}}(x_1+x_2+\ldots\ldots+x_n)}\right\}$$

$$=e^{-t\frac{\mu\sqrt{n}}{\sigma}}\,E\left\{e^{\frac{t}{\sigma\sqrt{n}}x_1}\cdot e^{\frac{t}{\sigma\sqrt{n}}x_2}\cdots\cdots e^{\frac{t}{\sigma\sqrt{n}}x_n}\right\}$$

$$=e^{-t\frac{\mu\sqrt{n}}{\sigma}}\,E\left\{e^{\frac{t}{\sigma\sqrt{n}}x_1}\right\}.E\left\{e^{\frac{t}{\sigma\sqrt{n}}x_2}\right\}\ldots E\left\{e^{\frac{t}{\sigma\sqrt{n}}x_n}\right\}$$

(as x's are independent)

Now since all x_i has same distribution, mean and *s.d.*, their m.g.f. will also be same.

$$\therefore\quad M_0(t) \text{ of } z=e^{-t\frac{\mu\sqrt{n}}{\sigma}}\left\{M_0\left(\frac{t}{\sigma\sqrt{n}}\right)\right\}^n$$

where $M_0\left(\frac{t}{\sigma\sqrt{n}}\right)$ is the m.g.f. of x_i

$$\therefore\quad \log\{M_0(t) \text{ of } z\}=-t\frac{\mu\sqrt{n}}{\sigma}+n\log\left\{M_0\left(\frac{t}{\sigma\sqrt{n}}\right)\right\}$$

$$=-t\frac{\mu\sqrt{n}}{\sigma}+n\log\left\{1+\mu_1'(0)\frac{t}{\sigma\sqrt{n}}+\frac{\mu_2'(0)}{2!}\left(\frac{t}{\sigma\sqrt{n}}\right)^2+\ldots\right\}$$

$$=-t\frac{\mu\sqrt{n}}{\sigma}+n\left[\left\{\mu_1'(0)\frac{t}{\sigma\sqrt{n}}+\frac{\mu_2'(0)}{2!}\left(\frac{t}{\sigma\sqrt{n}}\right)^2+\ldots\right\}-\frac{1}{2}\left\{\mu_1'(0)\frac{t}{\sigma\sqrt{n}}+\ldots\right\}^2+\ldots\ldots\right]$$

Now $\mu_1'(0)=\mu$.

$$\therefore\quad \log\{M_0(t) \text{ of } z\}=\frac{t^2}{2\sigma^2}\left[\mu_2'(0)-\{\mu_1'(0)\}^2\right]+\text{terms containing}$$

n in the denominator.

$$\therefore\quad \underset{n\to\infty}{\text{Lt}}\log\{M_0(t) \text{ of } z\}=\frac{t^2}{2\sigma^2}\left[\mu_2'(0)-\{\mu_1'(0)\}^2\right]=\frac{t^2}{2}$$

$$[\because\quad \mu_2'(0)-\{\mu_1'(0)\}^2=\mu_2=\sigma^2]$$

$$\therefore\quad \underset{n\to\infty}{\text{Lt}}\,M_0(t) \text{ of } z=e^{\frac{1}{2}t^2}$$

which is the m.g.f. of a $N(0, 1)$

$\therefore$ As $n\to\infty$ the distribution of z tends to the standard normal distribution.

14.3.2. The standard error of the mean of a random sample of size n from a population with variance σ^2.

Sol. Let $x_1, x_2, \ldots\ldots x_n$ be a random sample.

Then $\quad \bar{x}=\dfrac{x_1+x_2+\ldots\ldots x_n}{n}$

is the sample mean.

$$\therefore \quad \text{var}(\bar{x}) = \frac{1}{n^2}[\text{var}(x_1 + x_2 + \ldots\ldots + x_n)]$$

$$= \frac{1}{n^2}[\text{var}(x_1) + \text{var}(x_2) + \ldots\ldots + \text{var}(x_n)]$$

(as x's are independent)

$$= \frac{\sigma^2}{n} \quad (\because \ \text{var}(x_i) = \sigma^2 \text{ for all '}i\text{'})$$

$$\therefore \text{S.E. of } \bar{x} = \frac{\sigma}{\sqrt{n}}.$$

14.4. Sampling of variables

In this case population is the frequency distribution of the variable and its each member provides a value of the variable. Drawing of a sample is same as choosing certain values of the variable from those of the distribution.

14.4.1. Unbiased estimate of population mean

Let the values $X_1, X_2, \ldots\ldots X_N$ constitute a finite population with mean μ and variance σ^2.

Then $$\mu = \frac{X_1 + X_2 + \ldots\ldots + X_N}{N}$$

Let $x_1, x_2, \ldots\ldots x_n$ be a random sample from the population. The sample mean is

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\therefore \quad E(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n} E(x_i)$$

Now for fixed i, x_i can take any one of the values $X_1, X_2, \ldots X_N$ each with probability $\frac{1}{N}$.

$$\therefore \quad E(x_i) = \frac{1}{N}(X_1 + X_2 + \ldots X_N) = \mu$$

$$\therefore \quad E(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu$$

$\therefore$ Sample mean $\bar{x}$ is an unbiased estimate of the population mean.

14.4.2. Unbiased estimate of population variance

Sample *s.d.* is

$$s^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu+\mu-\bar{x})^2$$

$$=\frac{1}{n}\sum_{i=1}^{n}\{(x_i-\mu)^2+(\mu-\bar{x})^2+2(\mu-\bar{x})(x_i-\mu)$$

$$=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2-(\mu-\bar{x})^2$$

$$\therefore \qquad E(s^2)=\frac{1}{n}\sum_{i=1}^{n}E(x_i-\mu)^2-E(\bar{x}-\mu)^2.$$

Since $E(x_i)=\mu=E(\bar{x})$, $E(x_i-\mu)^2=\text{var}(x_i)=\sigma^2$

and $\quad E(\bar{x}-\mu)^2=\text{var}(\bar{x})=\dfrac{\sigma^2}{n}$

$$\therefore \qquad E(s^2)=\sigma^2-\frac{\sigma^2}{n}$$

$$=\left(\frac{n-1}{n}\right)\sigma^2$$

$$\therefore \qquad E\left(\frac{n}{n-1}s^2\right)=\sigma^2$$

Unbiased estimate of population variance

$$=\frac{n}{n-1}\cdot s^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2=S^2.$$

14.4.3. Test of significance of single mean

Consider a large random sample with mean $\bar{x}$ from a large population with mean μ and *s.d.* σ. Then $\bar{x}\sim N\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right)$

$$\therefore \qquad z=\frac{x-\mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

The hypothesis to be tested here is 'the sample has been drawn from a population with mean μ and *s.d.* σ'.

The significance is tested with the aid of normal curve and the rules of taking decisions are same as before.

14.4.4. Confidence Limits or Fiducial Limits

Consider a large random sample of size n with mean $\bar{x}$ from a population (not necessarily normal) with mean μ and s.d. σ. Then

$$z=\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$$

is nearly a N(0, 1). If σ be known but not μ, there is a range of possible values of μ for which $\bar{x}$ is not significant at any specified level of probability. If $\bar{x}$ is not significant at 5% level of probability, then since

$$P\{\,|z| > 1{\cdot}96\}=0{\cdot}05,$$

μ must be s t.

$$\left|\frac{\bar{x}-m}{\sigma/\sqrt{n}}\right| < 1{\cdot}96$$

$$\therefore \qquad \bar{x}-1{\cdot}96\,\frac{\sigma}{\sqrt{n}} < \mu < \bar{x}+1{\cdot}96\frac{\sigma}{\sqrt{n}}$$

The values $\bar{x}-1{\cdot}96\,\frac{\sigma}{\sqrt{n}}$ are called 95% **Fiducial Limits** *or* **Confidence Limits** *for the mean of the population corresponding to the given samnle. The interval $\bar{x}-1{\cdot}96\frac{\sigma}{\sqrt{n}}$ to $\bar{x}+1{\cdot}96\,\frac{\sigma}{\sqrt{n}}$ is called 95%* **Confidence Interval.**

Similarly since $P\{\,|z| > 2{\cdot}58\}=0{\cdot}01$, 99% Fiducial limits for the population mean are $\mp\bar{x}2{\cdot}58\frac{\sigma}{\sqrt{n}}$.

In general if $P\{\,|z| > z'\}=P'$, $100\,(1-P')$%. Fiducial limits are $\bar{x}\mp z'\,\frac{\sigma}{\sqrt{n}}$.

Evidently the limit vary from sample to sample. The totality of values of limits (for given P') for different samples determine the field within which μ is asserted to lie. This field is known as **Confidence Belt.**

Ex. 14·24. *A sample of 400 male students is found to have a mean height of 67·47". Can it be reasonably regarded as a sample from a large population with mean height 67·39" and s.d. 1·3" ?*

Sol. Here $n=400$, $\bar{x}=67·47''$, $\mu=67·39''$ and $\sigma=1·3''$

$\therefore$ $z=1·23(<1·96)$.

$\therefore$ The sample may be regarded as drawn from the population with mean 67·39" and *s.d.* 1·3".

Ex 14·25. *A sample of 900 members is found to have a mean of 3·4 cm. Can it be reasonably regarded as a simple sample from a large population with mean 3·25 cm. and s.d. 2·61 cm ?*

Sol. Here $n=900$, $\bar{x}=3·4$, $\mu=3·25$ and $\sigma=2·61$.

$\therefore$ $z=1·7\ (<1·96)$

$\therefore$ The sample can be regarded as drawn from a population with mean 3·25 and *s.d.* 2·61.

Ex. 14·26. *Mean of 10 readings on the length of a given rod is 20". The s.d of errors of measurement is known to be 0·1". Does the result contradict the assumption that the length of the rod is 19·9" ?*

Sol. Here $\mu=19·9$, $\sigma=0·1$, $n=10$, and $\bar{x}=20$,

$\therefore$ $z=3·162\ (>3)$

$\therefore$ Sample contradicts the given assumption.

Ex. 14.27. *A sample of 900 members is found to have mean 3·5 cms. Can it be reasonably regarded as a random sample from a large population with mean 3·3 cms. and s.d. 2·3 cms ?*

Sol. Here $n=900$, $\bar{x}=3·5$, $\mu=3·3$ and $\sigma=2·3$.

$\therefore$ $z=2·6\ (>2·58)$

At 1% level sample cannot be regarded as drawn from a population with mean 3·3 and *s.d.* 2·3.

Ex. 14·28. *The mean of a certain normal population is equal to the standard error of the mean of the samples of 100 from that distribution. Find the probability that the mean of the sample of 25 from the distribution will be negative.*

Sol. Let μ and σ be the mean and *s.d.* of the population.

Then $\mu=\dfrac{\sigma}{\sqrt{n}}=\dfrac{\sigma}{10}$.

Let $\bar{x}$ be the mean of the sample of size 25.

$$\therefore \quad z=\frac{\bar{x}-\mu}{\sigma/\sqrt{25}}=\frac{\bar{x}-\sigma/10}{\sigma/5}=\frac{5\bar{x}}{\sigma}-\frac{1}{2} \sim N(0,1)$$

$$\therefore \quad \bar{x}=\frac{\sigma}{5}z+\frac{\sigma}{10}$$

$$\therefore\ P(\bar{x}<0)=P\left(\frac{\sigma}{5}\,z+\frac{\sigma}{10}<0\right)=P\left(z<-\frac{1}{2}\right)$$

$$=P\left(z>\frac{1}{2}\right)=0{\cdot}5-P\left(0<z<\frac{1}{2}\right)$$

$$=0{\cdot}5-(0{\cdot}1915)=0{\cdot}3085 \qquad \text{(from tables).}$$

Ex. 14·29. *A normal population has mean 0·1 and s.d. 2·1. Find the probability that the mean of a simple sample of size 900 will be negative.*

Sol. Here $\mu=0{\cdot}1$, $\sigma=2{\cdot}1$, $n=900$.

$$\therefore \qquad z=\left(\frac{\bar{x}-0{\cdot}1}{2{\cdot}1}\right)30\sim N(0,\,1)$$

$$\therefore \qquad \bar{x}=0{\cdot}07\,z+0{\cdot}1$$

$$\therefore\ P(\bar{x}<0)=P\left(z<-\frac{10}{7}\right)$$

$$=P(z>1{\cdot}43)=0{\cdot}5-P(0<z<1{\cdot}43)$$

$$=0{\cdot}5-0{\cdot}4236=0{\cdot}0764.$$

Ex. 14·30. *A research worker wishes to estimate the mean of a population, using a sample sufficiently large, such that the probability will be 0·95 that the sample mean will not differ from the true mean by more than 25% of the s.d. How large a sample should be taken?*

Sol. Let n be the size of the sample.

Now $\qquad P\{\,|\,\bar{x}-\mu\,|\leqslant 0{\cdot}25\sigma\}=0{\cdot}95$

$\therefore \qquad P(\,|\,z\,|\leqslant 0{\cdot}25\sqrt{n}\,)=0{\cdot}95$

$\therefore \qquad P(0<z<0{\cdot}25\sqrt{n}\,)=0{\cdot}4750.$

$\therefore \qquad 0{\cdot}25\sqrt{n}=1{\cdot}96$

$\therefore \qquad n=62.$

Ex. 14·31. *If the mean breaking strength of copper wire is 574 lbs with a s.d. of 8·3 lbs, how large a sample must be used in order that there be chance $\frac{1}{100}$ that the mean breaking strength of the wire is less than 571 lbs?*

Sol. How $\mu=574$, $\sigma=8{\cdot}3$.

$$\therefore \qquad z=\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}<\left(\frac{571-574}{8{\cdot}3}\right)\sqrt{n}=-\frac{3}{8{\cdot}3}\sqrt{n}$$

Now $\qquad P\left\{z<-\frac{3}{8{\cdot}3}\sqrt{n}\right\}=0{\cdot}01$

$$\therefore \quad P\left\{ |z| < \frac{3}{8\cdot 3}\sqrt{n} \right\} = 1 - 2(0\cdot 01) = 0\cdot 98$$

$$\therefore \quad P\left\{ 0 < z < \frac{3}{8\cdot 3}\sqrt{n} \right\} = 0\cdot 49$$

$$\therefore \quad \frac{3}{8\cdot 3}\sqrt{n} \simeq 2\cdot 327 \qquad \text{(from normal tables)}$$

$$\therefore \quad n = \frac{(8\cdot 3)^2}{9} \cdot (2\cdot 327)^2 = 41\cdot 46.$$

$$\therefore \quad n \simeq 42.$$

Ex. 14·32. *The guaranteed average life of a certain type of electric light bulbs is 1,000 hours with a s.d. of 125 hours. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by more than 2·5%. What must be the sample size ?*

Sol. Here $\mu = 1{,}000$, $\sigma = 125$

$$\mu - \bar{x} < \frac{2\cdot 5}{100}\mu = 25 \text{ i.e., } \bar{x} > \mu - 25 = 975$$

$$\therefore \quad z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} > \frac{-25}{125/\sqrt{n}} = \frac{-\sqrt{n}}{5}$$

$$\therefore \quad P\left(z > -\frac{\sqrt{n}}{5} \right) = 0\cdot 9.$$

$$\therefore \quad P\left(0 < z < \frac{\sqrt{n}}{5} \right) = 0\cdot 9 - 0\cdot 5 = 0\cdot 4.$$

$$\therefore \quad \frac{\sqrt{n}}{5} \simeq 1\cdot 28 \qquad \text{(from normal tables)}$$

$$\therefore \quad n \simeq 40\cdot 96 \simeq 41.$$

Ex. 14·33. *It is known that the mean and s.d. of a variable are respectively 100 and 10 in the universe. It is however considered sufficient to draw a sample of sufficient size but such as to ensure that the mean of the sample would be, in all probabilities, within 0·01% of the true value. How much would be the cost (exclusive of overhead charges) if the charges for drawing 100 members of a sample be one rupee ?*

Sol. Assume the simple sampling conditions hold.

Here $\quad \mu = 100, \sigma = 10$

$$|\bar{x} - \mu| < \frac{0\cdot 01}{100}\mu = 0\cdot 01 \text{ in all probabilities.}$$

$$\therefore \quad |z| = \left| \frac{\bar{x} - \mu}{\sigma} \right| \sqrt{n} < \frac{0\cdot 01}{10}\sqrt{n} \text{ in all probabilities.}$$

$$\therefore \quad \frac{0.01}{10}\sqrt{n}=3$$

$$\therefore \quad n=9000,000$$

$\therefore$ Sampling charges $= Rs. \dfrac{9,000,000}{100} = Rs.\ 90,000.$

Ex. 14·34. *To know the mean weight of all 12-year old boys in a state, a sample of 225 is taken. The mean weight of this sample is found to be 67 ibs with a s.d. 9 ibs. Can you draw any inference from it about the mean weight of the universe ?*

Sol Here *s.d.* of the universe is not given but we can take in its place the sample *s.d.* as the sample is large.

$\therefore$ S.E. of the mean $= \dfrac{9}{\sqrt{225}} = 0{\cdot}6.$

Assuming simple sampling conditions, the mean *weight* μ of the universe would in all probability be *s.t.*

$$\sqrt{n}\left|\frac{\bar{x}-\mu}{\sigma}\right|<3$$

i.e., $\quad \bar{x}-\dfrac{3\sigma}{\sqrt{n}}<\mu<\bar{x}+\dfrac{3\sigma}{\sqrt{n}}$

i.e., $\quad 67-1{\cdot}8<\mu<67+1{\cdot}8$

i.e., $\quad 65{\cdot}2<\mu<68{\cdot}8.$

Ex. 14·35. *The mean height of 10,000 children of age 6 years is 41·26" and the s.d. is 2·24". Find the odds against the possibility that the mean of a random sample of 100 is greater than 41·7".*

Sol. $\mu=41{\cdot}26,\ \sigma=2{\cdot}24,\ n=100.$

S.E. of the sample mean $= \dfrac{\sigma}{\sqrt{n}} = 0{\cdot}224.$

The probability of

$$z=\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}>\frac{41{\cdot}7-41{\cdot}26}{0{\cdot}224}=1{\cdot}96$$

is needed.

Since $\quad z\sim N(0,\ 1).$

$$P(z>1{\cdot}96)=0{\cdot}5-P(0<z<1{\cdot}96)$$

$$=0{\cdot}5-0{\cdot}4750=0{\cdot}025=\frac{1}{40}$$

$\therefore$ Odds against are 39 : 1.

Ex. 14·36. *Suppose that the distribution of the statures of men is a normal distribution with s.d. 2·48". One hundred male students in a large university are measured and their average height is found to be 68·52". Determine the 98% confidence limits for the mean height of the men of the university.*

Sol. Here $\bar{x}=68\cdot52$, $n=100$ and $\sigma=2\cdot48$.

$\therefore$ S.E. of sample mean $=\dfrac{\sigma}{\sqrt{n}}=0\cdot248"$.

$$\therefore \qquad z=\frac{68\cdot52-\mu}{0\cdot248}.$$

Now since $P\{\,|\,z\,|<2\cdot33\}=0\cdot98$, (from normal tables)

μ is needed s.t.

$$\left|\frac{68\cdot52-\mu}{0\cdot248}\right|<2\cdot33$$

i.e., $$67\cdot9<\mu<69\cdot1.$$

$\therefore$ 98% confidence limits for μ are 67·9" and 69·1".

Ex. 14·37. *The data concerning height measurement for a random sample of individuals from a given population are as follows:*

$$mean=172,\ S.D.=12,\ n=100$$

If a large number of samples of the same size were selected at random from the given population, what would be the limits of 2% confidence interval for the true mean?

Sol. The limits of 2% confidence interval for the true mean means the same thing as 98% confidence limits for the true mean.

$\therefore$ Reqd. limits are

$$172\pm2\cdot33\left(\frac{12}{\sqrt{100}}\right)\ \textit{i.e.,}\ 169\cdot2 \text{ and } 174\cdot8.$$

Ex. 14·38. *An unbiased coin is thrown n times. It is desired that the relative frequency of the appearance of heads should lie between 0·49 and 0·51. Find the smallest value of n that will ensure this result with 90% confidence.*

Sol. Probability of head (tail) in a single toss $=\dfrac{1}{2}$

$\therefore$ S.E. of proportion of head $=\sqrt{\dfrac{1}{2}\cdot\dfrac{1}{2}\cdot\dfrac{1}{n}}=\dfrac{1}{2\sqrt{n}}$.

Let p be the observed proportion of heads and

$$z=\frac{p-0\cdot5}{1/2\sqrt{n}}.$$

Now since $P\{\,|\,z\,|\, < 1{\cdot}645\}=0{\cdot}9$, '$n$' is to determined *s.t.*

$$\left|\frac{p-0{\cdot}5}{1/2\sqrt{n}}\right| < 1{\cdot}645$$

i.e.,
$$0{\cdot}5-\frac{1{\cdot}645}{2\sqrt{n}} < p < 0{\cdot}5+\frac{1{\cdot}645}{2\sqrt{n}}$$

$\therefore$
$$0{\cdot}5-\frac{1{\cdot}645}{2\sqrt{n}}=0{\cdot}49$$

and
$$0{\cdot}5+\frac{1{\cdot}645}{2\sqrt{n}}=0{\cdot}51$$

Subtracting
$$\frac{1{\cdot}645}{\sqrt{n}}=0{\cdot}02$$

$\therefore$
$$n=6765.$$

Ex. 14·39. *If p is the observed proportion of success in n independent Bernoullian trials, prove that the 95% fiducial limits for the population proportion p', for large samples, are,*

$$p \pm 1{\cdot}96\sqrt{\frac{pq}{n}}.$$

Also show that 99% fiducial limits are the roots of quadratic equation

$$(p-p^2)(2{\cdot}58)^2=n(p'-p)^2$$

Sol. S.E. of proportion of successes$=\sqrt{\dfrac{pq}{n}}$

Now
$$z=\frac{p-p'}{\sqrt{\dfrac{pq}{n}}} \sim N(0,\,1)$$

Now since $P\{\,|\,z\,|\, < 1{\cdot}96\}=0{\cdot}95$, 95% fiducial limits are given by

$$\left|\frac{p-p'}{\sqrt{\dfrac{pq}{n}}}\right| < 1{\cdot}96$$

i.e.,
$$p-1{\cdot}96\sqrt{\frac{pq}{n}} < p' < p+1{\cdot}96\sqrt{\frac{pq}{n}}$$

Similarly, since $P\{\,|\,z\,|\, < 2{\cdot}58\}=0{\cdot}99$, 99% fiducial limits are given by

$$p'=p \pm 2{\cdot}58\sqrt{\frac{pq}{n}}$$

i.e.,
$$n(p'-p)^2=(2\cdot58)^2\,pq=(2\cdot58)^2\,p(1-p)$$
$$=(2\cdot58)^2(p-p^2).$$

EXERCISES

1. If p_1, p_2 are observed proportion of successes in two independent sets of trials of large sizes n_1 and n_2, show that 95% fiducial limits for the difference $(p_1'-p_2')$ of the proportions of successes in the population are
$$p_1-p_2\pm2\cdot58\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}$$
Also find 95% fiducial limits.

2. A sample of 900 members is found to have a mean of 3·4 cm. Can it be reasonably regarded as a simple sample from a large population with mean 3·2 cm. and s.d. 2·3 cm ?
[**Ans.** $z=2\cdot6$, No.]

3. A simple sample of 1000 members is found to have a mean 3·5 cm. Could it be reasonably regarded as a simple sample from a large population with mean 3·2 cm and s.d. 2·6 cm. ?
[**Ans.** $z=3\cdot6$, No.]

4. The standard deviation of a population is 2·7". Find the probability that in samples of size 100 (*i*) the sample mean will differ from the population mean by 0·75 or more and (*ii*) the sample mean will exceed the population mean by 0·75" or more.
[**Ans.** 0·0054 ; 0·0027]

5. A sample of 900 members is found to have a mean of 3·47 cm. Can it be reasonably regarded as a simple sample from a population with mean 3·23 cm and s.d. 2·31 cm. [**Ans.** No.]

14·4·5 Test of significance of the difference between the means of two large samples.

Let $\bar{x}_1, \bar{x}_2$ be the means of two independent samples of sizes n_1 and n_2 (both n_1 and n_2 are large) from two different populations with means μ_1, and μ_2 and s.d. σ_1 and σ_2 respectively. Then
$$\bar{x}_1\sim N\left(\mu_1, \frac{\sigma_1}{\sqrt{n_1}}\right) \text{ and } \bar{x}_2\sim N\left(\mu_2, \frac{\sigma_2}{\sqrt{n_2}}\right)$$
$$\therefore \quad \bar{x}_1-\bar{x}_2\sim N\left(\mu_1-\mu_2, \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\right)$$
$$\therefore \quad z=\frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}}\sim N(0, 1)$$

The hypothesis to be tested is 'Are population means same *i.e.*, $\mu_1=\mu_2$'. Assuming this hypothesis the test statistic becomes

$$z=\frac{\bar{x}_1-\bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$$

The significance is tested with the aid of normal curve.

Note 1. If $\sigma_1^2=\sigma_2^2=\sigma^2$, then

$$z=\frac{\bar{x}_1-\bar{x}_2}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}.$$

Note 2. If σ is not known, then it is to be estimated from the samples. An unbiased estimate of σ^2 based upon two samples is

$$\sigma^2=\frac{(n_1-1)\,S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$$

where $\quad S_1^2=\frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_{1i}-\bar{x}_1)^2$ and $S_2^2=\frac{1}{n_2-1}\sum_{j=1}^{n_2}(x_{2j}-\bar{x}_2)^2$

σ^2 is unbiased because

$$E(\sigma^2)=\frac{1}{n_1+n_2-2}\ \{n_1-1)E(S_1^2)+(n_2-1)E(S_2^2)\}$$

$$=\frac{1}{n_1+n_2-2}\ \{(n_1-1)\sigma^2+(n_2-1)\sigma^2\}=\sigma^2$$

$$(\because\quad E(S_1^2)=E(S_2^2)=\sigma^2)$$

Since n_1 and n_2 are large, $n_1-1\sim n_1$ and $n_2-1\sim n_2$.

$\therefore \qquad S_1^2\sim s_1^2$ and $S_2^2\sim s_2^2$

where $\qquad s_1^2=\frac{1}{n_1}\sum_{i=1}^{n_1}(x_{1i}-\bar{x}_1)^2$ and $s_2^2=\frac{1}{n_2}\sum_{j=1}^{n_2}(x_{2j}-\bar{x}_2^2)$

are sample variances.

$$\therefore \qquad \sigma^2\simeq\frac{n_1s_1^2+n_2s_2^2}{n_1+n_2}.$$

Note 3. If $\sigma_1^2\neq\sigma_2^2$ and σ_1 and σ_2 are unknown, these are to be estimated from respective samples. Respective estimates are $S_1^2\sim s_1^2$ and $S_2^2\sim s_2^2$.

Ex. 14-40. *The mean of simple samples of 1,000 and 2,000 are 67·5" and 68 0" respectively. Can the samples be regarded as drawn from the same population of s.d. 2·5". ?*

Sol. $n_1=1{,}000$, $n_2=2{,}000$, $\bar{x}_1=67{\cdot}5$, $\bar{x}_2=68{\cdot}0$ and $\sigma=2{\cdot}5$.

$\therefore \quad z \simeq 5{\cdot}2 (>3)$

$\therefore$ Hypothesis is wrong and hence the samples cannot be regarded as drawn from the same population of s.d. 2·5".

Ex. 14·41. *A simple sample of heights of 6,400 Englishmen has a mean of 67·85" and s.d. of 2·56", while a simple sample of heights of 1,600 Australians has a mean 68·55" and a s.d. of 2·52". Do the data indicate that Australians are on the average taller than Englishmen?*

Sol. Here population standard deviations are not known so these are to be estimated from samples. As samples are large, sample standard deviations can be taken as estimates of population standard deviations.

$\therefore \quad \sigma_1=2{\cdot}56, \ \sigma_2=2{\cdot}52,$

$\therefore \quad z \simeq 10 (>3).$

$\therefore$ Difference between sample means is significant.

$\therefore$ Englishmen are on the average smaller than Australians.

Ex. 14·42. *A random sample of 1200 men from one state gives their mean pay as Rs. 40 p.m. with a s.d. of Rs. 24 p.m. and a random sample of 1600 men from another state gives their mean pay as Rs. 36 p.m. with a s.d. of Rs. 32 p.m. Discuss whether the mean levels of pay of men from the two states differ.*

Sol. Here $n_1=1200$, $\bar{x}_1=40$, $\sigma_1=24$ and
$n_2=1600$, $\bar{x}_2=36$, $\sigma_2=32$.

$$\therefore \quad z=\frac{4}{\sqrt{\dfrac{(24)^2}{1200}+\dfrac{(32)^2}{1600}}} \simeq 3{\cdot}78\ (>3)$$

$\therefore$ Difference between means is significant and hence mean levels of pay in two states differ.

Ex. 14·43. *Mean and standard deviations calculated from the weights in kgm of students of two groups taken from two universities are given below:*

	Mean	*S.D.*	*Sample size*
University A	*55*	*10*	*400*
University B	*57*	*15*	*100*

Test the significance of the difference between the means.

Sol. Here
$$z=\frac{57-55}{\sqrt{\dfrac{(10)^2}{400}+\dfrac{(15)^2}{100}}} \simeq 1{\cdot}2648\ (<1{\cdot}96)$$

$\therefore$ Difference between the means is due to fluctuation of sampling only.

Ex. 14·44. *A random sample of 1000 farms in a certain year gives an average yield of rice 2000 lbs per acre with a s.d. of 192 lbs. A random sample of 1000 farms in the following year gives an average yield of 2100 lbs per acre with a s.d. of 224 lbs. Show that the data are inconsistent with the hypothesis that the average yield in the country as a whole was the same in the two years.*

Here $$|z|=\frac{2100-2000}{\sqrt{\frac{(192)^2}{1000}+\frac{(224)^2}{1000}}}\simeq 10{\cdot}7\ (>3)$$

$\therefore$ Data is inconsistent with the hypothesis.

Ex. 14·45. *A potential buyer of light bulbs bought 50 bulbs each of two brands. Upon testing these bulbs, he found that brand A had a mean life of 1282 hours with a s.d. of 80 hours whereas B had a mean life of 1208 hours with a s.d. of 94 hours. Can the buyer be quite certain that the two brands do differ in quality ?*

Sol. Here $n_1=50$, $\bar{x}_1=1282$, $\sigma_1=80$

and $n_2=50$, $\bar{x}_2=1208$, $\sigma_2=94$

$\therefore$ $$z\simeq 4{\cdot}2\ (>3)$$

$\therefore$ Difference is significant.

Ex. 14·46. *A random sample of 200 villages was taken from a certain district and the average population per village was found to be 485 with a s.d. of 50. Another random sample of 200 villages from the same district gave an average population of 510 per village with a s.d. of 40. Is the difference between the averages of the two samples significant ? Give reasons.*

Sol. Here $n_1=200$, $\bar{x}_1=485$, $\sigma_1=50$

and $n_2=200$, $\bar{x}_2=510$, $\sigma_2=40$

$\therefore$ $$|z|=\frac{510-485}{\sqrt{\frac{(50)^2+(40)^2}{200}}}\simeq 5{\cdot}5(>3)$$

$\therefore$ Difference is significant.

Ex. 14·47. *If 60 new entrants in a given university are found to have a mean ht of 68·60" and 50 seniors a mean height of 69·51", is the evidence conclusive that the mean height of the seniors is greater than that of the new entrants ? Assume the s.d. of the height to be 2·48".*

Sol. Here $n_1=60$, $\bar{x}_1=68{\cdot}6$, $\sigma_1=2{\cdot}48$

and $n_2=50$, $\bar{x}_2=69{\cdot}51$, $\sigma_2=2{\cdot}48$

$\therefore$ $$|z|\simeq\frac{0{\cdot}91}{2{\cdot}48\sqrt{\frac{1}{60}+\frac{1}{50}}}\simeq 1{\cdot}92\ (<1{\cdot}96)$$

$\therefore$ Difference is insignificant and hence it cannot be said that the mean height of the seniors is greater than that of the new entrants.

Ex. 14·48. *A certain psychological test was given to two groups (samples) of army prisoners: (a) first offenders (b) recidivists. The sample statistics were as follows:*

Population	*Sample size*	*Sample mean*	*Sample s.d.*
(*i*) *First offenders*	*580*	*34·45*	*8·83*
(*ii*) *Recidivists*	*786*	*28·02*	*8·81*

Find the 95% confidence limits of the difference of the means for the two populations.

Sol. $$z=\frac{(34{\cdot}45-28{\cdot}02)-(\mu_1-\mu_2)}{\sqrt{\dfrac{(8{\cdot}83)^2}{580}+\dfrac{(8{\cdot}81)^2}{786}}}$$

$$=\frac{6{\cdot}43-(\mu_1-\mu_2)}{0{\cdot}48}\sim N(0,1)$$

$\therefore$ 95% confidence limits are given by

$$\left|\frac{6{\cdot}43-(\mu_1-\mu_2)}{0{\cdot}48}\right|<1{\cdot}96$$

i.e., $6{\cdot}43-(0{\cdot}48)(1{\cdot}96)<\mu_1-\mu_2<(0{\cdot}48)(1{\cdot}96)+6{\cdot}43.$

i.e., $5{\cdot}4892<\mu_1-\mu_2<7{\cdot}3708$

Ex. 14·49. *Two populations have the same mean, but the s.d. of one is twice that of the other. Show that in samples of 4500 each drawn under simple sampling conditions the difference of means will in all probability not exceed (0·1)σ where σ is the smaller s.d. and find the probability that the difference exceeds half that amount.*

Sol. Let μ be the common mean and σ, 2σ be the standard deviations of two populations. Let $\bar{x}_1$ and $\bar{x}_2$ be the sample means.

Then $$z=\frac{\bar{x}_1-\bar{x}_2}{\dfrac{\sigma^2}{4500}+\dfrac{(2\sigma)^2}{4500}}$$

$$=\frac{\bar{x}_1-\bar{x}_2}{\dfrac{\sigma}{30}}$$

Now $\quad |z| < 3 \quad$ (for all probability)

i.e., $\quad |\bar{x}_1 - \bar{x}_2| < \dfrac{\sigma}{10} = 0 \cdot 1\sigma.$

$$\begin{aligned} \text{Now } P\{|\bar{x}_1-\bar{x}_2| > 0{\cdot}05\sigma\} &= P\{|z| > 1{\cdot}5\} \\ &= 1 - P\{|z| < 1{\cdot}5\} \\ &= 1 - 2P\{0 < z < 1{\cdot}5\} \\ &= 1 - 2(0{\cdot}4332) = 0{\cdot}1336 \end{aligned}$$

Ex. 14-50. *In an intelligence test administrated to 60 boys and 100 girls, the following results were obtained :*

	Mean score	*S.D.*
Boys	*114*	*13*
Girls	*110*	*11*

Assuming the correlation coefficient between the two to be 0·75, test whether the difference between the means is significant.

Sol. S.E. of the difference between the means

$$= \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} - 2\frac{2r\sigma_1\sigma_2}{\sqrt{n_1 n_2}}}$$

$$= \sqrt{\frac{(13)^2}{60} + \frac{(11)^2}{100} - 2\frac{(0{\cdot}75)(13)(11)}{\sqrt{100 \cdot 60}}}$$

$$\simeq \sqrt{1{\cdot}2577} \simeq 1{\cdot}12$$

$$\therefore \quad |z| = \frac{4}{1{\cdot}12} \simeq 3{\cdot}6 (>3)$$

$\therefore$ Difference is significant.

EXERCISES

1. The data given below gives the mean and s.d. of stature of two groups of boys taken from a certain city :

Sample size	*Sample mean*	*Sample s.d.*
1145	48·6	2·416
654	50·79	2·53

Find whether the difference between the means significant.

[**Ans.** Significant]

2. 64 senior boys from college A and 81 senior boys from college B had mean heights of 68·2″ and 67·3″ respectively. If the s.d. for heights of all senior boys is 2·43″, is the difference between the two groups significant ?

($z = 2{\cdot}21$, significant at 5% level).

3. Two random samples of sizes 1000 and 1500 give following values of mean and s.d. :

Sample size	*Sample mean*	*Sample s.d.*
1000	47	28
1500	49	40

Test whether the difference between means is significant.
[**Ans.** No.]

4. Intelligence test on two groups of boys and girls, give the following results. Examine if the difference between means is significant.

	Sample mean	*Sample s.d.*	*Sample size*
Girls	84	10	121
Boys	81	12	81

[**Ans.** Not-significant]

5. Two samples of bricks, produced at two different works, were tested for transverse strength with the following results :

	Sample size	*Sample mean*	*Sample s.d.*
1st sample	300	990	240
2nd sample	200	1000	202

Is the difference between the means significant ?
[**Ans.** Not-significant]

6. Two populations have the same mean, but the standard deviation of one is twice that of the other. Show that in samples of 500 each drawn under simple random conditions, the difference of the means will in all probability not exceed $0{\cdot}3\sigma$ where σ is the smaller s.d. and assuming the distribution of the difference of the means to be normal, find the probability that it exceeds half that amount. [**Ans.** 0·1336]

14.4.6. Test of significance of difference between the standard deviations of two large samples.

Consider two large independent samples of sizes n_1, n_2 and standard deviations s_1 ; s_2 from two populations with standard deviations σ_1, σ_2 respectively.

The hypothesis to be tested is 'Are population standard deviations same '*i.e.*, $\sigma_1=\sigma_2$'. Assuming this hypothesis, the statistic

$$z=\frac{s_1-s_2}{\text{S.E.}(s_1-s_2)}\sim N(0,1)$$

for large samples. Now for large samples drawn from normal populations,

$$\text{var}(s_1)=\frac{\sigma_1^2}{2n_1} \text{ and var } (s_2)=\frac{\sigma_2^2}{2n_2}$$

$$\therefore \quad \text{var}(s_1-s_2)=\text{var}(s_1)+\text{var}(s_2)=\frac{\sigma_1^2}{2n_1}+\frac{\sigma_2^2}{2n_2}$$

$$\therefore \quad \text{S.E.}(s_1-s_2)=\sqrt{\frac{\sigma_1^2}{2n_1}+\frac{\sigma_2^2}{2n_2}}$$

$\therefore$ For large samples, $z=\dfrac{s_1-s_2}{\sqrt{\dfrac{\sigma_1^2}{2n_1}+\dfrac{\sigma_2^2}{2n_2}}}\sim N(0, 1)$

The significance is tested with the aid of normal curve.

Ex. 14·51. *Random samples drawn from two universities A and B gave the following data relating to the heights of male students :*

	Sample mean	*Sample s.d.*	*Sample size*
University A	*67·42*	*2·58*	*1000*
University B	*67·25*	*2·50*	*1200*

(*i*) *Is the difference between the means significant ?*

(*ii*) *Is the difference between the standard deviations significant ?*

Sol. Here $n_1=1000$, $\bar{x}_1=67{\cdot}42$, $s_1=2{\cdot}58$

and $\qquad n_2=1200$, $\bar{x}_2=67{\cdot}25$, $s_2=2{\cdot}50$

$$\therefore \qquad z=\frac{0{\cdot}17}{\sqrt{\dfrac{(2{\cdot}58)^2}{1000}+\dfrac{(2{\cdot}5)^2}{1200}}}\simeq 1{\cdot}56(<1{\cdot}96)$$

$\therefore$ There is no significant difference between sample means.

$$(ii) \qquad z=\frac{0{\cdot}08}{\sqrt{\dfrac{(2{\cdot}58)^2}{2000}+\dfrac{(2{\cdot}5)^2}{2400}}}\simeq 1{\cdot}04(<1{\cdot}96)$$

$\therefore$ Sample standard deviations are not significantly different.

Ex. 14·52. *The mean yields of two sets of plots and their variability are as given below. Examine* (*i*) *whether the difference in the mean yields of the two sets of plots is significant and* (*ii*) *whether the difference in the variability in yields is significant.*

	Set of 40 plots	*Set of 60 plots*
Mean yield per plot	*1258*	*1243*
S.D. per plot	*34*	*28*

Sol. (*i*) $$|z|=\frac{15}{\sqrt{\dfrac{(34)^2}{40}+\dfrac{(28)^2}{60}}}\simeq 2{\cdot}3(>2{\cdot}58)$$

Difference between the mean yields is significant at 1% level.

(*ii*) $$|z| = \frac{6}{\sqrt{\frac{(34)^2}{2(40)} + \frac{(28)^2}{2(60)}}} \simeq 1\cdot3 (< 1\cdot96)$$

Difference is not significant.

EXERCISES

1. Test whether the difference between the standard deviations is significant, given that

	Size	*s.d.*
Sample A	1,392	53·84
Sample B	630	56·56

[**Ans.** Not-significant]

2. Two samples of sizes 1000 and 800 gave the following results :

	Mean	*S.D.*
1st sample	17·5	2·5
2nd sample	18	2·7

Assuming that samples are independent, test whether the two samples may be regarded as drawn from the universes with same standard deviations. [**Ans.** Yes at 1% level]

Ex. 14·53. *Two samples of sizes 100 and 80 gave the following results :*

	Median	*S.D.*
1st sample	*85*	*7*
2nd sample	*100*	*8*

Test whether the difference between the medians is significant.

Sol. Let σ_1, σ_2 be the standard deviations of two samples of sizes n_1 and n_2. Then assuming the samples to be independent.

S.E. (e) of the difference between the medians

$$=(1\cdot25331)\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

Here $\sigma_1=7$, $\sigma_2=8$, $n_1=100$ and $n_2=80$

$\therefore$ $e \simeq 1\cdot42$

$\therefore$ $$|z| = \frac{\text{Difference between medians}}{e}$$

$$=\frac{15}{1\cdot42} \simeq 10\cdot6 (>3)$$

$\therefore$ Difference is highly significant.

□ □

15

Chi-Square Distribution

15.1. ψ^2-Distribution.

Let $x_1, x_2, \ldots x_n$ be n independent standard normal variates. Then, each one of

$$\frac{1}{2}x_1^2,\ \frac{1}{2}x_2^2, \ldots, \frac{1}{2}x_n^2$$

is gamma variate with parameter $\frac{1}{2}$.

$\therefore\quad \frac{1}{2}(x_1^2+x_2^2+\ldots+x_n^2)$ is a $\gamma\left(\frac{n}{2}\right)$ variate.

$\therefore$ If $\qquad \psi^2=x_1^2+x_2^2+\ldots+x_n^2,$

$\frac{\psi^2}{2}$ is a $\gamma\left(\frac{n}{2}\right)$ variate.

$\therefore$ Distribution of $\frac{\psi^2}{2}$ is

$$dP=\frac{1}{\Gamma\left(\frac{n}{2}\right)}e^{-\frac{\psi^2}{2}}\left(\frac{\psi^2}{2}\right)^{\frac{n}{2}-1}d\left(\frac{\psi^2}{2}\right)$$

i.e.,
$$dP=\frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}e^{-\frac{\psi^2}{2}}(\psi^2)^{\frac{n}{2}-1}d(\psi^2)$$

where $0\leqslant\psi^2<\infty$.

This distribution is known as chi-square distribution and ψ^2 is called chi-square variate. n is called the degrees of freedom associated with chi-square distribution.

Remark. (1) Normal distribution can be regarded as a particular case of chi-square distribution for $n=1$.

Hereafter chi-square distribution will be written as ψ^2-distribution.

(2) $x_1, x_2, \ldots, x_n$ can be represented by a sample point with co-ordinates $(x_1, x_2, \ldots, x_n)$ in Euclidean hyperspace of n dimensions. If these variates are subjected to a linear constraint, that constraint can be considered to represent a hyperplane. Thus the effect of this constraint is to lower the dimension by one and hence the number of degrees of freedom associated with ψ^2 will be $n-1$.

In general, if there are p independent linear constraints, the number of *d.f.* is $n-p$.

15.1.1. M.G.F. of ψ^2-distribution.

$$M_0(t)=E\{e^{t\psi^2}\}$$

$$=\frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\int_0^\infty e^{t\psi^2}\cdot e^{-\frac{1}{2}\psi^2}\,(\psi^2)^{\frac{n}{2}-1}\,d(\psi^2)$$

$$=\frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\int_0^\infty e^{-\frac{1}{2}(1-2t)\,\psi^2}\,(\psi^2)^{\frac{n}{2}-1}\,d(\psi^2)$$

Put $\frac{1}{2}(1-2t)\,\psi^2=y$

$$d(\psi^2)=\frac{2dy}{1-2t}$$

$$=\frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\int_0^\infty e^{-y}\left(\frac{2y}{1-2t}\right)^{\frac{n}{2}-1}\frac{2dy}{1-2t}$$

$$=\frac{1}{(1-2t)^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\int_0^\infty e^{-y}\,y^{\frac{n}{2}-1}\,dy$$

$$=\frac{1}{(1-2t)^{n/2}}\;\frac{1}{\Gamma\left(\frac{n}{2}\right)}\;\Gamma\left(\frac{n}{2}\right)$$

$$=(1-2t)^{-\frac{n}{2}},$$

which exists only when $|2t|<1$.

15.1.2. Moments and β, γ coefficients.

$$M_0(t)=(1-2t)^{-\frac{n}{2}}$$

$$=1+\frac{n}{2}(2t)+\frac{\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)}{2!}(2t)^2+\ldots\ldots$$

$$+\ldots+\frac{\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)\ldots\left(\frac{n}{2}+\overline{r-1}\right)}{r}(2t)^r+\ldots$$

$$\therefore \quad \mu'_r(0)=\text{coefficient of } \frac{t^r}{r!}$$

$$=2^r\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)\cdots\left(\frac{n}{2}+\overline{r-1}\right)$$

$$=n(n+2)\ldots(n+2\overline{r-1})$$

$$\therefore \quad \text{mean}=\mu_1'(0)=n$$

$$\mu_2'(0)=n(n+2)$$

$$\therefore \quad \mu_2=n(n+2)-n^2$$

$$=2n$$

which gives variance.

$$\mu_3'(0)=n(n+2)(n+4)$$

$$\mu_3=\mu_3'(0)-3\mu_2'(0)\mu_1'(0)+2\{\mu_1'(0)\}^3$$

$$=n(n+2)(n+4)-3n^2(n+2)+2n^3$$

$$=n(n^2+6n+8)-3(n^3+2n^2)+2n^3$$

$$=8n$$

$$\mu_4'(0)=n(n+2)(n+4)(n+6)$$

$$\mu_4=\mu_4'(0)-4\mu_3'(0)\mu_1'(0)+6\mu_2'(0)\{\mu_1'(0)\}^2-3\{\mu_1'(0)\}^4$$

$$=n(n+2)(n+4)(n+6)-4n^2(n+2)(n+4)+6n^3(n+2)-3n^4$$

$$=n^4+12n^3+44n^2+48n-4n^2(n^2+6n+8)+6n^3(n+2)-3n^4$$

$$=12n^2+48n$$

β, γ Coefficients

$$\therefore \quad \beta_1=\frac{\mu_3^2}{\mu_2^3}=\frac{8}{n}, \qquad \beta_2=\frac{\mu_4}{\mu_2^2}=3+\frac{12}{n}$$

$$\gamma_1=\sqrt{\beta_1}=\sqrt{\frac{8}{n}}, \qquad \gamma_2=\beta_2-3=\frac{12}{n}$$

15.1.3. Cumulative Function and Cumulants.

$$K_0(t)=\log M_0(t)$$

$$=\log (1-2t)^{-\frac{n}{2}}$$

$$=-\frac{n}{2}\log(1-2t)$$

$$=\frac{n}{2}\left\{2t+\frac{(2t)^2}{2}+\ldots+\frac{(2t)^r}{r}+\ldots\right\}$$

$\therefore\quad K_1(0)=\text{coeff. of } t=n$

$$K_r=\text{coefficient of } \frac{t^r}{r!}=2^{r-1}(r-1)!\,n,\ r\geqslant 2.$$

15.1.4. Mode.

The density function is

$$f(\psi^2)=\frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\,e^{-\frac{1}{2}\psi^2}\,(\psi^2)^{\frac{n}{2}-1}$$

$$f'(\psi^2)=\frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\left\{-\frac{1}{2}\,e^{-\frac{1}{2}\psi^2}\,(\psi^2)^{\frac{n}{2}-1}\right.$$

$$\left.+e^{-\frac{1}{2}\psi^2}\left(\frac{n}{2}-1\right)(\psi^2)^{\frac{n}{2}-2}\right\}$$

$$=\frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}\,e^{-\frac{1}{2}\psi^2}\,(\psi^2)^{\frac{n}{2}-2}\left\{-\frac{\psi^2}{2}+\frac{n}{2}-1\right\}$$

$\therefore\quad f'(\psi^2)=0\ \Rightarrow\ \psi^2=n-2,\ \ 0$

for $\psi^2=0$, $f(\psi^2)=0$ which is minimum value of $f(\psi^2)$.

$\therefore\quad$ for $\psi^2=n-2$, $f(\psi^2)$ is maximum.

$\therefore\quad$ Mode$=n-2$.

15.1.5. Limiting form of ψ^2 distribution.

Let $$z=\frac{\psi^2-n}{\sqrt{2n}}$$

Then $\quad M_0(t)_{\text{ of } z} = E\left\{ e^{tz} \right\}$

$$= e^{-\sqrt{\frac{n}{2}}\,t} \; E\left\{ e^{\frac{t\psi^2}{\sqrt{2n}}} \right\}$$

$$= e^{-\sqrt{\frac{n}{2}}\,t} \left\{ M\left(\frac{t}{\sqrt{2n}}\right) \text{ of } \psi^2 \right\}$$

$$= e^{-\sqrt{\frac{n}{2}}\,t} \left(1-\sqrt{\frac{2}{n}}\,t\right)^{-\frac{n}{2}}$$

$\therefore \quad \log\left\{M_0(t)_{\text{ of } z}\right\} = -\sqrt{\frac{n}{2}}\,t - \frac{n}{2}\log\left\{1-\sqrt{\frac{2}{n}}\,t\right\}$

$$= -\sqrt{\frac{n}{2}}\,t + \frac{n}{2}\left\{\sqrt{\frac{2}{n}}\,t + \frac{\left(\sqrt{\frac{2}{n}}\,t\right)^2}{2} + \frac{\left(\sqrt{\frac{2}{n}}\,t\right)^3}{3} + \ldots\right\}$$

$$= \frac{1}{2}t^2 + O\left(\frac{1}{\sqrt{n}}\right)$$

$$\to \frac{1}{2}t^2 \text{ as } n \to \infty$$

$\therefore \quad M_0(t)_{\text{ of } z} \to e^{\frac{1}{2}t^2}$ as $n \to \infty$.

$\therefore \quad z$ and hence ψ^2 tends to normal variate as $n \to \infty$.

15.1.6. Additive Property of ψ^2-variates.

Theorem. The sum of any finite number of independent ψ^2-variates is a ψ^2-variate.

Proof. Let $\psi_1^2, \psi_2^2, \ldots\ldots \psi_n^2$ be n independent ψ^2-variates with $n_1, n_2, \ldots\ldots n_n$ degrees of freedom respectively.

Then $\quad M_0(t)_{\text{ of } \psi_i^2} = (1-2t)^{n_i/2}, \; i=1, 2, \ldots, n$

Let $\qquad x = \psi_1^2 + \psi_2^2 + \ldots\ldots + \psi_n^2$

Then,

$M_0(t)_{\text{ of } \psi^2} = E\left\{ e^{t\psi^2} \right\}$

$$= E\left\{ e^{t(\psi_1^2 + \ldots\ldots + \psi_n^2)} \right\}$$

$$=E\{e^{t\psi_1^2}\}\,E\{e^{t\psi_2^2}\}\ldots\ldots E\{e^{t\psi_n^2}\}$$

$$=(1-2t)^{-\frac{n_1}{2}}\,.\,(1-2t)^{-\frac{n_2}{2}}\,\ldots\ldots(1-2t)^{-\frac{n_n}{2}}$$

$$=(1-2t)^{-\left(\frac{n_1+n_2+\ldots+n_n}{2}\right)}$$

which is the m.g.f. of a ψ^2-variate with $(n_1+n_2+\ldots+n_n)$ d.f.

$\therefore$ x^2 is a ψ^2-variate with $(n_1+\ldots+n_n)$ d.f.

Ex. 15·1. *If ψ_1^2 and ψ_2^2 are two independent ψ^2-variates with n_1 and n_2 d.f. respectively, then ψ_1^2/ψ_2^2 is a $\beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ variate.*

Sol. Distributions of ψ_1^2 and ψ_2^2 respectively are

$$dP=\frac{1}{2^{n_1/2}\,\Gamma\left(\frac{n_1}{2}\right)}\,e^{-\frac{\psi_1^2}{2}}\,(\psi_1^2)^{\frac{n_1}{2}-1}\,d(\psi_1^2)$$

$$0<\psi_1^2<\infty$$

and

$$dP=\frac{1}{2^{n_2/2}\,\Gamma\left(\frac{n_2}{2}\right)}\,e^{-\frac{1}{2}\psi_2^2}\,(\psi_2^2)^{\frac{n_2}{2}-1}\,d(\psi_2^2)$$

$$0<\psi_2^2<\infty$$

The joint distribution of ψ_1^2 and ψ_2^2 is

$$dP=\frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\,e^{-\frac{1}{2}(\psi_1^2+\psi_2^2)}$$

$$(\psi_1^2)^{\frac{n_1}{2}-1}\,(\psi_2^2)^{\frac{n_2}{2}-1}\,d\psi_1^2\,d\psi_2^2$$

$$0<\psi_1^2,\ \psi_2^2<\infty$$

Put $\qquad x=\dfrac{\psi_1^2}{\psi_2^2}, \qquad y=\psi_2^2$

$\therefore \qquad \psi_1^2=x\,y, \qquad \psi_2^2=y$

$$\therefore \qquad \frac{\partial(\psi_1^2, \psi_2^2)}{\partial(x, y)}=\begin{vmatrix}\dfrac{\partial\psi_1^2}{\partial x} & \dfrac{\partial\psi_1^2}{\partial y}\\[2ex] \dfrac{\partial\psi_2^2}{\partial x} & \dfrac{\partial\psi_2^2}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix}$$

$$=y$$

$\therefore$ The joint distribution of x and y is

$$dP=\frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\,e^{-\frac{1}{2}y(1+x)}\,(xy)^{\frac{n_1}{2}-1}\,.\,y^{\frac{n_2}{2}-1}\,.\,y\,dx\,dy$$

$$=\frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\,e^{-\frac{1}{2}y(1+x)}\,x^{\frac{n_1}{2}-1}\,y^{\frac{n_1+n_2}{2}-1}\,dx\,dy$$

The range of x and y are from 0 to ∞.

$\therefore$ Marginal distribution of x is

$$\frac{x^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}}\,\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\,dx\int_{y=0}^{\infty}e^{-\frac{1}{2}(1+x)\,y}\,y^{\frac{n_1+n_2}{2}-1}\,dy$$

$$=\frac{x^{\frac{n_1}{2}-1}\,dx}{2^{\frac{n_1+n_2}{2}}\,\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{2}{1+x}\right)^{\frac{n_1+n_2}{2}}\int_0^{\infty}e^{-u}\,u^{\frac{n_1+n_2}{2}-1}\,du$$

where $u=\frac{1}{2}(1+x)\,y$

$$=\frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\;\frac{x^{\frac{n_1}{2}-1}}{(1+x)^{\frac{n_1+n_2}{2}}}\,dx$$

$$=\frac{1}{\left(\frac{n_1}{2}\,,\,\frac{n_2}{2}\right)}\;\frac{x^{\frac{n_1}{2}-1}}{(1+x)^{\frac{n_1}{2}+\frac{n_2}{2}}}\,dx$$

$\Rightarrow$ x is a $\beta_2\left(\frac{n_1}{2}\,,\,\frac{n_2}{2}\right)$ variate.

Ex. 15-2. *If ψ_1^2 and ψ_2^2 are independent ψ^2-variates with n_1 and n_2 d.f. respectively, show that*

$$\frac{\psi_1^2}{\psi_1^2+\psi_2^2} \quad \textit{and} \quad \psi_1^2+\psi_2^2$$

are independent. Hence find their distributions.

Sol. The joint dist. of ψ_1^2 and ψ_2^2 is

$$dP=\frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\,e^{-\frac{1}{2}(\psi_1^2+\psi_2^2)}$$

$$(\psi_1^2)^{\frac{n_1}{2}-1}\,(\psi_2^2)^{\frac{n_2}{2}-1}\,d\psi_1^2\,d\psi_2^2$$

$$0<\psi_1^2,\,\psi_2^2<\infty$$

Put $\qquad x=\dfrac{\psi_1^2}{\psi_1^2+\psi_2^2}$ and $y=\psi_1^2+\psi_2^2$

$\therefore \qquad \psi_1^2=xy$ and $\qquad \psi_2^2=y(1-x)$

$$\frac{\partial(\psi_1^2,\,\psi_2^2)}{\partial(x,\,y)}=\begin{vmatrix}\dfrac{\partial\psi_1^2}{\partial x} & \dfrac{\partial\psi_1^2}{\partial y}\\[2ex] \dfrac{\partial\psi_2^2}{\partial x} & \dfrac{\partial\psi_2^2}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix} y & x\\ -y & 1-x\end{vmatrix}=y$$

$\therefore$ The joint dist. of x and y is

$$dP=\frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\,e^{-\frac{1}{2}y}\,(xy)^{\frac{n_1}{2}-1}$$

$$\{y(1-x)\}^{\frac{n_2}{2}-1}\,y\,dx\,dy$$

$$=\left\{\frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma\left(\frac{n_1+n_2}{2}\right)}e^{-\frac{1}{2}y}\,y^{\frac{n_1+n_2}{2}-1}\,dy\right\}$$

$$\left\{\frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}x^{\frac{n_1}{2}-1}(1-x)^{\frac{n_2}{2}-1}\,dx\right\}$$

$\Rightarrow$ x and y are independent. Marginal distributions of x and y respectively are

$$\frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}x^{\frac{n_1}{2}-1}(1-x)^{\frac{n_2}{2}-1}\,dx$$

and
$$\frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma\left(\frac{n_1+n_2}{2}\right)}e^{-\frac{1}{2}y}.\,y^{\frac{n_1+n_2}{2}-1}\,dy$$

$\therefore$ x is a $\beta_1\left(\frac{n_1}{2},\frac{n_2}{2}\right)$

and y is a ψ^2-variate with (n_1+n_2) d.f.

Ex. 15.3. *For a ψ^2-variate with n d. f. show that*

$$\mu_{r+1}=2r(\mu_r+n\mu_{r-1}),\quad r>1$$

Sol. For a ψ^2-variate with n *d. f.*

mean$=n$

$\therefore$ M.G.F. about mean is given by

$$\therefore\qquad M_{\overline{\psi^2}}(t)=E\{e^{t(\psi^2-n)}\}$$

$$=e^{-nt}M_0(t)$$
$$=e^{-nt}(1-2t)^{-n/2}$$

$$\therefore\ \log\left\{M_{\overline{\psi_2}}(t)\right\}=-nt-\frac{n}{2}\log(1-2t)$$

Differentiating w.r.t. 't'

$$\frac{1}{M_{\psi^2}(t)} M'_{\psi^2}(t) = -n + \frac{n}{2}\,\frac{2}{1-2t}$$

$$= \frac{2nt}{1-2t}$$

$$\Rightarrow \quad (1-2t)M'_{\psi^2}(t) = 2nt\, M_{\psi^2}(t)$$

Dtfferentiating r times w.r.t. 't' by Leibnitz's theorem

$$(1-2t)\, M^{r+1}_{\psi^2}(t) - 2r\, M^{r}_{\psi^2}(t) = 2n\{t\, M^{r}_{\psi^2}(t) + r\, M^{r-1}_{\psi^2}(t)\}$$

Now $\quad \mu_r = \{M^{r}_{\psi^2}(t)\}_{t=0}$ ...(1)

$\therefore$ Substituting $t=0$ in (1)

$$\mu_{r+1} - 2r\mu_r = 2nr\mu_{r-1}$$

$$\Rightarrow \quad \mu_{r+1} = 2r\{\mu_r + n\mu_{r-1}\}$$

15.1.7. Chief Features of the chi-square Probability curve.

The eq. of the ψ^2 probability curve is with n d.f. is

$$y = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}\, e^{-\frac{1}{2}\psi^2}\, (\psi^2)^{\frac{n}{2}-1}$$

$$\therefore \quad \log y = -\frac{1}{2}\psi^2 + \left(\frac{n}{2}-1\right)\log\psi^2 - \log 2^{n/2} - \log\Gamma\left(\frac{n}{2}\right)$$

Differentiating w.r.t. ψ^2

$$\frac{1}{y}\,\frac{dy}{d\psi^2} = -\frac{1}{2} + \left(\frac{n}{2}-1\right)\frac{1}{\psi^2}$$

$$= \frac{1}{2}\left\{\frac{(n-2)-\psi^2}{2\psi^2}\right\}.$$

Since $\psi^2 > 0$, $y > 0$, we have

for $\quad n = 1, 2, \qquad \dfrac{dy}{d\psi^2} < 0$

and for $n > 2$,

$$\frac{dy}{d\psi^2} > 0 \quad \text{if } 0 < \psi^2 < n-2$$

$$= 0 \quad \text{if } \psi^2 = n-2$$

$$< 0 \quad \text{if } \psi^2 > n-2$$

$\therefore$ For $n=1, 2$, y decreases continuously as ψ^2 increases

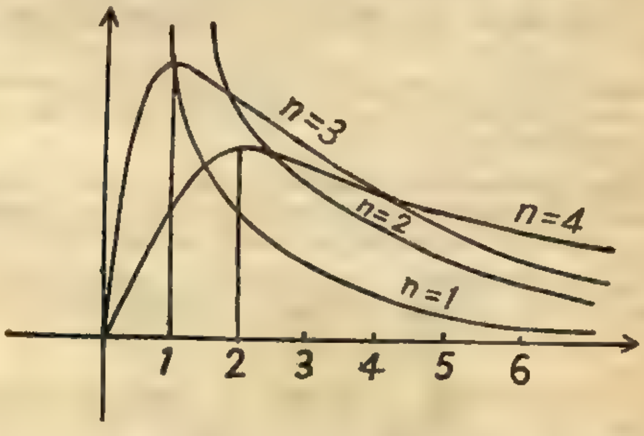
and for $n>2$, y increases or decreases as ψ^2 increases according as $\psi^2<n-2$ or $\psi^2>n-2$

and for $\psi^2=n-2$, $\dfrac{dy}{d\psi^2}=0$ which implies that y is maximum.

$\therefore$ For all values of n, $y \to 0$ as $\psi^2 \to \infty$.

$\therefore$ ψ^2-axis is an asymptote to the curve.

The shape of curve for $n=1, 2, 3\ldots$ is shown below :



Ex. 15·4. *If ψ^2 is a chi-square variate with n d.f., show that if n is large $\sqrt{2\psi^2}$ is normally distributed about mean $\sqrt{2n-1}$ with variance unity.*

Sol. Now $\sqrt{2\psi^2} \leqslant \sqrt{2n-1}+z$

if $$2\psi^2 \leqslant (2n-1)+z^2+2\sqrt{2n-1}\,z$$

i.e., $$\psi^2 \leqslant n-\frac{1}{2}+\frac{1}{2}z^2+\sqrt{2n-1}\,z$$

i.e., $$\frac{\psi^2-n}{\sqrt{2n}} \leqslant -\frac{1}{2\sqrt{2n}}+\frac{1}{2\sqrt{2n}}z^2+\sqrt{1-\frac{1}{2n}}\cdot z$$

i.e., $$\frac{\psi^2-n}{\sqrt{2n}} \leqslant z, \text{ as } n \text{ is large}$$

$$\therefore \quad P\left\{\frac{\psi^2-n}{\sqrt{2n}} \leqslant z\right\} \approx P\left\{\sqrt{2\psi^2} \leqslant \sqrt{2n-1}+z\right\}$$

$$=P\left\{\sqrt{2\psi^2}-\sqrt{2n-1} \leqslant z\right\}$$

Now $$\frac{\psi^2-n}{\sqrt{2n}}\sim N(0, 1)$$

$\therefore$ $$\sqrt{2\psi^2}-\sqrt{2n-1}\sim N(0, 1)$$

$\rightarrow$ $$\sqrt{2\psi^2}\sim N(\sqrt{2n-1}, 1).$$

Ex. 15·5. *If ψ^2 is a chi-square variate with n.d.f., show that if n is large $\sqrt{2\psi^2}$ is normally distributed about mean $\sqrt{2n}$ with variance unity.*

15·2. ψ^2-tests.

Tests of significance based on ψ^2-distribution are called ψ^2-tests.

Cells. *When a given data is arranged in compartments, the compartments are called cells and the corresponding frequency is called* **Cell Frequency**.

Linear Constraints. *Constraints which involve linear equations in the cell frequencies (i.e., equations containing no squares or higher powers of the frequencies) are called linear constraints.*

Degrees of Freedom. *It is the greatest number of cell frequencies which can be assigned arbitrarily. It is given by*

$$v=n-k$$

where n is the total number of cells and k the number of independent constraints.

Definition of ψ^2. *If O_i and e_i by the observed and expected frequencies, the variate ψ^2 is defined by*

$$\psi^2=\sum_i \frac{(O_i-e_i)^2}{e_i}$$

This variate follows ψ^2-distribution as seen below :

Let there be a random sample of size n whose members are distributed at random in k cells.

Let p_i=prob. that a member is in ith cell. Then, the prob. that 0_1 members are in 1st cell, 0_2 members in 2nd cell etc. is given by

$$P=\frac{n\,!}{0_1\,!\,0_2!\ldots 0_k\,!}\,p_1^{\,0_1}\,p_2^{\,0_2}\ldots\ldots p_k^{\,0_k}$$

Also $0_1+0_2+\ldots\ldots+0_k=n$ ...(1)

If n is sufficiently large so that $0_1, 0_2\ldots\ldots 0_k$ are not small, Stirling's approximation for factorials can be used.

$$\therefore \qquad P \approx \frac{\sqrt{2\pi}\, e^{-n} n^{n+\frac{1}{2}}}{\prod_{i=1}^{k} \left\{ \sqrt{2\pi}\, e^{-0_i}\, 0_i^{0_i+\frac{1}{2}} \right\}} p_1^{0_1} p_2^{0_2} p_k^{0_k}$$

$$= \frac{1}{(2\pi)^{\frac{k-1}{2}}} \frac{1}{n^{\frac{k-1}{2}}} \frac{1}{(p_1 p_2 \cdots p_k)^{\frac{1}{2}}} \left(\frac{np_1}{0_1} \right)^{0_1+\frac{1}{2}} \cdots\cdots \left(\frac{np_k}{0_k} \right)^{0_k+\frac{1}{2}}$$

$$= c \prod_{i=1}^{k} \left(\frac{np_i}{0_i} \right)^{0_i+\frac{1}{2}}$$

where $\qquad c = \dfrac{1}{(2\pi)^{\frac{k-1}{2}}\, n^{\frac{k-1}{2}}\, (p_1 p_2 \cdots\cdots p_k)^{\frac{1}{2}}}$

$$\therefore \quad \log P \approx \log c + \sum_{i=1}^{k} \left(0_i + \frac{1}{2} \right) \log \frac{np_i}{0_i}$$

Now $\qquad e_i =$ expected frequency of ith cell

$\qquad = np_i$

Let $\qquad \xi_i = \dfrac{(0_i - e_i)}{\sqrt{e_i}}$

$\therefore \qquad 0_i = \sqrt{e_i}\, \xi_i + e_i$

$$\therefore \quad \log P \approx \log c + \sum_{i=1}^{k} \left\{ e_i + \sqrt{e_i}\, \xi_i + \frac{1}{2} \right\} \log \left(\frac{e_i}{e_i + \xi_i \sqrt{e_i}} \right)$$

$$\therefore \quad \log \frac{P}{c} \approx \sum_{i=1}^{k} \left\{ e_i + \sqrt{e_i}\, \xi_i + \frac{1}{2} \right\} \log \left\{ \frac{1}{1 + \frac{\xi_i}{\sqrt{e_i}}} \right\}$$

$$\approx - \sum_{i=1}^{k} \left\{ e_i + \sqrt{e_i}\, \xi_i + \frac{1}{2} \right\} \log \left\{ 1 + \frac{\xi_i}{\sqrt{e_i}} \right\}$$

If e_i is large, ξ_i will be small as compared to $\sqrt{e_i}$ and hence the expansion of $\log \left\{ 1 + \dfrac{\xi_i}{\sqrt{e_i}} \right\}$ is valid.

$\therefore$ Assuming e_i large,

$$\log \frac{P}{c} \approx -\sum_{i=1}^{k}\left\{e_i+\sqrt{e_i}\,\xi_i+\frac{1}{2}\right\}\left\{\frac{\xi_i}{\sqrt{e_i}}-\frac{1}{2}\,\frac{\xi_i^2}{e_i}+\ldots\ldots\right\}$$

$$\approx -\sum_{i=1}^{k}\left\{\xi_i\sqrt{e_i}+\frac{1}{2}\xi_i^2+O\left(\frac{1}{\sqrt{e_i}}\right)\right\}$$

Now $$\sum_{i=1}^{k}\xi_i\sqrt{e_i}=\sum_{i=1}^{k}\left\{O_i-e_i\right\}$$

$$=\sum_{i=1}^{k}O_i-\sum_{i=1}^{k}e_i=0 \qquad \ldots(1)$$

$\therefore$ Neglecting small quantities $\sum_{i=1}^{k}O\left(\frac{1}{\sqrt{e_i}}\right)$,

$$\log \frac{P}{c} \approx -\frac{1}{2}\sum_{i=1}^{k}\xi_i^2$$

$$\Rightarrow \quad P \approx c e^{-\frac{1}{2}\sum_{i=1}^{n}\xi_i^2}$$

$\Rightarrow$ Each ξ_i is distributed as $N(0, 1)$.

Also ξ_i's are connected by linear relation (1).

$$\therefore \qquad \chi^2=\sum_{i=1}^{k}\frac{(O_i-e_i)^2}{e_i}=\sum_{i=1}^{k}\xi_i^2$$

is distributed as χ^2-variate with $(k-1)$ d.f.

Conditions for the Application of χ^2 test

(*i*) *The members of the sample must be independent.*

(*ii*) *Constraints on the cell-frequency, if any, should be linear.*

(*iii*) *N, the total frequency must be reasonably large. N should be at least 50, however few the number of cells.*

(iv) No expected or theoretical cell frequency should be less than 5. It is better if it is greater than or equal to 10.

Note. *If any expected cell frequency is less than 5, then to apply ψ^2-test this cell is to be merged with the proceeding or succeeding cells so that the new cell frequency (obtained on adding the cell frequencies of cells merged) is more than 5.*

Rules of Decision

Let $$P = P(\psi^2 > \psi_0^2)$$

For various fixed values of P and for degrees of freedom n ranging from 1 to 30, values of ψ_0^2 are tabulated in the form of ψ^2 table. For $n > 30$, a property that ψ^2 is a normal variate is used. Thus rule of decision is as below :

Values of ψ^2 at specified levels of significance for given degrees of freedom are seen from the tables. Generally 5% and 1% levels are taken. If $\psi^2_{cal} < \psi^2_{0.05}$, the hypothesis is acceptable at the 5% level of significance otherwise non-acceptable. Similarly for 1% level of significance.

Alternately, the probability P is determined. If this is small, the hypothesis is rejected and if this is not small the hypothesis is accepted.

Remarks. (1) If $\psi^2 = 0$, $0_i = e_i \; \forall \; i$

i.e., observed and expected frequencies coincide. On the other hand, if observed and expected frequencies differ greatly, ψ^2 is large. Thus ψ^2 gives a measure of correspondence between theory and experiment.

(2) Not only small values of P lead us to suspect the hypothesis but the value of P very near to unity may also lead to a similar result.

(3) ψ^2-test depends only on the set of observed and expected frequencies and on degrees of freedom. It does not make any assumptions regarding the parent population. Since ψ^2-variate does not involve any population parameter, this test is known as **Non-parametric test.**

(4) An alternate expression for ψ^2 is as below :

$$\psi^2 = \sum_i \frac{(0_i - e_i)^2}{e_i}$$

$$= \sum_i \left\{ \frac{0_i^2}{e_i} + e_i - 20_i \right\}$$

$$= \sum_i \frac{0_i^2}{e_i} + \sum_i e_i - 2\sum_i 0_i$$

$$=\sum_i \frac{0_i^2}{e_i}-\sum_i 0_i. \qquad (\because \; \Sigma e_i=\Sigma 0_i)$$

(5) The value ψ_0^2 is called critical value.

Uses of ψ^2-test

Some of the uses of the ψ^2-test are :

(*i*) *To test the goodness of fit.*

(*ii*) *To test the independence of attributes.*

(*iii*) *To test for variance of a normal population.*

(*iv*) *To test the homogeneity of several independent estimates of the population variance.*

(*v*) *To test the homogeneity of several independent estimates of population correlation co-efficient.*

(*vi*) *To combine various probabilities obtained from independent experiments to give a single test of significance.*

Note. *Here only* (*i*) *and* (*ii*) *will be considered.*

15.2.1. The Test of Goodness of Fit

One of the principal uses of ψ^2 distribution is to test how well an observed distribution fits a theoretical one. When ψ^2-test is used in this way it is called the test of "goodness of fit". The expression within inverted commas may be used in two ways. In the first place it may describe the "fit" of observed to the hypothetical data. In the second it may be used, without reference to a hypothesis, merely to test the merits of a particular formula or a particular curve in graduating a set of values or a series of points, *e.g.*, it may be tested how well a binomial distribution or normal distribution or Poisson distribution fits the given data. The calculations in both the cases are exactly on the same lines.

Ex. 15·6. *In experiments on pea-breeding, Mendal got the following frequencies of seeds : 315 round and yellow ; 101 wrinkled and yellow ; 108 round and green ; 32 wrinkled and green. Theory predicts that the frequencies should be in the proportions 9 : 3 : 3 : 1. Test the correspondence between theory and experiment.*

Sol. Total frequency$=315+101+108+32=556$.

$\therefore$ Expected number of round and yellow seeds$=\dfrac{9}{16}\,556\simeq 313$

Expected number of wrinkled and yellow seeds$=\dfrac{3}{16}\,556\simeq 104$

Expected number of round and green seeds$=\dfrac{3}{16}\,555\simeq 104$

Expected number of wrinkled and green seeds $= \frac{1}{16} \cdot 556 = 35$

$$\therefore \quad \psi^2 = \frac{(313-315)^2}{313} + \frac{(101-104)^2}{104} + \frac{(108-104)^2}{104} + \frac{(32-35)^2}{35}$$

$$= 0{\cdot}013 + 0{\cdot}087 + 0{\cdot}154 + 0{\cdot}257 = 0{\cdot}5.$$

Since there are four expected frequencies, number of *d. f.*

$$= 4 - 1 = 3.$$

From tables $\psi_{0\cdot 05}$ for 3 *d. f.* $= 7{\cdot}815$

Now $\qquad \psi_{cal}^2 < \psi_{0\cdot 05}^2$

$\therefore$ The difference between expected and observed frequencies is not significant at 5% level of significance.

$\therefore$ Experiment is in agreement with the theory.

Ex. 15·7. *A genetical law says that children having one parent of blood group M and the other parent of blood group N will always be one of the three blood groups M, MN and N, and that the average numbers of children in these groups will be in the ratio 1 : 2 : 1. The report on an experiment states as follows : of 162 children having one M parent and one N parent, 28·4% were found to be of group M, 42% of group MN and the rest of the group N.' Do the data in the report conform to the expected genetic ratio 1 : 2 : 1.*

Sol. Total freq. $= 162$.

Observed frequencies are

$$\frac{28{\cdot}4}{100} \cdot 162 = 46, \ \frac{42}{100} \cdot 162 = 68 \text{ and } 162 - 46 - 68 = 48$$

and expected frequencies are.

$$\frac{1}{4} \cdot 162 = 40{\cdot}5, \ \frac{2}{4} \cdot 162 = 81 \text{ and } 40{\cdot}5$$

$$\therefore \quad \psi^2 = \frac{(5{\cdot}5)^2}{40{\cdot}5} + \frac{(13)^2}{81} + \frac{(7{\cdot}5)^2}{40{\cdot}5}$$

$$= 4{\cdot}2.$$

No. of *d. f.* $= 3 - 1 = 2$

Now $\qquad \psi_{0\cdot 05}^2$ for 2 *d.f.* $= 5{\cdot}99$

$\therefore \qquad \psi_{cal}^2 < \psi_{0\cdot 05}^2$

$\therefore$ Hypothesis may be correct and hence genetical law appears to be correct.

Ex. 15·8. *300 digits were chosen at random and found to give the following distribution :*

Digit	0	1	2	3	4	5	6	7	8	9
Freq.	18	32	28	34	42	50	17	23	27	29

Test the hypothesis that the digits were distributed in equal numbers in the table from which the data were collected.

Sol. On the assumption that digits are distributed in equal numbers in the table, expected frequency of each class

$$=\frac{300}{10}=30$$

$$\therefore \quad \psi^2=\frac{1}{30}\{(12)^2+2^2+2^2+4^2+(12)^2+(20)^2+(13)^2+7^2+3^2+1^2\}$$

$$=31{\cdot}3$$

$$\text{No. of } d.f.=10-1=9$$

From tables, $\psi_{0\cdot05}^2$ for 9 $d.f.=16{\cdot}92$

$$\therefore \qquad \psi_{cal}^2 > \psi_{0\cdot05}^2$$

$\therefore$ Assumption is wrong.

Ex. 15.9. *200 digits were chosen at random from a set of tables. The frequencies of digits were :—*

Digits	0	1	2	3	4	5	6	7	8	9
Freq.	18	19	23	21	16	25	22	20	21	15

Use ψ^2 test to assess the correctness of hypothesis that the digits were distributed in equal numbers in the table. Given that the values of ψ^2 are respectively 16·9, 18·3 and 19·7 for 9, 10 and 11 degrees of freedom at 5% level of significance.

Sol. Set the hypothesis 'The digits were distributed in equal numbers in the table'. Then expected frequency of each digit

$$=\frac{200}{10}=20$$

$$\therefore \quad \psi^2=\frac{1}{20}\{4+1+9+1+16+25+4+0+1+25\}$$

$$=\frac{86}{20}=4{\cdot}3$$

No. of $d.f.=10-1=9$

Now $\psi_{0\cdot05}^2$ for 9 $d.f.=16{\cdot}9$

$$\therefore \qquad \psi_{cal}^2 < \psi_{0\cdot05}^2$$

$\therefore$ Data is consistent with the hypothesis and hence the hypothesis may be correct.

Ex. 15·10. *In 120 throws of a single die, the following distribution of faces were obtained :*

Faces	1	2	3	4	5	6	*Total*
Freq.	30	25	18	10	22	15	120

Test whether these results constitute a refutation of the 'equal probability' hypothesis.

Sol. Set the 'equal probability' hypothesis. Then expected frequency of each face

$$=\frac{120}{6}=20$$

$$\therefore \quad \psi^2=\frac{1}{20}\ \{100+25+4+100+4+25\}=12\cdot9$$

$$\text{No. of } d.f.=6-1=5$$

$$\therefore \quad \psi_{0\cdot05}^2=11\cdot07$$

$$\therefore \quad \psi_{cal}^2 > \psi_{0\cdot05}^2$$

$\therefore$ The hypothesis is wrong.

Ex. 15.11. *The following figures show the distribution of digits in numbers chosen at random from a telephone directory :*

Digit	0	1	2	3	4	5	6	7	8	9	
Freq.	1026	1107	997	966	1075	933	1107	972	964	853	=10,000

Test whether the digits may be taken to occur equally frequently in the directory.

Sol. Set the hypothesis 'Digits occur equally frequently in the directory'. Then expected frequency of each digit

$$=\frac{10,000}{10}=1,000$$

$$\therefore \quad \psi^2=\frac{1}{1000}\ \{(26)^2+(107)^2+(3)^2+(34)^2+(75)^2+(67)^2$$
$$+(107)^2+(28)^2+(36)^2+(47)^2\}=39\cdot142$$

$$\text{No. of } d.f.=10-1=9$$

Now from tables, $\psi_{0\cdot05}^2$ for 9 $d.f.=16\cdot92$

$$\psi_{cal}^2 > \psi_{0\cdot05}^2$$

$\therefore$ Hypothesis is certainly wrong and hence digits can't be taken to occur equally frequently in the directory.

Ex. 15-12. *In the construction of a table of random numbers, 15,000 digits were taken from some logarithm tables and the numbers of each digit obtained were as follows :*

Digit.	0	1	2	3	4	5	6	7	8	9
Freq.	1439	1441	1461	1452	1494	1454	1613	1491	1482	1519

Use ψ^2-test to assess the correctness of the hypothesis that each digit had an equal chance of being chosen.

Sol. Assuming that each digit had an equal chance of being chosen, expected frequency of each digit

$$=1500$$

$$\therefore \quad \psi^2=\frac{1}{1500}\{(61)^2+(59)^2+(39)^2+(48)^2+6^2+(46)^2+(113)^2$$
$$+9^2+(18)^2+(19)^2\}=17{\cdot}8$$

No. of *d. f.* $=10-1=9$.

Now $\psi_{0\cdot05}^2$ for 9 *d. f.* $=16{\cdot}92$.

$$\therefore \quad \psi_{cal}^2>\psi_{0\cdot05}^2$$

$\therefore$ Hypothesis is wrong.

Ex. 15.13. *Five dice were thrown 96 times and the number of times 4, 5 or 6 was thrown are given below :*

No. of dice showing 4, 5 or 6	5	4	3	2	1	0
Freq.	8	18	35	24	10	1

Calculate ψ^2.

Sol. The probability of getting a 4, 5 or 6 in a throw of a single die

$$=\frac{1}{2}.$$

$\therefore$ By B.D., the expected frequencies are the successive terms in the binomial expansion of

$$96\left(\frac{1}{2}+\frac{1}{2}\right)^5$$

$\therefore$ Expected frequencies are

3, 15, 30, 30, 15, 3

Since the border frequencies are small these are to be combined with the adjacent ones. Doing so

Observed freq.	26	35	24	11
Expected freq.	18	30	30	18

$$\therefore \quad \psi^2=\frac{64}{18}+\frac{25}{30}+\frac{36}{30}+\frac{49}{18}=8{\cdot}31$$

Ex. 15.14. *Twelve dice were thrown 4096 times and a throw of 6 was reckoned as a success ; the observed frequencies are given below :*

No. of successes	0	1	2	3	4	5	6	7 *and over*
Freq.	447	1145	1181	796	380	115	24	8

Find the value of ψ^2 *on the hypothesis that dice were unbiased and hence show that the data are consistent with the hypothesis so far as the* ψ^2*-test is concerned.*

Sol. On the hypothesis of unbiased dice the theoretical frequencies are the successive terms in the binomial expansion of

$$4096\left(\frac{5}{6}+\frac{1}{6}\right)^{12}$$

as the probability of success with a throw of one die is $\frac{1}{6}$

$\therefore$ Expected frequencies are

459 ; 1102 ; 1212 ; 808 ; 364 ; 116 ; 27 and 8.

$$\therefore\ \psi^2=\frac{(12)^2}{459}+\frac{(43)^2}{1102}+\frac{(31)^2}{1212}+\frac{(12)^2}{808}+\frac{(16)^2}{364}+\frac{1^2}{116}+\frac{3^2}{27}+\frac{(8-8)^2}{8}$$

$\simeq 4{\cdot}00$

No. of $d.f.=8-1=7$

Now $\psi_{0\cdot05}{}^2$ for 7 $d.f.=14{\cdot}07$

$\therefore$ $\psi_{cal}{}^2<\psi_{0\cdot05}{}^2$

$\therefore$ The data are consistent with the hypothesis.

Ex. 15.15. *A set of 6 similar coins is tossed 640 times with the following results :*

No. of heads	0	1	2	3	4	5	6
Freq.	7	64	140	210	132	75	12

Calculate the binomial frequencies on the assumption that the coins are symmetrical and test the hypothesis.

Sol. On the assumption that coins are unbiased, the expected frequencies are given by the successive terms in the binomial expansion of

$$640\left(\frac{1}{2}+\frac{1}{2}\right)^6=10(1+1)^6$$

$$=10\left(1+6+\frac{6.5}{2}+\frac{6.5.4}{3.2}+\frac{6.5.4.3}{4.3.2.1}+\frac{6.5.4.3.2}{5.4.3.2.1}+1\right)$$

$\therefore$ Expected frequencies are :

10, 60, 150, 200, 150, 60, 10

$$\therefore\ \psi^2=\frac{3^2}{10}+\frac{4^2}{60}+\frac{(10)^2}{150}+\frac{(10)^2}{200}+\frac{(18)^2}{150}+\frac{(15)^2}{60}+\frac{2^2}{10}=8{\cdot}6$$

No. of $d.f.=7-1=6$

Now $\psi_{0\cdot05}{}^2$ for 6 $d.f.=12{\cdot}59$

$\therefore$ $\psi_{cal}{}^2<\psi_{0\cdot05}{}^2$

$\therefore$ Assumption may be correct.

Ex. 15.16. *12 dice were rolled 26306 times and each time the number of dice which had 5 or 6 on the uppermost face was recorded. The results are given in the form of the following table :*

No. of dice showing 5 or 6	0	1	2	3	4	5	6	7	8
Freq.	185	1149	3265	5475	6114	5194	3067	1331	403

9	10	11	12
105	14	4	—

Fit a binomial dist and test for goodness of fit.

Sol. From the data,

$$\text{A.M.} = \frac{1}{26306}\{1149+6530+16425+24456+25970+18402+9317+3224+945+140+44\}$$

$$= \frac{106602}{26306}$$

Let p be the probability of occurrence of 5 or 6 in a throw of single die. Then since for binomial distribution mean$=np$, estimate of p is given by

$$np = \frac{106602}{26306}$$

where n=no. of dice$=12$

$$\therefore \quad p = 0{\cdot}3377$$

$$\therefore \quad q = 1-p = 0{\cdot}6623$$

$\therefore$ Expected frequencies are successive terms in the binomial expansion of

$$26306(0{\cdot}6623+0{\cdot}3377)^{12}$$

$\therefore$ Expected frequencies are :

187, 1146, 3215, 5465, 6269, 5115, 3043, 1330, 424, 96, 15, 1, 0.

Since expected frequencies of last two classes are less than 5, last three classes are to be merged.

$$\therefore \quad \psi^2 = \frac{2^2}{187} + \frac{3^2}{1146} + \frac{(50)^2}{3215} + \frac{(10)^2}{5465} + \frac{(155)^2}{6269} + \frac{(79)^2}{5115} + \frac{(24)^2}{3043}$$
$$+ \frac{1^2}{1330} + \frac{(21)^2}{424} + \frac{9^2}{96} + \frac{(18-16)^2}{16} \simeq 8{\cdot}201$$

Now since mean and total frequency have been used from the data to obtain expected frequencies,

$$\text{No. of } d.f. = 11-2 = 9$$

Now $$\psi_{0\cdot05}^2 \text{ for } 9\ d.f. = 16{\cdot}92$$

$$\therefore \quad \psi_{cal}^2 < \psi_{0\cdot05}^2$$

$\therefore$ Fit is good.

Ex. 15.17. *The following data shows the suicides of 1096 women in 8 Punjab cities during 14 years.*

No. of suicides in a state per year	0	1	2	3	4	5	6	7
Freq.	364	376	218	89	33	13	2	1

Fit a Poisson distribution to the data and show that the fit is not good. ($e^{-1\cdot18} = 0{\cdot}3075$).

Sol. The parameter m of the Poisson distribution is to be obtained from the data itself. Since it is equal to the mean of distribution, we have

$$m=\frac{1}{1096}\{0(364)+1(376)+2(218)+3(89)+4(33)+5(13)+6(2)+7(1)\}\simeq 1\cdot 18$$

$\therefore$ The theoretical frequencies are

$$1096.e^{-1\cdot 18}.\frac{(1\cdot 18)^x}{x\,!},\ x=0,1,\ldots\ldots 7$$

i.e., 337, 398, 235, 92, 27, 6, 1, 0

$$\therefore\quad \psi^2=\frac{(27)^2}{337}+\frac{(22)^2}{398}+\frac{(17)^2}{235}+\frac{3^2}{92}+\frac{6^2}{27}+\frac{(16-7)^2}{7}\simeq 17\cdot 6$$

merging last three classes as the expected frequencies of last two classes are less than 5.

Here no. of classes$=6$(as last three classes have been merged)

$\therefore$ No. of $d.f.=6-2=4$ (as mean and total freq. are kept same for expected and observed frequencies).

Now $\psi_{0\cdot 05}^2$ for 4 $d.f.=9\cdot 49$

$$\therefore\quad \psi_{cal}^2>\psi_{0\cdot 05}^2$$

$\therefore$ Fit is not good.

Ex. 15.18. *Fit a normal distribution to the data given below and test the goodness of fit :*

Height (inches)	60—62	63—65	66—68	69—71	72—74
Freq.	5	18	42	27	8

Sol. The A.M. m and $s.d.$ σ of the given data can be easily shown to be 67·45″ and 2·92″ respectively.

The calculations are arranged in the table below :

Heights	*Class boundaries (X)*	*Z*	*Areas under normal curve from 0 to Z*	*Areas for each class*	*Expected Freq.*	*Observed freq.*
60—62	59·5	−2·72	0·4967	0·0413	4·13	5
63—65	62·5	−1·70	0·4554	0·2068	20 68	18
66—68	65·5	−0 67	0·2486 add	0·3892	38·92	42
69—71	68·5	0·36	0·1406	0·2771	27·71	27
72—74	71·5	1·39	0·4177	0·0743	7·43	8
	74·5	2·41	0·4920			

In 2nd column class boundaries (X) are written, in 3rd column the values of $Z = \frac{X - 67{\cdot}45}{2{\cdot}92}$ are written and in 4th column areas under the normal curve from $Z=0$ to the various values of Z are written. In 5th column areas for each class are written. These are obtained by subtracting the successive areas in the 4th column when the corresponding Z's have the same sign and adding them when Z's have opposite signs (which occurs only once in the table above). In 6th column expected frequencies are written by multiplying the entries in 5th column by total frequency 100.

$$\therefore \quad \psi^2 = \frac{(5-4{\cdot}13)^2}{4{\cdot}13} + \frac{(18-20{\cdot}68)^2}{20{\cdot}68} + \frac{(42-38{\cdot}92)^2}{38{\cdot}92} + \frac{(27-27{\cdot}71)^2}{27{\cdot}71} + \frac{(8-7{\cdot}43)^2}{7{\cdot}43} \simeq 0{\cdot}84$$

Since mean, *s.d.* and total frequency have been used from the data to obtain expected frequencies, number of *d. f.* $=5-3=2$

Now $\qquad \psi_{0\cdot05}^2$ for 2 *d. f.* $=5{\cdot}99$

$\therefore \qquad \psi_{cal}^2 < \psi_{0\cdot05}^2$

$\therefore$ Fit is good.

EXERCISES

1. In a sample of peas from coffee plants the number of round peas is 336 and the number of angular peas is 101. Is this in agreement with the Mendalian hypothesis that the ratio in which they should occur is 3 : 1 ? [Ans. $\psi^2=0{\cdot}8$]

2. In a Mendalian experiment on pea-breeding the four possible seed varieties are expected to occur in the proportion 9 : 3 : 3 : 1. In one experiment involving 720 trials the actual observed frequencies were respectively 396, 139, 129 and 56. Examine whether these results correspond with the theory. [Ans. $\psi^2=3{\cdot}27$]

3. Genetic theory states that children having one parent of blood type M and the other of blood type N will always be one of the three type M, MN, N and that the proportions of three types will on average be 1 : 2 : 1. A report states that out of 300 children having one M parent and one N parent, 30% were found to be type M, 45% type MN and the remainder type N. Test the hypothesis by ψ^2-test.
[Ans. $\psi^2=4{\cdot}5$. Hypothesis may be correct]

4. Find the value of ψ^2 for the following table :

Class	A	B	C	D	E
Observed	8	29	44	15	4
Expected freq.	7	24	38	24	7

[Ans. 6·8]

5. Find the value of ψ^2 for the following table :

Class	A	B	C	D	E
Observed freq.	8	29	47	16	4
Expected freq.	7	24	38	24	7

[Ans. 7·3]

6. The following table gives the number of aircraft accidents that occurred during the various days of the week. Find whether the accidents are uniformly distributed over the week.

Days	Sun.	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.	Total
No. of accidents	14	16	8	12	11	9	14	84

[Ans. Uniformaly distributed]

7. Five coins are tossed 320 times and the following results are obtained .

No. of heads	0	1	2	3	4	5
Freq.	8	57	110	90	0	5

Test the hypothesis that coins are unbiased.

8. 12 dice were rolled 4,096 times and a throw of 4, 5 or 6 is reckoned as a success, the observed frequencies are given below:

No. of successes	0	1	2	3	4	5	6	7	8
Freq.	0	7	60	198	430	731	948	847	536
	9	10	11	12					
	257	71	11	0					

Apply ψ^2-test to test whether dice can be regarded as unbiased.
[Ans. Dice can't be regarded unbiased]

9. Five dice were thrown 192 times and the number of times 3, 4 or 5 were thrown are given below :

No. of dice throwing 3, 4 or 5	5	4	3	2	1	0
Freq.	6	46	70	48	20	3

Calculate ψ^2. [Ans. 16·6]

10. The following is the distribution of 106 eight pig litters according to the number of males in the litters :

No. of males	0	1	2	3	4	5	6	7	8	Total
No. of litters	6	5	8	22	23	25	12	1	4	=106

Fit a binomial distribution under the hypothesis that the sex ratio is 1 : 1 and test the goodness of fit.
($\psi_{0.05}^2$ for 8 $d.f.$=15·51)

11. Records taken of the number of male and female births in 800 families.

No. of male births	No. of female births	No of families
0	4	32
1	3	178
2	2	290
3	1	236
4	0	64
		800

Test whether the data are consistent with the hypothesis that the binomial law holds and that the chance of a male birth is equal to that of a female birth, namely $q=p=\frac{1}{2}$. You may use the table given below :

D.f.	1	2	3	4	5
5% value of ψ^2	3·84	5·99	7·82	9·49	11·07

[**Ans.** Binomial law does not hold]

12. One hundred and ninety-two families (for each of which the possibility of an albino child being born is otherwise established) had the following distribution of albinos among the first three children.

No. of albinos	0	1	2	3	Total
No. of families	77	90	20	5	192

Find the expected frequencies on the basis of a theoretical probability 0·25 of a child being born an albino and test the goodness of fit.

[**Ans.** Fit is good]

13. Fit binomial distribution to the following data and test the goodness of fit.

x :	0	1	2	3	4	5	6	7	8	9	Total
f :	3	8	11	15	16	14	12	11	9	1	100

14. In 1,000 extensive sets of trials for an event of small probability the frequencies 'f' of the number x of successes are found to be

x :	0	1	2	3	4	5	6	7
f :	305	365	210	80	28	9	2	1

Fit a Poisson distribution to the data and test the goodness of fit.

15. A systematic sample of 100 pages was taken from the concise Oxford Dictionary and the observed frequency distribution of foreign words per page was found to be as follows :

No. of foreign words per page	0	1	2	3	4	5	6
Freq.	48	27	12	7	4	1	7

Graduate the data by a Poisson distribution and judge the goodness of your graduation by ψ^2-test.

16. The table below gives the number of mistakes committed per page in typing a manuscript of 584 pages :

Mistakes per page	0	1	2	3	4	5	6	7 and above
No. of pages	238	208	97	30	9	0	2	0

Graduate the data by *a* Poisson distribution and test the goodness of fit. Present your results in a tabular form.

[Below are given values of ψ^2 with probability P of being exceeded in random sampling; n being the number of degrees of freedom :

$P \rightarrow$ / $n \downarrow$	0·95	0·05	0·01
4	0·71	9·49	13·28
5	1·14	11·07	15·09
6	1·64	12·59	16·81
7	2·17	14·07	18·481

15.2.2. Test of Independence of Attributes

Consider for example the attribute-heights of individuals. Then it may be divided into a large number of parts, *e.g.*, very-tall, tall, medium-sized, short and very short. Thus the given attribute A can be divided into a number of classes A_1, A_2, ...A_t. Similarly any other given attribute B can be divided into classes B_1, B_2......B_s. Evidently when both attributes A and B are taken into account each one of the classes A_1, A_2, ... A_t would be divided into a large number of subclasses according to B_1, B_2,.......B_s. Such a classification is called **manifold classification** and a table of the following type is obtained.

Attributes			A	
		A_1	$A_2 \ldots\ldots A_j \ldots\ldots A_t$	Total
	B_1	O_{11}	$O_{12} \ldots\ldots O_{1j} \ldots\ldots O_{1t}$	(B_1)
	B_2	O_{21}	$O_{22} \ldots\ldots O_{2j} \ldots\ldots O_{2t}$	(B_2)
B	$\vdots$	$\vdots$	$\vdots \qquad \vdots \qquad \vdots$	$\vdots$
	B_i	O_{i1}	$O_{i2} \ldots\ldots O_{ij} \ldots\ldots O_{it}$	(B_i)
	$\vdots$	$\vdots$	$\vdots \qquad \vdots \qquad \vdots$	$\vdots$
	B_s	O_{s1}	$O_{s2} \ldots\ldots O_{sj} \ldots\ldots O_{st}$	(B_s)
	Totals	(A_1)	$(A_2) \ldots\ldots (A_j) \ldots\ldots (A_t)$	N

Such a table is called $s \times t$ **contingency table.** Here N is the total frequency, O_{ij} is the frequency of (i, j)th cell (*i.e.*, a place common to ith row and jth column), $(B_1), (B_2) \ldots\ldots (B_s)$ are totals of rows and $(A_1), (A_2) \ldots\ldots (A_t)$ are column totals. Evidently

$$N = (A_1) + (A_2) + \ldots + (A_t) = (B_1) + (B_2) + \ldots + (B_s) \qquad \ldots(1)$$

To test whether there is any relationship between A and B, the independence of two attributes is assumed (Null hypothesis). On the basis of this hypothesis expected frequencies of various cells are obtained by keeping the rows and the column totals for expected frequencies same as for observed frequencies.

Now proportion of individuals belonging to class B_i in the entire data

$$= \frac{(B_i)}{N}.$$

Since A has no influence on B, this proportionality is expected to be maintained in all the classes $A_1, A_2, \ldots\ldots A_t$.

$\therefore$ Expected number of individuals belonging to (i, j)th cell

$$= \frac{(A_j)(B_i)}{N} \qquad i = 1, 2, \ldots\ldots s$$

$$j = 1, 2, \ldots\ldots t.$$

Knowing expected frequencies independence is tested by applying ψ^2-test as usual.

No. of degrees of freedom associated with a $s \times t$ contingency table.

There are in all $s.t.$ cells. Since row and column totals are kept same for expected and observed frequencies, there are $(s+t)$ constraints. Because of (1) there are only $(s+t-1)$ independent linear constraints.

$$\therefore \quad \nu = \text{The no. of } d.f. = s.t. - (s+t-1)$$

$$= (s-1)(t-1).$$

This is the number of cells whose frequencies can be arbitrarily assigned.

Test of Independence

Ex. 15·19. *An opinion poll was conducted to find the reaction to a proposed civic reform in 100 members of each of the two political parties. The information is tabulated below :*

	Favourable	*Unfavourable*	*Indifferent*
Party A	*40*	*30*	*30*
Party B	*42*	*28*	*30*

Test for independence of reactions with the party affiliations. (Given that : $\psi_{0.05}^2$ for 2 d.f.$=5{\cdot}99$).

Sol. Assuming the independence of reactions with the party affiliations, the table of expected frequencies is as below :

	Favourable	Unfavourable	Indifferent
Party A	$\frac{82\times100}{200}=41$	$\frac{58\times100}{200}=29$	$\frac{60\times100}{200}=30.$
Party B	$\frac{82\times100}{200}=41$	$\frac{58\times100}{200}=29$	$\frac{60\times100}{200}=30.$

$$\therefore \quad \psi^2=\frac{1^2}{41}+\frac{1^2}{41}+\frac{1^2}{29}+\frac{1^2}{29}=\frac{2}{41}+\frac{2}{29}=\frac{140}{1189}\simeq 0{\cdot}12.$$

No. of $d.f.=(2-1)(3-1)=2.$

Now $\quad \psi_{0.05}^2$ for 2 $d.f.=5{\cdot}99$

$\therefore \quad \psi_{cal}^2 < \psi_{0.05}^2$

$\therefore$ Hypothesis of independence of reactions with the party affiliations may be correct.

Ex. 15·20. *The following table shows the result of inoculation against cholera :*

	Not-attacked	*Attacked*
Inoculated	*431*	*5*
Not-inoculated	*291*	*9*

Is there any significant association between inoculation and attack ? Given that

$$\nu=1\begin{cases}P=0{\cdot}074 \text{ for } \psi^2=3{\cdot}2\\ P=0{\cdot}069 \text{ for } \psi^2=3{\cdot}3\end{cases}$$

Sol. Assuming the independence between inoculation and attack, expected frequency table is :

	Not-attacked	Attacked
Inoculated	$\frac{722\times436}{736}=427{\cdot}7$	$\frac{14\times436}{736}=8{\cdot}3$
Not-inoculated	$\frac{722\times300}{736}=294{\cdot}3$	$\frac{14\times300}{736}=5{\cdot}7$

$$\therefore \quad \psi^2=(3)^2\left\{\frac{1}{427\cdot7}+\frac{1}{8\cdot3}+\frac{1}{294\cdot3}+\frac{1}{5\cdot7}\right\}\simeq 3\cdot28$$

$\nu=$ No. of $d.f.=(2-1)(2-1)=1$

Now for $\psi^2=3\cdot2$, $\quad P=0\cdot074$

Now for $\psi^2=3\cdot3$, $\quad P=0\cdot069$

Now when ψ^2 increases by $0\cdot1$, P decreases by $0\cdot005$

$\therefore$ when ψ^2 increases by $0\cdot08$, P decreases by $\dfrac{0\cdot005}{0\cdot1}\times 0\cdot08$

$$=\cdot0040.$$

$\therefore$ For $\psi^2=3\cdot28$, $\quad P=0\cdot074-0\cdot004=0\cdot07.$

Thus, if the hypothesis is true, the data give results which would be obtained about 7 times in hundred trials. This is infrequent but not very infrequent. Moreover the theoretical frequencies in the 'attacked' column are not very large. It will therefore be unjustified in rejecting the hypothesis but it can be said that data lead us somewhat to believe that hypothesis is not correct *i.e.*, inoculation and attack are associated.

Ex. 15·21. *From the following table*

		Eye colour in sons	
		Not light	*Light*
Eye colour in fathers	*Not light*	*230*	*148*
	Light	*151*	*471*

test the association between the eye colours of fathers and sons.

Sol. Assuming that there is no association, the expected frequency table is

		Eye colour in sons	
		Not light	Light
Eye colour in fathers	Not light	144	234
	Light	237	385

$$\therefore \quad \psi^2=\frac{(230-144)^2}{144}+\frac{(148-234)^2}{234}+\frac{(151-237)^2}{237}+\frac{(471-385)^2}{385}$$

$$\simeq 133.$$

No. of $d.f.=(2-1)(2-1)=1$

$\therefore \qquad \psi_{0\cdot05}^2=3\cdot84$

$\therefore \qquad \psi_{cal}^2>\psi_{0\cdot05}^2$

$\therefore$ Assumption is wrong.

Ex. 15·22. *In an experiment on immunization of cattle from tuberculosis the following results were obtained :*

	Affected	*Unaffected*
Inoculated	*12*	*28*
Not inoculated	*13*	*7*

Examine the effect of vaccine in controlling the incidence of the disease.

Sol. Assuming that vaccine has no effect on disease, expected frequencies table is,

	Affected	Unaffected
Inoculated	17	23
Not inoculated	8	12

$$\therefore \quad \psi^2=\frac{(17-12)^2}{17}+\frac{(23-28)^2}{23}+\frac{(8-13)^2}{8}+\frac{(12-7)^2}{12}$$

$$=25\left\{\frac{1}{17}+\frac{1}{23}+\frac{1}{8}+\frac{1}{12}\right\}=7\cdot 8.$$

No. of $d.f.=(2-1)(2-1)=1$

$$\therefore \quad \psi_{0\cdot 05}{}^2=3\cdot 84$$

$$\therefore \quad \psi_{cal}{}^2>\psi_{0\cdot 05}{}^2$$

$\therefore$ **Assumption is wrong.**

Ex. 15·23. *Examine by any suitable method, whether the nature of area is related to voting preference in the election for which the data are tabulated below :*

Vote for → *Area* ↓	*A*	*B*	*Total*
Rural	620	480	1,100
Urban	380	520	900
Total	1,000	1,000	2,000

($\psi_{0\cdot 05}{}^2$ *for 1 d.f.*$=3\cdot 84$)

Sol. Assuming the independence of voting preference and the nature of the area, the table of expected frequencies is :

Vote for → Area ↓	A	B
Rural	550	550
Urban	450	450

$$\therefore \quad \psi^2=\frac{(70)^2}{550}+\frac{(70)^2}{550}+\frac{(70)^2}{450}+\frac{(70)^2}{450}=39\cdot 6>\psi_{0\cdot 05}{}^2 \text{ for 1 } d.f.$$

$\therefore$ **Assumption is wrong.**

Ex. 15-24. *An investigator into chocolate consumption divided the United Kingdom into eight areas and took a random sample from each, the individuals so obtained being classified as consumers or non-consumers of chocolate. His results were as follows :*

Area number :	1	2	3	4	5	6	7	8	*Total*
Consumers :	56	87	142	71	88	72	100	142	758
Non-consumers :	17	20	58	20	31	23	25	48	242
Total	73	107	200	91	119	95	125	190	1,000

Do these results suggest that the consumption of chocolate varies from place to place.

Sol. On the assumption that areas and chocolate consumption are independent *i.e.*, chocolate consumption does not vary from place to place, the expected frequencies table is

Area number :	1	2	3	4	5	6	7	8
Consumers :	55	81	152	69	90	72	95	144
Non-consumers :	18	26	48	22	29	23	30	46

$\therefore \qquad \psi^2=6{\cdot}28$

No. of $d.f.=(2-1)(8-1)=7$.

From tables, $\psi_{0\cdot05}{}^2$ for 7 $d.f.=14{\cdot}07$

$\therefore \qquad \psi_{cal}{}^2 < \psi_{0\cdot05}{}^2$

$\therefore$ Assumption may be correct.

Ex. 15-25. *Deduce that for a $s\times t$ contingency table. $\psi^2 \leqslant N(s-1)$ or $\psi^2 \leqslant N(t-1)$ whichever is less.*

Sol. Let e_{ij} be the expected frequency of (i, j)th cell.

Then $$e_{ij}=\frac{(A_i)(B_j)}{N}.$$

Now $$\psi^2=\sum_{i=1}^{s}\sum_{j=1}^{t}\frac{(O_{ij}-e_{ij})^2}{e_{ij}}=\sum_i\sum_j\left\{\frac{O_{ij}^2}{e_{ij}}-2O_{ij}+e_{ij}\right\}$$

$$=\sum_i\sum_j\frac{O_{ij}^2}{e_{ij}}-2\sum_i\sum_j O_{ij}+\sum_i\sum_j e_{ij}$$

$$=N\sum_i\sum_j\left\{\frac{O_{ij}}{(A_i)}\right\}\left\{\frac{O_{ij}}{(B_j)}\right\}-2N+N$$

Now since $O_{ij}\leqslant(A_i)$,

$$\frac{O_{ij}}{(A_i)}\leqslant 1$$

$$\therefore \qquad \psi^2 \leqslant N \sum_i \sum_j \frac{O_{ij}}{(B_i)} - N$$

$$= N\left\{\sum_i \frac{(\sum_j O_{ij})}{(B_i)} - 1\right\} = N\left(\sum_i \frac{(B_i)}{(B_i)} - 1\right)$$

$$[\because \ \sum_j O_{ij} = (B_i)]$$

$$= N(s-1).$$

Similarly $\psi^2 \leqslant N(t-1)$,

$$\therefore \qquad \psi^2 \leqslant \text{min. } [N(s-1),\ N(t-1)].$$

Co-efficient of Contingency.

The co-efficient of contingency (C) is given by

$$C = \sqrt{\frac{\psi^2}{N+\psi^2}}$$

where $\qquad N = \text{total freq.}$

Yates Correction of Continuity. *This correction consists in modifying the definition of* ψ^2 *as below :*

$$\psi^2 = \sum \frac{\{\,|\,O_i - e_i\,|\, - 0{\cdot}5\}^2}{e_i}$$

In general correction is made only when the number of degrees of freedom is $v=1$. *For large samples this yields practically the same results as the uncorrected* ψ^2, *but difficulties can arise near critical values.*

Ex. 15.25. *Show that in a* 2×2 *contingency table wherein the frequencies are* $\dfrac{a \mid b}{c \mid d}$, ψ^2 *calculated from the hypothesis of independence is*

$$\frac{(a+b+c+d)(ad-bc)^2}{(a+b)(c+d)(a+c)(b+d)}$$

Sol. Let $\dfrac{a' \mid b'}{c' \mid d'}$ be the expected frequencies obtained on the hypothesis of independence.

Then $\quad a' = \dfrac{(a+b)(a+c)}{a+b+c+d}, \quad b' = \dfrac{(a+b)(b+d)}{a+b+c+d}$

$$c' = \frac{(a+c)(c+d)}{a+b+c+d} \text{ and } d' = \frac{(b+d)(c+d)}{a+b+c+d}$$

$$\therefore \quad (a-a')^2=\left\{a-\frac{(a+b)(a+c)}{a+b+c+d}\right\}^2=\frac{(ad-bc)^2}{(a+b+c+d)^2}$$

Similarly $(b-b')^2=(c-c')^2=(d-d')^2=\dfrac{(ad-bc)^2}{(a+b+c+d)^2}$

$$\therefore \quad \psi^2=\sum\frac{(a-a')^2}{a'}=\frac{(ad-bc)^2}{a+b+c+d}\left\{\frac{1}{(a+b)(a+c)}+\frac{1}{(a+b)(b+d)}\right.$$

$$\left.+\frac{1}{(a+c)(c+d)}+\frac{1}{(b+d)(c+d)}\right\}$$

$$=\frac{(ad-bc)^2}{a+b+c+d}\left\{\frac{a+b+c+d}{(a+b)(a+c)(b+d)}+\frac{a+b+c+d}{(a+c)(b+d)(c+d)}\right\}$$

$$=(ad-bc)^2\left\{\frac{a+b+c+d}{(a+b)(a+c)(b+d)(c+d)}\right\}.$$

Ex. 15.26. *Show that for a $2\times n$ contingency table,*

$$\psi^2=\sum_r\left\{\frac{N_1 N_2\left(\dfrac{\mu_{1r}}{N_1}-\dfrac{\mu_{2r}}{N_2}\right)^2}{\mu_{1r}+\mu_{2r}}\right\}$$

where μ_{1r}, μ_{2r} are the 2 frequencies in the rth column and N_1, N_2 are the marginal sums of the 2 rows.

Sol. Let η_{1r} and η_{2r} be the expected frequencies in rth column.

Then $\quad \eta_{1r}=\dfrac{(\mu_{1r}+\mu_{2r})\,N_1}{N_1+N_2}$ and $\eta_{2r}=\dfrac{(\mu_{1r}+\mu_{2r}).N_2}{N_1+N_2}$

$$\therefore \quad \psi^2=\sum_r\left[\frac{\left\{\mu_{1r}-\dfrac{(\mu_{1r}+\mu_{2r})N_1}{N_1+N_2}\right\}^2}{\eta_{1r}}\right.$$

$$\left.+\frac{\left\{\mu_{2r}-\dfrac{(\mu_{1r}+\mu_{2r})\,N_2}{N_1+N_2}\right\}^2}{\eta_{2r}}\right]$$

$$=\sum_r\left[\frac{(\mu_{1r}N_2-\mu_{2r}N_1)^2}{(N_1+N_2)(\mu_{1r}+\mu_{2r})}\left\{\frac{1}{N_1}+\frac{1}{N_2}\right\}\right]$$

$$=\sum_r\left\{\frac{N_1 N_2\left(\dfrac{\mu_{1r}}{N_1}-\dfrac{\mu_{2r}}{N_2}\right)^2}{\mu_{1r}+\mu_{2r}}\right\}$$

Ex. 15.27. *Show that for entries in 2×r contingency table,*

							Total
	a_1	a_2	...	a_i	...	a_r	a
	b_1	b_2	...	b_i	...	b_r	b
Total	n_1	n_2	...	n_i	...	n_r	n

$$\psi^2 = \sum_{i=1}^{r} w_i (p_i - p)^2$$

where $p_i = \dfrac{a_i}{n_i}$, $p = \dfrac{a}{n}$, $w_i = \dfrac{n_i}{pq}$, $q = \dfrac{b}{n}$, $q_i = 1 - p_i$

Sol. Two expected frequencies of ith column are $\dfrac{n_i a}{n}$ and $\dfrac{n_i b}{n}$.

$$\therefore \quad \psi^2 = \sum_{i=1}^{r} \left[\frac{n}{n_i a} \left\{ a_i - \frac{n_i a}{n} \right\}^2 + \frac{n}{n_i b} \left\{ b_i - \frac{n_i b}{n} \right\}^2 \right]$$

$$\text{Now } q_i = 1 - p_i = 1 - \frac{a_i}{n_i} = \frac{n_i - a_i}{n_i} = \frac{b_i}{n_i} \qquad (\because \; n_i = a_i + b_i)$$

$$\therefore \quad \psi^2 = \sum_{i=1}^{r} \left[\frac{n_i n}{a} \left\{ \frac{a_i}{n_i} - \frac{a}{n} \right\}^2 + \frac{n n_i}{b} \left\{ \frac{b_i}{n_i} - \frac{b}{n} \right\} \right]$$

$$= \sum_{i=1}^{r} \left[\frac{n_i n}{a} (p_i - p)^2 + \frac{n n_i}{b} (q_i - q)^2 \right]$$

$$= \sum_{i=1} \left[\frac{n n_i}{a} (p_i - p)^2 + \frac{n n_i}{b} \{(1 - p_i) - (1 - p)\}^2 \right]$$

$$= \sum_{i=1}^{r} n n_i (p_i - p)^2 \left\{ \frac{a+b}{ab} \right\}$$

$$=\sum_{i=1}^{r}\left(\frac{n}{a}\right)\left(\frac{n}{b}\right)n_i(p_i-p)^2=\sum_{i=1}^{r}\frac{n_i}{pq}(p_i-p)^2$$

$$=\sum_{i=1}^{r}w_i(p_i-p)^2$$

Ex. 15.28. *In Ex. 15.27 show that*

$$\psi^2=\frac{1}{pq}\left\{\sum_{i=1}^{r}(a_ip_i)-ap\right\}$$

Sol. From Ex. 15.27.

$$\psi^2=\sum_{i=1}^{r}w_i(p_i-p)^2=\frac{1}{pq}\sum_{i=1}^{r}n_i\{p_i^2-2p_ip+p^2\}$$

$$=\frac{1}{pq}\sum_{i=1}^{r}\{(n_ip_i)p_i-2(p_in_i)p+n_ip^2\}$$

$$=\frac{1}{pq}\sum_{i=1}^{r}\{a_ip_i-2pa_i+n_ip^2\}$$

$$=\frac{1}{pq}\left[\sum_{i=1}^{r}(a_ip_i)-2p\left(\sum_{i=1}^{r}a_i\right)+p^2\left(\sum_{i=1}^{r}n_i\right)\right]$$

$$=\frac{1}{pq}\left[\sum_{i=1}^{r}(a_ip_i)-2pa+p^2n\right]$$

$$=\frac{1}{pq}\left[\sum_{i=1}^{r}(a_ip_i)-2pa+ap\right]$$

$$=\frac{1}{pq}\left\{\sum_{i=1}^{r}(a_ip_i)-ap\right\}$$

EXERCISES

1. Find the value of ψ^2 for 2×2 contingency table

Hair colour→ Eye colour ↓	Light	Dark
Blue	26	9
Brown	7	18

[**Ans.** 12·6]

2. In a locality 100 persons were randomly selected and asked about their educational achievements. The results are given as below :

		Education		
		Middle	High School	College
Sex	Male	10	15	25
	Female	25	10	15

Can you say that education depends on sex ?

[**Ans.** 9·9, Education depends on sex]

3. From the following table find whether the hair colour and sex are associated :

		Hair Colour				
		Fair	Red	Medium	Dark	Jet black
Sex	Boys	592	119	849	504	36
	Girls	544	97	677	451	14

[**Ans.** Associated]

4. In an experiment with immunization of cattle from tuberculosis, the following results were obtained :

	Affected	Unaffected
Inoculated	12	26
Not inoculated	16	6

Examine the effect of vaccine in controlling the susceptibility to tuberculosis.

[**Ans.** Vaccine is effective in controlling the susceptibility to tuberculosis]

5. In an experiment on the immunization of goats from anthrox the following results were obtained. Derive your inference on the efficiency of the vaccine :

	Died	Survived
Inoculated	2	10
Not Inoculated	6	6

[Ans. Survival is not associated with inoculation of vaccine]

6. In experiments on the Spahlinger anti-tuberculosis vaccine the following results were obtained :

	Died or seriously affected	Unaffected or not seriously affected	Total
Inoculated	6	13	19
Not-inoculated or inoculated with control media	8	3	11
Total	14	16	30

Find the value of ψ^2 and test the independence. [Ans. 4·7]

7. The table below gives the data obtained during an epidemic of cholera :

	Attacked	Not attacked
Inoculated	31	469
Not inoculated	185	1,315

Test the effectiveness of inoculation in preventing the attack of cholera. [Ans. Inoculation is effective]

8. Can vaccination be regarded as a preventive measure for small pox as evidenced by the following data :

'of 1,482 persons exposed to small-pox in a locality, 368 in all were attacked. Of these 1,482 persons, 343 were vaccinated and of these only 35 were attacked.

9. From the following data test whether there is any association between intelligency and economic conditions :

		Intelligency			
		Excellent	Good	Medium	Dull
Economic conditions	Good	48	200	150	80
	Not Good	52	180	190	100

10. A producer of a certain film claimed that his movie was not liked equally by men and women. Accordingly a sample of

men and women was collected. The following are the number of men and women falling into each of the five classes :

	Most liked	More liked	Liked	Not much liked	Disliked
Men	110	591	840	500	30
Women	90	549	670	450	20

Is producer's remark supported by data ?

11. The following table shows the association among 1000 school boys, their general ability and their mathematical ability. Calculate the co-efficient of contingency between the two.

		General ability		
		Good	Fair	Poor
	Good	44	22	4
Maths ability	Fair	265	257	178
	Poor	41	91	98

12. The following data observed for hybrids of Datura

		Flowers	
		Violet	White
Fruits	Prickly	47	21
	Smooth	12	3

Apply ψ^2-test to test the association between colour of flowers and character of fruits. Given that

$$v=1 \begin{cases} P=0{\cdot}402 \text{ for } \psi^2=0{\cdot}7 \\ P=0{\cdot}399 \text{ for } \psi^2=0{\cdot}71 \end{cases}$$

[**Ans.** There is no association]

13. From the following table, test the hypothesis that the flower colour is independent of flatness of leaf :

	Flat leaves	Curled leaves	Total
White flowers	99	36	135
Red flowers	20	5	25

Use the following table giving the values of ψ^2 for 1 *d.f.* for different values of *P*.

P :	0·5	0·1	0·05
ψ^2 :	0·455	2·706	3·841

[**Ans.** $\psi^2=0{\cdot}5$]

14. Candidates for a degree in Mathematics are required to pass a subsidiary examination in Physics. The table below gives the number of candidates classified according to the grading awarded in two subjects. Test if the performances in the two subjects are independent.

		Class in Maths.			
		I	II	Pass	Fail
	I	38	60	50	11
Class in Physics	II	24	70	100	27
	Pass	12	36	91	25

15. Sixteen pieces of photographic paper were printed down to different depths of colour from nearly white to a very deep blackish brown. Small scraps were cut from each sheet and pasted on cards, two scraps on each card one above the other, combining scraps from the several sheets in all possible ways, so that there were 256 cards in the pack. Twenty observers then went through the pack independently, each one naming each tint either 'light', 'medium' or 'dark'.

The following table shows the name assigned to each of the two pieces of paper :—

		Name assigned to upper tint			
		Light	Medium	Dark	Total
Name assigned to lower tint	Light	850	571	580	2001
	Medium	618	593	455	1666
	Dark	540	456	457	1453
	Total	2008	1620	1492	5120

Show that there is a significant association between the name assigned to one piece and the name assigned to the other.

16

t, F and Z Distributions and Small Sample Tests

16-1. Introduction

Let $x_1, x_2 .. x_n$ be the members of a random sample drawn from a normal population with mean μ and s.d. σ.

Let
$$\bar{x}=\frac{1}{n}\sum_{i=1}^{n} x_i$$

and
$$s^2=\frac{1}{n}\sum_{i=1}^{n} (x_i-\bar{x})^2$$

The joint distribution of $x_1, x_2 \ldots\ldots x_n$ is

$$dP=\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \text{exp.} \left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} (x_i-\mu)^2\right\} dx_1, dx_2 \ldots\ldots dx_n.$$

Now
$$\sum_{i=1}^{n} (x_i-\mu)^2= \sum_{i=1}^{n} \{(x_i-\bar{x})+(\bar{x}-\mu)\}^2$$

$$= \sum_{i=1}^{n} \{(x_i-\bar{x})^2+(\bar{x}-\mu)^2+2(\bar{x}-\mu)(x_i-\bar{x})\}$$

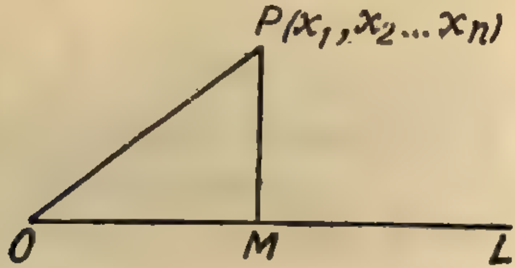
$$= \sum_{i=1}^{n} (x_i-\bar{x})^2+n(\bar{x}-\mu)^2$$

$$\{\because \sum_{i=1}^{n} (x_i-\bar{x})=0\}$$

$$=ns^2+n(\bar{x}-\mu)^2$$

(16.1)

Represent the sample values $(x_1, x_2 \ldots\ldots x_n)$ by a pt P with Co-ordinates $(x_1, x_2 \ldots x_n)$ in Euclidean hyperspace of n dimension. Let O be the origin. Let OL be the line through O with direction ratios $(1, 1, \ldots\ldots 1)$. Draw $PM \perp OL$.



Let co-ordinates of M be $(\alpha, \alpha, \ldots\ldots \alpha)$ (where $\alpha \neq 0$).

Than $d.r.$'s of OM are $\alpha, \alpha, \ldots\ldots, \alpha$

and $d.r.$'s of PM are $x_1-\alpha, x_2-\alpha, \ldots\ldots x_n-\alpha$.

Since $PM \perp OM$,

$$\alpha(x_1-\alpha)+\alpha(x_2-\alpha)+\ldots\alpha(x_n-\alpha)=0$$

$$\Rightarrow \qquad \alpha=\frac{x_1+\ldots\ldots+x_n}{n}=\bar{x}$$

$\therefore$ Co-ordinates of M are $(\bar{x}, \bar{x}, \ldots\ldots \bar{x})$.

$$\therefore \qquad PM^2=(x_1-\bar{x})^2+\ldots\ldots+(x_n-\bar{x})^2$$

$$=\sum_{i=1}^{n}(x_i-\bar{x})^2=ns^2$$

$$\therefore \qquad PM=\sqrt{n}\,s$$

and $OM^2=\bar{x}^2+\ldots\ldots+\bar{x}^2=n\bar{x}^2 \Rightarrow OM=\sqrt{n}\,\bar{x}$.

If $\bar{x}$ and s are kept fixed, P moves in $(n-1)$ dimensional space orthogonal to OE on the surface of a hypersphere of radius PM and center M.

$\therefore$ The spherical shell in which P moves has thickness $d(PM)$ and suffers a displacement of length $d(OM)$.

$\therefore$ As $\bar{x}$ increases by $d\bar{x}$ and s by ds, P describes an element of volume proportional to

$$(PM)^{n-2}\,.\,d(PM)\,.\,d(OM)$$

$$=(s\sqrt{n})^{n-2}\,.\,\sqrt{n}\,ds\,.\,\sqrt{n}\,d\bar{x}$$

$$=\text{constant } s^{n-2}\,ds\,d\bar{x}$$

$$\therefore \quad dP = \text{const. exp.} \left\{ -\frac{1}{2\sigma^2} (ns^2 + n(\bar{x}-\mu)^2) \right\} s^{n-2}\, ds\, d\bar{x}$$

$$= \left\{ c_1\, e^{-\frac{1}{2}\frac{ns^2}{\sigma^2}} s^{n-2}\, ds \right\} \left\{ c_2\, e^{-\frac{1}{2}\frac{(\bar{x}-\mu)^2}{\sigma^2/n}} d\bar{x} \right\}$$

where c_1 and c_2 are constants.

$\Rightarrow$ s and $\bar{x}$ are independent.

Dist. of $\bar{x}$ is

$$dP = c_2\, e^{-\frac{1}{2}\frac{(\bar{x}-\mu)^2}{\sigma^2/n}} d\bar{x}$$

$\bar{x}$ varies from $-\infty$ to ∞.

$\therefore$ c_2 is given by

$$c_2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{(\bar{x}-\mu)^2}{\sigma^2/n}} d\bar{x} = 1$$

Put $$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = y$$

$$\therefore \quad c_2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \frac{\sigma}{\sqrt{n}}\, dy = 1$$

or $$c_2 \frac{\sigma}{\sqrt{n}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy = 1$$

$\Rightarrow$ $$c_2 \frac{\sigma}{\sqrt{n}} \cdot \sqrt{2\pi} = 1$$

$\therefore$ $$c_2 = \sqrt{\frac{n}{2\pi}} \cdot \frac{1}{\sigma}$$

$\therefore$ Dist of $\bar{x}$ is

$$dP = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma/\sqrt{n}}\, e^{-\frac{1}{2}\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2} d\bar{x}$$

$\Rightarrow$ $$\bar{x} \text{ is } N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

16.4

Dist of s is

$$dP = c_1 e^{-\frac{1}{2}\frac{ns^2}{\sigma^2}} s^{n-2}\, ds$$

s varies from 0 to ∞.

$\therefore$ c_1 is given by

$$c_1 \int_0^\infty e^{-\frac{1}{2}\frac{ns^2}{\sigma^2}} s^{n-2}\, ds = 1$$

Put $$\frac{1}{2}\frac{ns^2}{\sigma^2} = y$$

$\therefore$ $sds = \frac{\sigma^2}{n} dy$

$$\therefore \quad c_1 \cdot \frac{\sigma^2}{n} \int_0^\infty e^{-y} \left(\frac{2\sigma^2}{n} y\right)^{\frac{n-3}{2}} dy = 1$$

i.e., $$\frac{c_1}{2}\left(\frac{2\sigma^2}{n}\right)^{\frac{n-1}{2}} \int_0^\infty e^{-y} \cdot y^{\frac{n-1}{2}-1}\, dy = 1$$

$$\Rightarrow \quad \frac{c_1}{2}\left(\frac{2\sigma^2}{n}\right)^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right) = 1$$

$$\Rightarrow \quad c_1 = 2\left(\frac{n}{2\sigma^2}\right)^{\frac{n-1}{2}} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)}$$

$\therefore$ Dist. of s is

$$P = 2\left(\frac{n}{2\sigma^2}\right)^{\frac{n-1}{2}} \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{1}{2}\cdot\frac{ns^2}{\sigma^2}} s^{n-2}\, ds$$

$$= \frac{1}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{1}{2}\frac{ns^2}{\sigma^2}} \left(\frac{ns^2}{\sigma^2}\right)^{\frac{n-3}{2}} d\left(\frac{ns^2}{\sigma^2}\right)$$

Or

$$\frac{1}{\Gamma\left(\frac{n-1}{2}\right)} e^{-\frac{1}{2}\frac{ns^2}{\sigma^2}} \left(\frac{1}{2}\frac{ns^2}{\sigma^2}\right)^{\frac{n-1}{2}-1} d\left(\frac{ns^2}{2\sigma^2}\right)$$

$\therefore$ $\frac{ns^2}{\sigma^2}$ is a ψ^2 variate with $(n-1)$ d.f.

and $\frac{1}{2}\frac{ns^2}{\sigma^2}$ is a $\gamma\left(\frac{n-1}{2}\right)$ variate.

Remark.

$$E(s^2)=\frac{1}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\int_0^\infty s^2 e^{-\frac{1}{2}\frac{ns^2}{\sigma^2}}\left(\frac{ns^2}{\sigma^2}\right)^{\frac{n-3}{2}} d\left(\frac{ns^2}{\sigma^2}\right).$$

Put $\frac{ns^2}{2\sigma^2}=y$

$$=\frac{1}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\int_0^\infty \frac{2\sigma^2}{n} y \,.\, e^{-y}(2y)^{\frac{n-3}{2}}(2dy)$$

$$=\frac{2}{\Gamma\left(\frac{n-1}{2}\right)}\cdot\frac{\sigma^2}{n}\int_0^\infty e^{-y}\, y^{\frac{n+1}{2}-1} dy$$

$$=\frac{2\sigma^2}{n}\cdot\frac{1}{\Gamma\left(\frac{n-1}{2}\right)}\Gamma\left(\frac{n+1}{2}\right)$$

$$=\frac{2\sigma^2}{n}\cdot\frac{\frac{n-1}{2}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$=\sigma^2\cdot\frac{n-1}{n}$$

$$\therefore \quad E\left\{\frac{ns^2}{n-1}\right\}=\sigma^2$$

$\therefore \quad \dfrac{ns^2}{n-1}$ is an unbiased estimate of σ^2.

16.2. Student's t-Distribution

Student's t statistic is defined by

$$t=\left(\frac{\bar{x}-\mu}{S}\right)\sqrt{n}$$

where
$$S^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2=\frac{ns^2}{n-1}$$

$$\therefore \qquad \frac{t^2}{\nu}=\frac{(\bar{x}-\mu)^2}{s^2}, \text{ where } \nu=n-1$$

$$=\left\{\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right\}^2\Big/\frac{ns^2}{\sigma^2}$$

Now, $\bar{x}$ is $N\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right)$

$\therefore \quad \left(\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2$ is a ψ^2 variate with 1 d.f.

and $\dfrac{ns^2}{\sigma^2}$ is a ψ^2 variate with $n-1=\nu$ d.f.

$\therefore \quad \dfrac{t^2}{\nu}$ is a $\beta_2\left(\dfrac{\nu}{2}, \dfrac{1}{2}\right)$ variate

$\therefore$ Dist. of t is

$$dP=\frac{1}{\beta\left(\frac{\nu}{2}, \frac{1}{2}\right)}\frac{\left(\frac{t^2}{\nu}\right)^{\frac{1}{2}-1}}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}d\left(\frac{t^2}{\nu}\right), 0<t^2<\infty$$

$$=\frac{1}{\sqrt{\nu}\,\beta\left(\frac{\nu}{2}, \frac{1}{2}\right)}\frac{dt}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}, -\infty<t<\infty$$

This distribution is known as student's t-distribution with ν *d.f.*

Remark. t-distribution was first found by W. S. Gosset in 1908 in his paper entitled **'the probable error of the mean'** written under the name of his student. Student defined his statistic as

$$t = \frac{\bar{x} - \mu}{S}$$

and investigated its sampling distribution. Later on in 1926, Prof. R.A. Fisher defined his own statistic and gave a rigorous proof for its sampling distribution. He defined his statistic as

$$t = \frac{\xi}{\sqrt{\frac{\psi^2}{n}}}$$

where ξ is a $N(0, 1)$, ψ^2 is a chi-square variate with n.d.f. and ξ, ψ^2 are independent.

Distribution of Fisher's t.

Since ξ and ψ^2 are independent, their joint distribution is

$$dP = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2} \cdot \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} e^{-\frac{1}{2}\psi^2} (\psi^2)^{\frac{n}{2}-1} d\xi\, d\psi^2$$

$$-\infty < \xi < \infty,\ 0 < \psi^2 < \infty$$

Put $\qquad t = \dfrac{\xi}{\sqrt{\psi^2/n}}, \qquad u = \psi^2$

$\Rightarrow \qquad \xi = t\dfrac{\sqrt{u}}{\sqrt{n}}, \qquad \psi^2 = u$

$$\therefore \quad \frac{\partial(\xi, \psi^2)}{\partial(t, u)} = \begin{vmatrix} \frac{\partial \xi}{\partial t} & \frac{\partial \xi}{\partial u} \\ \frac{\partial \psi^2}{\partial t} & \frac{\partial \psi^2}{\partial u} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\sqrt{u}}{\sqrt{n}} & \frac{t}{2\sqrt{u}\sqrt{n}} \\ 0 & 1 \end{vmatrix} = \frac{\sqrt{u}}{\sqrt{n}}$$

$\therefore$ The joint distribution of t and u is

$$dP = \frac{1}{\sqrt{2\pi} \cdot 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} e^{-\frac{1}{2}u\left(1+\frac{t^2}{n}\right)} u^{\frac{n}{2}-1} \cdot \frac{\sqrt{u}}{\sqrt{n}}\, du\, dt$$

$$=\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{2\pi}.2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\,e^{-\frac{1}{2}u\left(1+\frac{t^2}{n}\right)}.u^{\frac{n}{2}-\frac{1}{2}}\,du\,dt$$

$$-\infty<t<\infty,\ 0<u<\infty$$

$\therefore$ Marginal distribution of t is

$$\frac{1}{\sqrt{n}.\sqrt{2\pi}}\cdot\frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}(dt)\int_0^\infty e^{-\frac{1}{2}u\left(1+\frac{t^2}{n}\right)}.u^{\frac{n}{2}-\frac{1}{2}}\,du$$

$$=\frac{dt}{\sqrt{n}\sqrt{2\pi}\,2^{n/2}\Gamma\left(\frac{n}{2}\right)}\int_0^\infty e^{-y}\left(\frac{2y}{1+\frac{t^2}{n}}\right)^{\frac{n-1}{2}}\left(\frac{2dy}{1+\frac{t^2}{n}}\right)$$

$$\text{where } y=\frac{1}{2}u\left(1+\frac{t^2}{n}\right)$$

$$=\frac{dt}{\sqrt{n}\sqrt{2\pi}\;2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}\cdot\frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,2^{\frac{n+1}{2}}$$

$$\int_0^\infty e^{-y}.y^{\frac{n+1}{2}-1}\,dy$$

$$=\frac{dt}{\sqrt{n}.\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\Gamma\left(\frac{n+1}{2}\right)$$

$$=\frac{1}{\sqrt{n}}\;\frac{1}{\beta\left(\frac{n}{2},\frac{1}{2}\right)}\;\frac{dt}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

which is t-distribution with n.d.f.

(2) Taking

$$\xi=\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$$

$$\psi^2 = \frac{ns^2}{\sigma^2}$$

which is chi-square variate with $(n-1)$ d.f. Fisher's statistics t takes the form

$$t = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \frac{1}{\sqrt{\dfrac{ns^2}{\sigma^2(n-1)}}}$$

$$= \left(\frac{\bar{x}-\mu}{S}\right)\sqrt{n}$$

which is student's t statistic. Thus, student's t can be regarded as a particular case of Fisher's t.

16.2.1. Properties of t-distribution

$$\bar{t} = \text{Mean} = E(t)$$

$$= \frac{1}{\sqrt{\nu}\ \beta\left(\frac{\nu}{2} \cdot \frac{1}{2}\right)} \int_{-\infty}^{\infty} \frac{t}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}\, dt$$

$$= 0$$

$$\therefore \qquad \mu_{2r+1} = E(t-\bar{t})^{2r+1}$$

$$= E(t^{2r+1})$$

$$= \frac{1}{\sqrt{\nu}\ \beta\left(\frac{r}{2}, \frac{1}{2}\right)} \int_{-\infty}^{\infty} \frac{t^{2r+1}}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}\, dt$$

$$= 0$$

and $\qquad \mu_{2r} = E(t^{2r})$

$$= \frac{1}{\sqrt{\nu}\ \beta\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_{-\infty}^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}\, dt$$

$$= \frac{2}{\sqrt{\nu}\ \beta\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_{0}^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}\, dt$$

This converges if $2r < \nu$. So if $r < \frac{\nu}{2}$, μ_{2r} exist.

Put $\qquad \frac{t^2}{\nu} = x$

$\therefore \qquad 2t\,dt = \nu\,dx$

$$\therefore \qquad \mu_{2r} = \frac{\nu^{r+\frac{1}{2}}}{\sqrt{\nu}\;\beta\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_0^\infty \frac{x^{r-\frac{1}{2}}}{(1+x)^{\frac{\nu+1}{2}}}\,dx$$

$$= \frac{\nu^r}{\beta\left(\frac{\nu}{2}, \frac{1}{2}\right)} \int_0^\infty \frac{x^{r+\frac{1}{2}-1}}{(1+x)^{\left(\frac{\nu}{2}-r\right)+\left(r+\frac{1}{2}\right)}}\,dx$$

$$= \frac{\nu^r}{\beta\left(\frac{\nu}{2}, \frac{1}{2}\right)}\;\beta\left(\frac{\nu}{2}-r,\ r+\frac{1}{2}\right)$$

$$= \frac{\nu^r}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}\right)}\;\Gamma\left(\frac{\nu}{2}-r\right)\Gamma\left(r+\frac{1}{2}\right)$$

$$= \frac{\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\ldots\ldots\frac{1}{2}}{\left(\frac{\nu}{2}-1\right)\left(\frac{\nu}{2}-2\right)\ldots\left(\frac{\nu}{2}-r\right)}\;\nu^r$$

$$= \frac{(2r-1)(2r-3)\ldots1}{(\nu-2)(\nu-4)\ldots(\nu-2r)}\;\nu^r$$

Put $\qquad r = 1, 2$

$$\therefore \qquad \mu_2 = \frac{1}{\nu-2}\cdot\nu = \frac{\nu}{\nu-2} > 1$$

$$\therefore \qquad \mu_4 = \frac{3.1}{(\nu-2)(\nu-4)}\;\nu^2 = \frac{3\nu^2}{(\nu-2)(\nu-4)}$$

$$\therefore \qquad \beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{3(\nu-2)}{\nu-4} \to 3 \text{ as } \nu\to\infty$$

$$\therefore \qquad \gamma_2 = \beta_2 - 3 = \frac{6}{\nu-4} \to 0 \text{ as } \nu\to\infty.$$

Also $\qquad \beta_1 = \frac{\mu_3^{\,2}}{\mu_2^{\,3}} = 0,\quad \gamma_1 = \sqrt{\beta_1} = 0$

Recurrence formula for moments

$$\mu_{2r}=\frac{(2r-1)\ldots\ldots(1)}{(\nu-2)\ldots(\nu-2r)}\,\nu^r$$

and
$$\mu_{2r-2}=\frac{(2r-3)\ldots\ldots1}{(\nu-2)\ldots(\nu-2r+2)}\,\nu^{r-1}$$

Dividing

$$\frac{\mu_{2r}}{\mu_{2r-2}}=\frac{2r-1}{\nu-2r}\,.\,\nu$$

16.2.2. Chief Features of the t-Probability Curve

The equation of the t-probability curve is

$$y=\frac{1}{\sqrt{\nu}\;\beta\left(\frac{\nu}{2},\frac{1}{2}\right)}\,\frac{1}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}$$

(1) Since on changing t to $-t$ y does not change, curve is symmetrical about $t=0$

Median$=0$

(2) $y\to0$ as $|t|\to\infty$.

$\therefore$ Curve is asymptotic to t-axis at both ends

(3) y decreas rapidly as $|t|$ increases.

$\therefore$ y is maximum for $t=0$.

$\therefore$ Mode$=0$

Mean, Mode and Median coincide.

16.3. t-tests

Tests of significance based on t-distribution are called t-tests. Various t-tests are :

(*i*) Test for single proportion.

(*ii*) Test for the difference of means.

(*iii*) Test for the significance of an observed sample correlation co-efficient.

(*iv*) Test for the significance of an observed regression co-efficient.

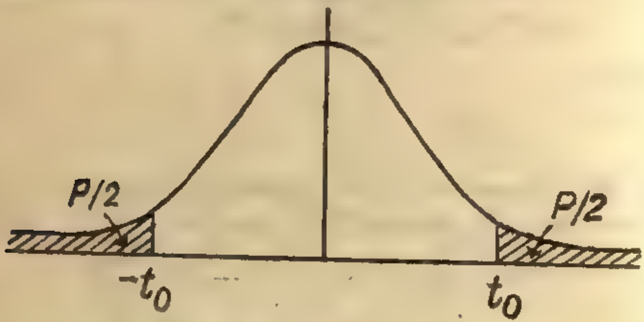
(v) Test for the significance of a rank correlation coefficient.

All these tests are for small samples and are based an fundamental assumption that the parent population is normal.

Rules of Decision

Let
$$P=P\{|t|>t_0\}$$
$$=2\,P\{t>t_0\}$$

For various fixed values of P and for ν ranging from 1 to 60 ; values of t_0 have been tabulated in the form of t-tables. For $\nu > 60$, t is considered as a standard normal variate. The value t_0 is called the critical value of t at level of significance P and $d.f.\ \nu$.



To test the significance the calculated value of t is compared with tabulated value at certain specified level of significance. Generally 5% or 1% levels are taken.

If calculated value of $|t|$ exceeds tabulated value, the null hypothesis is rejected and the difference is said to be significant and if it is less than tabulated value, the hypothesis is accepted at the level of significance adopted.

Remark

In above rules both the ends of t curve are considered and hence tests with these rules are called two-tailed tests. If however, one tail is used tests are called single-tailed tests.

Since t-curve is symmetrical about $t=0$,

$$P\{t > t_0(\alpha)\} = P\{t \leqslant -t_0(\alpha)\}$$

where α is the level of significance and $t_0(\alpha)$ the critical value of t at level of significance α.

$$\therefore \quad \alpha = P\{|t| > t_0(\alpha)\} = 2P\{t \geqslant t_0(\alpha)\}$$

$$\Rightarrow \quad \frac{\alpha}{2} = P\{t \geqslant t_0(\alpha)\}$$

changing α to 2α

$$\Rightarrow \quad \alpha = P\{t > t_0(2\alpha)\}$$

Hence for a single tailed test, the critical values of t for level of significance α can be obtained from those of two tailed test by looking the values at level of significance 2α.

15.3.1. Test for Single Mean

Let $x_1, x_2, \ldots\ldots x_n$ be a random sample from a normal population with mean μ. The problem here is to test "is the sample mean differs significantly from the population mean μ"? Assuming the

null hypothesis. "There is no significant difference between the sample mean and the population mean", the statistic.

$$t=\frac{\overline{x}-\mu}{S/\sqrt{n}}$$

where $\overline{x}$=sample mean

and
$$S^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2=\frac{n}{n-1}s^2$$

(where s^2 is sample $s.d.$)

which follows student's—distribution with $(n-1)$ $d.f.$, is calculated.

Ex. 16·1. *A mechanist is making engine parts with axle diameters 0·700". A random sample of 10 parts shows a mean diameter of 0·742" with a.s.d. of 0·04". Test whether work is meeting the specification.*

Sol. Here $\overline{x}=0{\cdot}742$, $n=10$, $s=0{\cdot}04$ and $\mu=0{\cdot}7''$.

$$\therefore \quad t=\frac{\overline{x}-\mu}{S/\sqrt{n}}=\left(\frac{\overline{x}-\mu}{s}\right)\sqrt{n-1}\simeq 3{\cdot}15$$

No. of $d.f.=10-1=9$

From tables $t_{0.05}$ for 9 $d.f.=2{\cdot}26$

$$\therefore \quad t_{cal}>t_{0.05}$$

$\therefore$ $\overline{x}$ differs from μ significantly and hence the work is not meeting the specification.

Ex. 16·2. *Ten individuals are chosen at random from a normal population and the heights are found to be in inches 63, 63, 66, 67, 68, 69, 70, 70, 71, and 71. In the light of these data discuss the suggestion the mean height in the universe is 66" having given that*

$$\left.\begin{array}{ll} \textit{for } t=1{\cdot}8 & P=0{\cdot}947 \\ \textit{for } t=1{\cdot}9 & P=0{\cdot}955 \end{array}\right\} \textit{ for 9 d.f.}$$

where P is the area to the left of the ordinate at t.

Sol. **Calculation of mean and s.d.**

											Total
x :	63	63	66	67	68	69	70	70	71	71	
$X=x-68$:	−5	−5	−2	−1	0	1	2	2	3	3	−2
X^2 :	25	25	4	1	0	1	4	4	9	9	82

$$\therefore \quad \overline{x}=68+\left(-\frac{2}{10}\right)=67{\cdot}8,\ s^2=\frac{82}{10}-\left(-\frac{2}{10}\right)^2=8{\cdot}2-0{\cdot}04$$
$$=8{\cdot}16.$$

Assuming population mean to be 66, $\mu=66$

$$t=\left(\frac{\bar{x}-\mu}{s}\right)\sqrt{n-1}=\frac{(1\cdot8)\sqrt{9}}{\sqrt{8\cdot16}}=\frac{5\cdot4}{\sqrt{8\cdot16}}\simeq1\cdot89$$

when t increases by 0·1, P increases by 0·008

when t increases by 0·09, P increases by $\left(\frac{0\cdot008}{0\cdot1}\right)(0\cdot09)=0\cdot0072$

$\therefore$ For $t=1\cdot89$, $P=0\cdot9542$

$\therefore$ $P_T=2(1-P)=2(1-0\cdot9542)=0\cdot0916>0\cdot05$

$\therefore$ Difference is not significant at 5% level of significance and hence test provides no evidence against the population mean being 66".

Ex. 16·3. *Nine patients, to whom a certain drug was administrated, registered the following increments in blood pressure :—*

7, 3, —1, 4, —3, 5, 6, —4, 1

Show that the data do not indicate that the drug was responsible for these increments. The values of t for 10, 9 and 8 d.f. at 5% level of significance are 2·23, 2·26 and 2·31 respectively.

Sol. Let x be the variable for the increment in blood pressure.

										Total
x :	7	3	-1	4	-3	5	6	-4	1	18
$x-\bar{x}$:	5	1	-3	2	-5	3	4	-6	-1	
$(x-\bar{x})^2$:	25	1	9	4	25	9	16	36	1	126

$$\bar{x}=\frac{18}{9}=2,\qquad S^2=\frac{1}{9-1}(126)\approx15\cdot75.$$

Assuming that the drug was not responsible for the increments in blood pressure, $\mu=0$

$$\therefore\qquad t=\frac{2}{\sqrt{15\cdot75}}\cdot\sqrt{9}=\frac{6}{\sqrt{15\cdot75}}\simeq1\cdot51.$$

No. of $d.f.=9-1=8$

$\therefore$ $t_{0\cdot05}=2\cdot31$

$\therefore$ $t_{cal}<t_{0\cdot05}$.

$\therefore$ The data do not indicate that the drug was responsible for increment in blood pressure.

Ex. 16·4. *10 patients to whom a drug was administered registered the following additional hours of sleep :*

0·7, $-1\cdot1$, $-0\cdot2$, 1·2, 0·1, 3·4, 3·7, 0·8, 1·8, 2 0

Compute the statistic you would use to determine whether these data justify the claim that the drug does produce additional sleep.

Sol. **Calculation of mean and s.d.**

											Total
x :	0·7	$-1·1$	$-0·2$	1·2	0·1	3·4	3·7	0·8	1·8	2·0	12·4
$(x-\bar{x})$:	$-0·54$	$-2·34$	$-1·44$	$-0·04$	$-1·14$	2·16	2·46	$-0·44$	0·56	0·76	
$(x-\bar{x})^2$:	0·2916	5·4756	2·0736	0·0016	1·2996	4·6656	6·0516	0·1936	0·3136	0·5776	20·9440

$$\bar{x}=\frac{12·4}{10}=1·24,\ s^2=2·0944$$

Assuming that drug does not produce additional sleep, $\mu=0$

$$\therefore \qquad t=\frac{(1·24)\sqrt{9}}{\sqrt{2·0944}}\approx 2·6$$

Ex. *A certain stimulus administered to each of the 12 patients resulted in the following increases of blood pressure :*

5, 2, 8, –1, 3, 0, 6, –2, 1, 5, 0, 4

Can it be concluded that stimulus will be, in general, accompanied by an increase in blood pressure? Given that $t_{0·05}(10)=2·23$, $t_{0·05}(11)=2·20$, $t_{0·05}(12)=2·18$ where $t_{\alpha}(n)$ denotes the value of 't' for 'n' d.f. at 'α' level of significance.

Sol. **Calculation of mean and s.d.**

													Total
x :	5	2	8	-1	3	0	6	-2	1	5	0	4	31
x^2 :	25	4	64	1	9	0	36	4	1	25	0	16	185

$$\bar{x}=\frac{31}{12}\text{ and } s^2=\frac{185}{12}-\left(\frac{31}{11}\right)^2=\frac{1259}{144}$$

Assuming that stimulus will not be accompanied by an increase in blood pressure, $\mu=0$.

$$\therefore \qquad t=\frac{31}{12}\times\frac{\sqrt{11}}{\sqrt{\frac{1259}{144}}}=\frac{31\sqrt{11}}{\sqrt{1259}}\approx 2·9$$

No. of $d.f.=12-1=11$

Now $\qquad t_{0·05}(11)=2·20$

$\therefore \qquad t_{cal}>t_{0·05}$

$\therefore$ Assumption is wrong and hence the stimulus will be, in general, accompanied by an increase in blood pressure.

Ex. 16·6. *Show that the 95% fiducial limits for the mean μ of the population are $\bar{x} \pm \frac{S t_{0.05}}{\sqrt{n}}$. Deduce that for a random sample of 16 values with mean 41·5″ and the sum of the squares of the deviations from the mean 135 (inches)^2 and drawn from a normal population, 95% fiducial limits for the mean of the population are 39·9″ and 43·1″.*

Sol. Since $P\{\,|t| < t_{0.05}\} = 0.95$, 95%, fiducial limits for the mean μ are given by

$$|t| = \left|\frac{\bar{x}-\mu}{S}\sqrt{n}\right| < t_{0.05}$$

i.e. $$|\bar{x}-\mu| < \frac{S t_{0.05}}{\sqrt{n}}$$

i.e. $$\bar{x} - \frac{S t_{0.05}}{\sqrt{n}} < \mu < \bar{x} + \frac{S t_{0.05}}{\sqrt{n}}$$

Here $n = 16$, $\bar{x} = 41.5$, $S^2 = \frac{135}{16-1} = 9$

No. of *d.f.* $= 16 - 1 = 15$.

$\therefore$ $t_{0.05} = 2.13$

$\therefore$ 95% fiducial limits are

$$41.5 \mp \frac{3}{4}(2.13) \qquad \textit{i.e.,} \quad 39.9 \text{ and } 43.1$$

EXERCISE

1. A machine which produces mica insulating washers of use in electric devices is set to turn out washers having a thickness of 10 mils (1 mil$=$0·001 inch). A sample of 10 washers has an average thickness of 9·52 mils with a s.d. of 0·60 mil. Test the significance of such a deviation. [**Ans.** $t = 2.4$]
2. Find the "student's t" for the following variate values in a sample of eight :

   $$-4, -2, -2, 0, 2, 2, 3, 3$$

   taking the mean of the universe to be zero. [**Ans.** 0·3]
3. Find student's t for the following variate values in a sample of ten :

   $$-6, -4, -3, -2, -2, 0, 1, 1, 3, 5$$

   taking μ to be zero. [**Ans.** 0·7]
4. Ten individuals are chosen at random from a population, their heights are found to be 63, 63, 66, 67, 68, 69, 70, 70, 71 and 71

inches respectively. Test whether the mean height is 69·6" in the population, given that for 9 *d f.* $P\{|t|>2{\cdot}262\}=0{\cdot}05$.

[Ans. $|t|=1{\cdot}89$]

5. The nine items of a sample had the following values :

45, 47, 50, 52, 48, 47, 49, 53, 51

Does the mean of the nine items differ significantly from the assumed population mean of 47·5 ? Given that

$$\nu=8\begin{cases}P=0{\cdot}945 \text{ for } t=1{\cdot}8\\ P=0{\cdot}953 \text{ for } t=1{\cdot}9\end{cases}$$

[Ans. $t=1{\cdot}84$]

6. The gain (in bushels per acre) in yield due to the use of a variety of wheat in nine plots is as follows :

16·3, 13·4, 3·8, 7·9, 2·6, 2·5, 9·6, 7·2 and 3·3

Are the observations consistent with the hypothesis that the average gain is 7·5 bushels per acre ? [Ans. Yes]

7. Ten individuals are chosen at random from a population and their heights are found to be inches 63, 63, 64, 65, 66, 69, 69 70, 70 and 71. Discussion the suggestion that the mean height in the universe is 65 inches. (Given that for 9 *d.f.* $t_{0{\cdot}05}=2{\cdot}262$)

[Ans. $t=2{\cdot}02$]

8. The table signifies additional hours of sleep gained by 10 patients in an experiment with a sleeping drug :

Patient	1	2	3	4	5	6	7	8	9	10
Hours gained	0·7	−1·1	−0·2	−1·2	0·1	3·4	3·7	0·8	1·9	2·0

Assuming that the hours of sleep is a normally distributed variable, calculate 't' for the above table. [Ans. 1·9]

9. A certain drug caused the following increases in blood pressure of 12 patients :

3, 0, 6, −2, 1, 5, 2, 8, 0, −1, 1, 5

Can it be concluded that the stimulus does not effect blood pressure ? [Ans. $t=2{\cdot}6$]

10. The mean weekly sale of the ice cream bar was 146·3 bars. After an advertising compaign the mean weekly sale in 22 shops for a typical week increased to 153·7 and showed a standard deviation 17·2. Is this evidence that the advertising was successful ? (Given that for $d.f.=21$, $t_{0{\cdot}05}=2{\cdot}08$]

[Ans. $t=1{\cdot}97$]

11. A certain colliery is supposed to supply coal of ash content about 15. To test this 20 random samples of the colliery's coal are selected and tested. The null hypothesis is that the

ash content is in fact 15. The results of 20 tests gave an average ash content of 16·8 with a standard deviation of 3·6. Is this sufficient evidence for rejecting the hypothesis?

(Given that for 19 *d.f.* $t_{0.05}=2{\cdot}09$) [Ans. $t=2{\cdot}18$]

12. A sample of 11 rats from a central population had an average blood viscosity of 3·92 with a *s.d.* of 0·61. On the basis of this sample establish 95% confidence limits of μ, the mean blood viscosity of central population. [Ans. 3·51 and 4·33]

13. The average breaking strength of steel rods is specified to be 17·5 lbs. To test this a sample of 14 rods was tested and gave the following results (in unit of 1,000 lbs)

15, 18, 16, 21, 19, 21, 17, 17, 15, 17, 20, 19, 17, 18

Is the result of the experiment significant? Also obtain the 95% fiducial limits from the sample for the average breaking strength of steel rods.

[Ans. $t=0{\cdot}68$]

14. The mean of I.Q.'s of 10 boys is 97·2 with the sum of the squares of the deviations from the mean of 1833·6. Do these data support the assumption of a population mean I.Q.'s of 100? Find the 95% confidence limits for the population mean.

[Ans. $t=0{\cdot}62$, 107·41 and 86·99]

16.3.2 Test for the difference of means

Let $x_1, x_2, \ldots x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ be two independent random samples with means $\bar{x}, \bar{y}$ and standard deviations s_1, s_2 respectively from two normal populations with the *same variance* σ. The problem is to test the hypothesis that the population means are μ_1 and μ_2. Assuming the populations means μ_1 and μ_2, the statistic is defined as

$$t=\frac{\bar{x}-\bar{y}-(\mu_1-\mu_2)}{S\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}$$

where
$$S^2=\frac{1}{n_1+n_2-2}\left[\sum_{i=1}^{n_1}(x_i-\bar{x})^2+\sum_{j=1}^{n_2}(y_j-\bar{y})^2\right]$$

$$=\frac{1}{n_1+n_2-2}\left[n_1 s_1^2+n_2 s_2^2\right]$$

Now $\bar{x}\sim N(\mu_1, \sigma/\sqrt{n_1})$

and $\bar{y}\sim N(\mu_2, \sigma/\sqrt{n_2})$

$$\therefore \quad \bar{x}-\bar{y}\sim N\left(\mu_1-\mu_2, \sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right)$$

$$\therefore \quad \left\{ \frac{(\bar{x}-\bar{y})-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \right\}^2 \text{ is a } \psi^2\text{-variate with 1 } d.f.$$

Also $\quad n_1 \dfrac{s_1^2}{\sigma^2}$ is a ψ^2-variate with (n_1-1) $d.f.$

and $\quad n_2 \dfrac{s_2^2}{\sigma^2}$ is a ψ^2-variate with (n_2-1) $d.f.$

$$\therefore \quad \frac{n_1 s_1^2}{\sigma^2}+\frac{n_2 s_2^2}{\sigma^2}=\frac{n_1 s_1^2+n_2 s_2^2}{\sigma^2}=\frac{\nu S^2}{\sigma^2},$$

where $\nu=n_1+n_2-2$, is a ψ^2 variate with $\nu=(n_1+n_2-2)$ $d.f.$

$$\therefore \quad \left\{ \frac{(\bar{x}-\bar{y})-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \right\}^2 \Big/ \frac{\nu S^2}{\sigma^2}=\frac{t^2}{\nu}$$

is $\beta_2\left(\dfrac{\nu}{2}, \dfrac{1}{2}\right)$ variate.

$\therefore$ Statistic t follows t-distribution with $\nu=n_1+n_2-2$ $d.f.$

If the hypothesis to be tested is "Are the two population means same or the two sample means differ significantly", under the null hypothesis "population means are same *i.e.*, $\mu_1=\mu_2$ or the two sample means do not differ significantly" the statistic to be calculated becomes

$$t=\frac{\bar{x}-\bar{y}}{S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

which also follows t-distribution with (n_1+n_2-2) $d.f.$

(*iii*) If (*i*) $n_1=n_2=n$ (say) and (*ii*) the samples are not independent but the sample observations are paired together *i.e.*, the pair of observations (x_i, y_i) $(i=1, 2, ...n)$ correspond to the same (ith) sample unit. The problem here again is to test "Are the sample means differ significantly".

Under the null hypothesis "sample means do dot differ significantly" the statistic

$$t=\frac{\bar{d}}{S/\sqrt{n}}$$

where
$$d_i=x_i-y_i$$

$$\bar{d}=\frac{1}{n}\sum_{i=1}^{n} d_i \text{ and}$$

$$S^2=\frac{1}{n-1}\sum_{i=1}^{n}(d_i-\bar{d})^2$$

which follows t-distribution with $(n-1)$ $d.f.$ is calculated.

Ex. 16-7. *For a random sample of 10 pigs fed on diet A the increases in weight in pounds in a certain period were*

10, 6, 16, 17, 13, 12, 8, 14, 15. 9 lbs.

For another sample of 12 pigs, fed on diet B the increases in the same period were

7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 lbs.

Test whether diets A and B differ significantly as regard the effect on increases in weight (or test whether the mean increases in the two samples are significantly different) you may use the fact 5% value of t for 20 d. f. is 2·09.

Sol. **Calculation of mean and s.d.**

x :	10	6	16	17	13	12	8	14	15	9			Total
$X=x-13$:	-3	-7	3	4	0	-1	-5	1	2	-4			-10
X^2 :	9	49	9	16	0	1	25	1	4	16			130
y :	7	13	22	15	12	14	18	8	21	23	10	17	
$Y=y-14$:	-7	-1	8	1	-2	0	4	-6	7	9	-4	3	12
Y^2 :	49	1	64	1	4	0	16	36	49	81	16	9	326

$\therefore \quad \bar{x}=13+\frac{(-10)}{10}=12$, and $\bar{y}=14+\left(\frac{12}{12}\right)=15$

$\therefore \quad s_1^2=\mu_2$ for $x=\mu_2$ for $X=\frac{\Sigma X^2}{10}-\left(\frac{\Sigma X}{10}\right)^2=13-1=12$

and $\quad s_2^2=\mu_2$ for $y=\mu_2$ for $Y=\frac{326}{12}-1=\frac{314}{12}$

$$\therefore \quad S^2=\frac{1}{10+12-2}\left\{(10)(12)+(12)\frac{(314)}{12}\right\}=21{\cdot}7$$

Assume the diets A and B do not differ significantly as regard the effect on increases in weight *i.e.*, the mean increases in two samples are not significantly different.

$$\text{Now } |t|=\frac{12\sim15}{\sqrt{21{\cdot}7}\sqrt{\frac{1}{10}+\frac{1}{12}}}=\frac{3\sqrt{120}}{\sqrt{21{\cdot}7}\sqrt{22}}$$

$$\simeq 1{\cdot}5$$

No. of $d.f.=10+12-2=20$

Now $\quad t_{0.05}$ for 20 $d.f.=2{\cdot}09$

$\therefore \qquad t_{cal} < t_{0.05}$

Assumption may be correct.

Ex. 16-8. *The following data represent the yield in bushels of Indian corn on ten sub-divisions of equal areas of two agricultural plots, in which Plot I was a central plot treated the same as Plot II except for the amount of phosphorus applied as a fertilizer* :

Plot I : 6·2 5·7 6·5 6·0 6·3 5·8 5·7 6·0 6·0 5·8

Plot II : 5·6 5·9 5·6 5·7 5·8 5·7 6·0 5·5 5·7 5·5

Is there a significant difference between the yields on the two plots, using the difference between their means as a criterion of judgment ?

Sol. Let x and y be variables for plots I and II.

											Tatal
x :	6·2	5·7	6·5	6·0	6·3	5·8	5·7	6·0	6·0	5·8	60
$X=x-\bar{x}$.	0·2	−0·3	0·5	0	0·3	−0·2	−0·3	0	0	−0·2	
Y^2 :	0·04	0·09	0·25	0	0·09	0·04	0·09	0	0	0·04	0·64
y :	5·6	5·9	5·6	5·7	5·8	5·7	6·0	5·5	5·7	5·5	57
$Y=y-\bar{y}$:	−0·1	0·2	−0·1	0	0·1	0	0·3	−0·2	0	−0·2	
Y^2 :	0·01	0·04	0·01	0	0·01	0	0·09	0·04	0	0·04	0·24

$$\bar{x}=\frac{\Sigma x}{10}=6 \text{ and } \bar{y}=\frac{57}{10}=5{\cdot}7$$

$$\therefore \quad S^2=\frac{1}{10+10-2}\,(0{\cdot}64+0{\cdot}24)=\frac{0{\cdot}44}{9}$$

$$\therefore \quad t=\frac{0{\cdot}3}{\sqrt{\frac{0{\cdot}44}{9}}\sqrt{\frac{1}{10}+\frac{1}{10}}}\simeq 3{\cdot}03$$

No. of $d.f.=10+10-2=18$

Now $\quad t_{0.05}$ for 18 $d.f.=2{\cdot}10$

$\therefore \qquad t_{cal} > t_{0.05}$

$\therefore$ The difference between the yields on the two plots is significant.

16-9. *Two independent samples of 8 and 7 items respectively had the following values :*

Sample I :	9	11	13	11	15	9	12	14
Sample II :	10	12	10	14	9	8	10	

Is the difference between the means of the samples significant ? Given that if $P\{\,|\,t\,|>t_0\}=0{\cdot}05$, $t_0=2{\cdot}16$ *for* 13 *d.f. and* $t_0=2{\cdot}13$ *for* 15 *d.f.*

Sol. **Calculation of mean and s.d.**

									Total
x :	9	11	13	11	15	9	12	14	
$X=x-11$:	-2	0	2	0	4	-2	1	3	6
X^2 :	4	0	4	0	16	4	1	9	38
y :	10	12	10	14	9	8	10		
$Y=y-10$:	0	2	0	4	-1	-2	0		3
Y^2 :	0	4	0	16	1	4	0		25

$$\therefore \quad \bar{x}=11+\frac{6}{8}=\frac{47}{4}\quad \bar{y}=10+\frac{3}{7}=\frac{73}{7}$$

$$s_1^2=\frac{38}{8}-\left(\frac{6}{8}\right)^2=\frac{67}{16} \text{ and } s_2^2=\frac{25}{7}-\left(\frac{3}{7}\right)^2=\frac{166}{49}$$

$$\therefore \; S^2=\frac{1}{8+7-2}\left\{8\left(\frac{67}{16}\right)+7\left(\frac{166}{49}\right)\right\}=\frac{801}{(14)(13)}$$

$$\therefore |t|=\frac{\frac{47}{4}\sim\frac{73}{7}}{\sqrt{\frac{801}{14.13}}\sqrt{\frac{1}{8}+\frac{1}{7}}}=\frac{37}{\sqrt{(801)(15)}}\sqrt{13}\approx 1{\cdot}2$$

No. of $d.f.=8+7-2=13$

$\therefore \qquad t_{0\cdot05}(13)=2{\cdot}16$

$\therefore \qquad t_{cal}<t_{0.05}$

$\therefore$ Difference is not significant.

Ex. 16-10. *Two horses A and B were tested according to the time (in seconds) to run a particular track with the following result :*

Horse A :	*28*	*30*	*.32*	*33*	*33*	*29*	*34*
Horse B :	*29*	*30*	*30*	*24*	*27*	*29*	

Test whether you can discriminate between the two horses. You can use the fact that 5% value of t for 11 d.f. is 2·20.

Sol. Let x and y be the time variable for horses A and B respectively.

Calculation of mean and s.d.

								Total
x :	28	30	32	33	33	29	34	
$X=x-32$:	-4	-2	0	1	1	-3	2	-5
X^2 :	16	4	0	1	1	9	4	35
y :	29	30	30	24	27	29		
$Y=y-29$:	0	1	1	-5	-2	0		-5
Y^2 :	0	1	1	25	4	0		31

$$\therefore \quad \bar{x}=32-\frac{5}{7}=\frac{219}{7},\ \bar{y}=29-\frac{5}{6}=\frac{169}{6}$$

$$s_1^2=\mu_2 \text{ of } x=\frac{35}{7}-\left(\frac{-5}{7}\right)^2=5-\frac{25}{49}=\frac{220}{49}$$

$$s_2^2=\mu_2 \text{ of } y=\frac{31}{6}-\left(\frac{-5}{6}\right)^2=\frac{161}{36}$$

$$\therefore \quad S^2=\frac{1}{7+6-2}\left\{\frac{220}{7}+\frac{161}{6}\right\}=\frac{2447}{462}$$

$$\therefore \quad |t|=\frac{\frac{219}{7}\sim\frac{169}{6}}{\sqrt{\frac{2447}{462}}\sqrt{\frac{1}{7}+\frac{1}{6}}}=\frac{131\sqrt{11}}{\sqrt{(13)(2447)}}\simeq 2{\cdot}4$$

No. of $d.f.=7+6-2=11$

$$\therefore \quad t_{0\cdot05}=2{\cdot}20$$

$$\therefore \quad t_{cal}>t_{0\cdot05}$$

$\therefore$ Difference is significant and hence two horses can be discriminated.

Ex. 16·11. *The following table shows the mean number of bacterial colonies per plate obtainable by four slightly different methods from soil samples taken at 4 P.M. and 8 P.M. respectively.*

Methods	*A*	*B*	*C*	*D*
4 P.M.	*29·75*	*27·50*	*30·25*	*27·80*
8 P.M.	*39·20*	*40·60*	*36·20*	*42·40*

Are there significantly more bacteria at 8 P.M. than at 4 P.M.?

Sol **Calculation of mean and s.d. of the difference**

Method	4 P.M. (x)	8 P.M. (y)	$d=y-x$	$d-\bar{d}$	$(d-\bar{d})^2$
A	29·75	39·20	9·45	−1·325	1·756
B	27·50	40·60	13·10	2·325	5·406
C	30·25	36·20	5·95	−4·825	23·281
D	27·80	42·40	14·60	3·825	14·631
Total			43·10		45·074

$$\bar{d}=\frac{43{\cdot}10}{4}=10{\cdot}775,\quad S^2=\frac{45{\cdot}074}{3}$$

$$\therefore \quad t=\frac{(10{\cdot}775)}{\sqrt{45{\cdot}074}}\sqrt{3(4)}=\frac{21{\cdot}55\sqrt{3}}{\sqrt{45{\cdot}074}}$$

$$\simeq 5{\cdot}56$$

No. of $d.f.=4-1=3$

Now $\qquad t_{0\cdot05}(3)=3{\cdot}18 < t_{cal}$

$\therefore$ Difference is highly significant and hence there are significantly more bacteria at 8 P.M. than at 4 P.M.

Ex. 16-12. *From the data given below test whether there is a significant difference between the effects of two drugs, on the assumption that different random samples of patients were used to test the two drugs A and B.*

Additional hours of sleep gained by use of soporfic drugs.

Patient	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Drug A	*0·7*	*−1·6*	*−0·2*	*−1·2*	*−0·1*	*3·4*	*3·7*	*0·8*	*0*	*2·0*
Drug B	*1·9*	*0·8*	*1·1*	*0·1*	*−0·1*	*4·4*	*5·5*	*1·6*	*4·6*	*3·6*

Sol. Calculation of mean and s.d. of the difference

Patient	x	y	$d=y-x$	$(d-\bar{d})$	$(d-\bar{d})^2$
1	0·7	1·9	1·2	−0.4	0·16
2	−1·6	0·8	2·4	0·8	0·64
3	−0·2	1.1	1·3	−0·3	0·09
4	−1·2	0·1	1·3	−0·3	0·09
5	−0·1	−0·1	0	−1·6	2·56
6	3·4	4·4	1·0	−0·6	0·36
7	3·7	5·5	1·8	0·2	0·04
8	0·8	1·6	0·8	−0·8	0·64
9	0	4·6	4·6	3·0	9·00
10	2·0	3·6	1·6	0	0·00
Total			16·0		13·58

$$\bar{d}=\frac{16}{10}=1{\cdot}6,\ S^2=\frac{13{\cdot}58}{9}$$

$$\therefore \quad t=\frac{1{\cdot}6}{\sqrt{\frac{13{\cdot}58}{9}}}\cdot\sqrt{10}=\frac{4{\cdot}8\sqrt{10}}{\sqrt{13{\cdot}58}}\simeq 4{\cdot}12$$

No. of $d.f.=10-1=9$

$t_{0\cdot05}(9)=2{\cdot}26<t_{cal}$

$\therefore$ Difference is highly significant.

Ex. 16-13. *In a certain experiment to compare two types of pig-foods A and B, the following results of increase in weights were observed in pigs :*

Pig No.		: 1	2	3	4	5	6	7	8
Increase in	*Food A*	: 49	53	51	52	47	50	52	53
wt. in lbs.	*Food B*	: 52	55	52	53	50	54	54	53

(a) Assuming that the two samples of pigs are independent can we conclude that food B is better than food A ? (b) Examine the case when the same set of eight pigs were used in both the foods.

Sol. (*a*) In this case the two samples of pigs are independent and hence the difference of means test for unpaired data can be applied.

Let x and y be the increase in weights due to foods A and B respectively.

Food A			*Food B*		
x	$X=x-50$	X^2	y	$X=y-52$	Y^2
49	−1	1	52	0	0
53	3	9	55	3	9
51	1	1	52	0	0
52	2	4	53	1	1
47	−3	9	50	−2	4
50	0	0	54	2	4
52	2	4	54	2	4
53	3	9	53	1	1
Total	7	37		7	23

$$\therefore \quad \bar{x}=50+\frac{7}{8}=\frac{407}{8},\ \bar{y}=52+\frac{7}{8}=\frac{423}{8}$$

$$\Sigma(x-\bar{x})^2=\Sigma(X-\bar{X})^2 =\Sigma X^2-\frac{1}{n_1}(\Sigma X)^2 =37-\frac{49}{8}=\frac{247}{8}$$

$$\Sigma(y-\bar{y})^2=\Sigma(Y-\bar{Y})^2 =23-\frac{49}{8}=\frac{135}{8}$$

$$\therefore \quad S^2=\frac{\frac{247}{8}+\frac{135}{8}}{8+8-2}=\frac{382}{112}$$

$$\therefore \quad |t| = \frac{\frac{407}{8} \sim \frac{423}{8}}{\sqrt{\frac{382}{112}}\sqrt{\frac{1}{8}+\frac{1}{8}}} \simeq 2{\cdot}17$$

$$\text{No. of } d.f. = 8+8-2=14$$

$$\therefore \quad t_{0{\cdot}05} = 2{\cdot}14 < |t_{cal}|$$

$\therefore$ At 5% level, difference between sample means is significant. Since $\bar{y} > \bar{x}$, food B is better.

(*b*) In this case the two samples cannot be regarded as independent. So the difference of means test for paired data will be applied.

									Total
x :	49	53	51	52	47	50	52	53	
y :	52	55	52	53	50	54	54	53	
$d=x-y$:	−3	−2	−1	−1	−3	−4	−2	0	$=-16$
d^2 :	9	4	1	1	9	16	4	0	$=44$

$$\bar{d} = -\frac{16}{8} = -2$$

$$\Sigma(d-\bar{d})^2 = 44 - \frac{1}{8}(-16)^2 = 12$$

$$\therefore \quad S^2 = \frac{12}{7}$$

$$\therefore \quad |t| = \frac{2\sqrt{8}}{\sqrt{\frac{12}{7}}} \simeq 4{\cdot}32$$

$$\text{No. of } d.f. = 8-1=7$$

$$\therefore \quad t_{0{\cdot}05} = 2{\cdot}36 \text{ and } t_{0{\cdot}01} = 3{\cdot}50$$

$$\therefore \quad |t_{cal}| > t_{0{\cdot}05} \text{ and } t_{0{\cdot}01}$$

$\therefore$ Difference between means is significant and hence food B is better.

EXERCISE

1. To compare the price of a certain commodity in two towns, ten shops were selected at random in ecah town. The following figures give the prices found :

Town A :	61	60	56	63	56	63	59	56	44	61
Town B :	55	54	47	59	51	61	57	54	62	58

Test whether the average price can be said to be the same in the two towns. (Ans. Yes)

2. Eight pots growing three wheat plants each were exposed to a high-tension discharge while nine similar pots were enclosed in an earthenware cage. The number of tillers in each pot were as follows :

Caged	: 17	26	18	25	27	28	26	23	17
Electrified	: 16	16	22	16	21	18	15	20	

Discuss whether electrification exercises any real effect on tillering. Given that $t_{0.05}(15)=2{\cdot}131$. (Ans. $|t|=2{\cdot}75$)

3. In a test given to two groups of students the marks obtained were as follows :

First group	: 18	20	36	50	49	36	34	41
Second group	: 29	28	26	35	30	44	46	

Calculate student's t and state the relevant null hypothesis.
(Ans. $|t|=0{\cdot}28$)

4. The heights of six randomly chosen sailors are, in inches : 63, 65, 68, 69, 71 and 72. Those of ten randomly chosen soldiers are 61, 62, 65, 66, 69, 69, 70, 71, 72 and 73. Discuss the light that these data throw on the suggestion that soldiers are on the average taller than sailors ; given that

$$\nu=14\begin{cases} P=0{\cdot}539 \text{ for } t=0{\cdot}1 \\ P=0{\cdot}527 \text{ for } t=0{\cdot}08 \end{cases}$$

(Ans. $|t|=0{\cdot}099$)

5. The means of two random samples of sizes 9 and 7 respectively are 196·42 and 198·82 respectively. The sum of the squares of the deviations from the means are 26·94 and 18·73 respectively. Can the samples be considered to have been drawn from the same normal populations, it being given that $t_{0.05}(14)=2{\cdot}145$?
(Ans. 2·6, No)

6. Two independent samples of 8 and 7 items respectively had the following values :

Sample 1 :	9	11	13	11	15	9	12	14
Sample 2 :	10	12	10	14	9	8	10	

Is the difference between the means of the samples significant ? Given that

$$\nu=13\begin{cases} P=0{\cdot}874 \text{ for } t=1{\cdot}2 \\ P=0{\cdot}892 \text{ for } t=1{\cdot}3 \end{cases}$$

(Ans. $|t|=1{\cdot}22$)

7. Two types of batteries A and B are tested for their length of life and the following results are obtained :

	No. in sample	*Mean*	*Variance*
A	10	500 hours	100
B	10	560 hours	121

Is there a significant difference in two means ? (Ans. Yes)

8. Intelligence test of two groups of boys and girls gave the following results. Examine if the difference of the means is significant.

Group of 12 girls : mean=84, $s.d.=10$

Group of 8 boys : mean=81, $s.d.=12$ (**Ans.** No)

9. A farmer grows crops on two fields A and B. On A he puts Rs. 10 worth of manure per acre and on B Rs. 20 worth. The net returns per acre, exclusive of the cost of manure, on the two fields are :

Year	1	2	3	4	5
Yield A Rs. per acre	34	28	42	37	44
Yield B Rs. per acre	36	33	48	38	50

Other things beings equal, discuss the question whether it is likely to pay the former to continue the more expensive dressing ? Given that $t_{0.05}(4)=2.78$. (**Ans.** $t=3.8$)

10. Calculate the value of 't' in case of two characteristics A and B whose corresponding frequencies are :

A :	16	10	8	9	9	8
B :	8	4	5	9	12	4

(**Ans.** $|t|=1.66$)

11. The yields of two types 'Type 17' and 'Type 51' of grains in pounds per acre in a replications are given below. What comment would you make on the differences in the mean yields ? You may assume that if there be 5 d.f. and $P=0.2$, t is approximately 1·476.

Replication :	1	2	3	4	5	6
Yield in lbs. : (Type 17)	20·50	24·60	23·06	29·98	30·37	23·83
Yield in lbs. : (Type 51)	24·86	26·39	28·19	30.75	29·97	22·04

(**Ans.** $t=1.49$)

12. The following figures show the additional hours of sleep gained by 10 patients when each was administered two soporifics A and B (with an adequate period between the two)

Patient :	1	2	3	4	5	6	7	8	9	10
Soporific A :	1·2	1·8	−0·3	−0·7	0·1	3·1	2·2	−1·5	0·0	2·1
Soporific B :	1·5	1·6	0·4	0·0	−0·6	2·5	4·5	1·9	2·2	3·0

Apply an appropriate test to see whether the soporifics really differ in average effect. (**Ans.** $t=2.1$)

13. In each of 10 pairs of rats, one receives protein from raw peanut while the other receives it from roasted peanuts.

The following table gives the gain in weights :

Pair	1	2	3	4	5	6	7	8	9	10
Raw	61	60	56	63	56	63	59	56	44	61
Roasted	55	54	47	69	51	61	57	54	62	58

Test whether or not roasting the peanuts had any effect on their protein value. (**Ans.** Roasting had no effect on protein value)

14. In a rat-feeding experiment, the following results were obtained :

Diets	Gain in wts. in gms.											
High Protein	13	14	10	11	12	16	10	9	11	12	9	12
Low Protein	7	11	10	8	10	13	9					

Investigate if there is any evidence of superiority of one diet over the other. [**Ans.** $|t| = 1{\cdot}93$]

15. Mitchel concluded a paired feeding experiment with pigs on the relative value of limestone and bonemeal for bone development. The results are given below :

Pair	1	2	3	4	5	6	7	8
Lime stone	49·2	53·3	50·6	52·0	46·8	50·5	52·1	53·0
Bonemeal	51·5	54·9	52·2	53·3	51·6	54·1	54·2	54·3

Determine the significance of the difference between the means in two ways (1) by assuming that the values are paired (2) by assuming that the values are not paired.

[**Ans.** Difference is significant]

16. Ten soldiers visit a rifle range for two consecutive weeks. For the first week their scores are :

67, 24, 57, 55, 63, 54, 56, 68, 33, 43

and during the second week they score, in the same order :

70, 38, 58, 58, 56, 67, 68, 72, 42, 38

Examine if there is any significant difference in their performance. [**Ans.** $|t| = 2{\cdot}04$]

17. Eleven school boys were given a test in mathematics. They were given a month's further tution and a second test was held at the end of it. Do the marks give evidence that the students benefitted by extra coaching ?

Roll No.	1	2	3	4	5	6	7	8	9	10	11
Marks: First test	23	20	19	21	18	20	18	17	23	16	19
Marks: Second test	24	19	22	18	20	22	20	20	23	20	17

[**Ans.** $|t| = 1{\cdot}48$]

18. To test the desirability of a certain modification in typists desks, 9 typists were given two tests of as nearly as possible the same nature, one on the desk in use and the other on the new type. The following differences in the number of words typed per minutes were recorded :

Typist	A	B	C	D	E	F	G	H	I
Increased no. of words per min	2	3	0	3	−1	4	−3	2	5

Do the data indicate that the modification in desk promoted speed in typing ? [Ans. $t=1{\cdot}96$]

16.3.3. Test for the significance of an Observed Correlation Coefficient

Consider a random sample (x_1, y_1), (x_2, y_2),...... (x_n, y_n) from a bi-variate normal population. Let the correlation coefficient calculated from the sample be r and ρ the population correlation coefficient. The hypothesis to be tested is :

'Whether population correlation coefficient is zero *i.e.*, $\rho=0$'.

Assuming this hypothesis, the statistic

$$t=\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

follows t-distribution with $(n-2)$ $d.f.$*

To test the significance, calculated value of 't' is compared with the tabulated value and the significance is tested as usual.

Ex. 16·14. *Test whether the correlation is significant if $r=0{\cdot}6$ $n=18$.*

Sol. $$t=\frac{(0{\cdot}6)\sqrt{16}}{\sqrt{0{\cdot}64}}=3$$

No. of $d.f.=18-2=16$

$\therefore$ $t_{0{\cdot}05}=2{\cdot}12 < t_{cal}$

$\therefore$ Correlation is significant.

Ex. 16-15. *A random sample of 18 pairs from a bivariate normal population showed a correlation co-efficient of 0·3. Is this value significant of correlation in the population ?*

Sol. $$t=\frac{(0{\cdot}3)\sqrt{18-2}}{\sqrt{1-0{\cdot}09}}=1{\cdot}26$$

No. of $d.f.=18-2=16$

$\therefore$ $t_{0{\cdot}05}=2{\cdot}12 > t_{cal}$

$\therefore$ Correlation is not significant.

*The proof of this is beyond the scope of this book.

Ex. 16.16. *Find the least value of 'r', in a sample of 18 pairs of observations from a bivariate normal population, significant at 5% level.*

Sol. No. of $d.f.=18-2=16$

$\therefore \qquad t_{0.05}=2.12$

The least value of 'r' significant at 5% level is given by

$$\left|\frac{r\sqrt{18-2}}{\sqrt{1-r^2}}\right| > 2.12$$

i.e., $\qquad 16r^2 > (2.12)^2(1-r^2)=4.4944(1-r^2)$

i.e., $\qquad 20.4944r^2 > 4.4944$

i.e., $\qquad r^2 > 0.2193$

i.e., $\qquad |r| > 0.4682$

$\therefore$ Reqd. value of $|r|=0.4682$

Ex. 16.17. *A random sample of 15 from a normal population gives a correlation co-efficient of -0.5. Is this significant of the existence of correlation in the population?*

Sol. $$t=\frac{(-0.5)\sqrt{13}}{\sqrt{0.75}}\simeq-2.08$$

$t_{0.05}$ for 13 $d.f.=2.16 > |t_{cal}|$

$\therefore$ Sample correlation co-efficient is not significant.

6.3.4. Test for the significance of an observed Regression Co-efficient

Consider a random sample (x_1, y_1)..... (x_n, y_n) from a bivariate normal population. Let the equation of line of regression of y on x (obtained from the sample) be

$$y-\bar{y}=b(x-\bar{x})$$

where b=regression coeff. of y on x.

Let $\qquad Y_i=\bar{y}+b(x_i-\bar{x})$

The hypothesis to be tested is :

"The regression co-efficient of y on x in the population is β".

Assuming this hypothesis, the statistic

$$t=(b-\beta)\left\{\frac{(n-2)\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(y_i-Y_i)^2}\right\}^{1/2}$$

follows t-distribution with $(n-2)$ $d.f.$

16.3.5. Test for the Significance of a Rank Correlation Coefficient

Let ρ be the rank correlation coefficient obtained from a sample of size n. The hypothesis to be tested is :

"Population rank correlation coefficient is zero."

Assuming this hypothesis, the statistic

$$t=\rho\left\{\frac{n-2}{1-\rho^2}\right\}^{1/2}$$

follow t-distribution with $(n-2)$ $d.f.$

Ex. 16.18. *12 pictures submitted in a competition were ranked by two judges with results as shown in the table below :*

Pictures	:	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Rank assigned by 1st judge	:	5	9	6	7	1	3	4	12
Rank assigned by 2nd judge	:	5	8	9	11	3	1	2	10

		I	*J*	*K*	*L*
		2	11	10	8
		4	12	7	6

Calculate ρ *the rank correlation co-efficient. Is there a lack of independence in these ranking ? (Assume that on the hypothesis of independence of two sets of n readings,* $t=\rho\left(\frac{n-2}{1-\rho^2}\right)^{1/2}$ *follows t-distribution with* $(n-2)$ $d.f.$ *Given that for* 10 $d.f.$, $t_{0.05}=2\cdot23$.

Sol. Let d be the difference in ranks assigned to the same individuals.

Now $\Sigma d^2=0+1+9+16+4+4+4+4+4+1+9+4=60$

$$\therefore \quad \rho=1-\frac{(6)(60)}{(12)(143)}\simeq 0\cdot79$$

$$\therefore \quad t=(0\cdot79)\,\frac{\sqrt{10}}{\sqrt{1-(0\cdot79)^2}}=\frac{(0\cdot79)\sqrt{10}}{\sqrt{0\cdot3759}}$$

$$\simeq 4\cdot075$$

No. of $d.f.=12-2=10$

$$\therefore \quad t_{0\cdot05}=2\cdot23 < t_{cal}$$

$\therefore$ Ranking is significant and hence there is lack of independence.

EXERCISE

1. Is a correlation co-efficient of 0·5 significant, if obtained from a random sample of 12 pairs of values from a normal population? [Ans. No.]

2. Find the least value of r, in a sample of 25 pairs from normal population, which is significant at 5% level. (Given that for 23 $d.f.$, $t_{0.05}=2{\cdot}07$). [Ans. 0·4]

16.4. F-distribution

Let $x_1, x_2, \ldots\ldots x_{n_1}$ and $y_1, y_2 \ldots\ldots y_{n_2}$

be two independent random samples drawn from the same normal population with variate σ^2.

Let $\bar{x}$ and $\bar{y}$ be the sample means and

$$S_1^2=\frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_i-\bar{x})^2,\quad S_2^2=\frac{1}{n_2-1}\sum_{i=1}^{n_2}(y_i-\bar{y})^2$$

The statistic F is defined by

$$F=\frac{S_1^2}{S_2^2}\qquad (S_1^2>S_2^2)$$

Let $\quad \nu_1=n_1-1,\quad \nu_2=n_2-1$

$$\therefore\quad \frac{\nu_1 F}{\nu_2}=\frac{(n_1-1)\,S_1^2}{(n_2-1)\,S_2^2}$$

$$=(n_1s_1^2/\sigma^2)/(n_2s_2^2/\sigma^2)$$

Now $n_2s_1^2/\sigma^2$ is a ψ^2 variate with ν_1 $d.f.$ and $n_2s_2^2/\sigma^2$ is a ψ^2-variate with ν_2 $d.f.$

$$\therefore\quad \frac{\nu_1 F}{\nu_2} \text{ is a } \beta_2\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right) \text{ variate.}$$

$\therefore$ Distribution of F is

$$dP=\frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\,\frac{(\nu_1F/\nu_2)^{\frac{\nu_1}{2}-1}}{(1+\nu_1F/\nu_2)^{\frac{\nu_1+\nu_2}{2}}}\,d\left(\frac{\nu_1F}{\nu_2}\right)$$

$$=\frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\,\frac{\nu_1^{\frac{\nu_1}{2}}\cdot\nu_2^{\frac{\nu_2}{2}}\cdot F^{\frac{\nu_1}{2}-1}}{(\nu_2+\nu_1F)^{\frac{\nu_1+\nu_2}{2}}}\,dF\qquad 0<F<\infty$$

This distribution is called the distribution of the variance ratio F with ν_1 and ν_2 *d. f.*

Ex. *Let ψ_1^2 and ψ_2^2 be two independent chi-square variates with n_1 and n_2 d. f. Find the distribution of* $F=\dfrac{\psi_1^2/n_1}{\psi_2^2/n_2}$.

16·4·1. Constants of F-Distribution

$\mu_r'(0)=E\{F^r\}$

$$=\frac{\nu_1^{\frac{\nu_1}{2}}\nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\int_0^\infty F^r\frac{F^{\frac{\nu_1}{2}-1}}{(\nu_2+\nu_1F)^{\frac{\nu_1+\nu_2}{2}}}\,dF$$

$$=\frac{\nu_1^{\frac{\nu_1}{2}}\nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\int_0^\infty \frac{F^{\frac{\nu_1}{2}+r-1}}{(\nu_2+\nu_1F)^{\frac{\nu_1+\nu_2}{2}}}\,dF$$

Put $\nu_1F=\nu_2x \Rightarrow dF=\dfrac{\nu_2dx}{\nu_1}$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\int_0^\infty\frac{x^{\frac{\nu_1}{2}+r-1}}{(1+x)^{\left(\frac{\nu_1}{2}+r\right)+\left(\frac{\nu_2}{2}-r\right)}}\,dx$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\beta\left(\frac{\nu_1}{2}+r,\frac{\nu_2}{2}-r\right)$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{\Gamma\left(\frac{\nu_1}{2}+r\right)\Gamma\left(\frac{\nu_2}{2}-r\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{\left(\frac{\nu_1}{2}+r-1\right)\cdots\cdots\left(\frac{\nu_1}{2}\right)}{\left(\frac{\nu_2}{2}-1\right)\cdots\cdots\left(\frac{\nu_2}{2}-r\right)}$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{\{\nu_1+2(r-1)\}\cdots\cdots(\nu_1)}{(\nu_2-2)\cdots\cdots(\nu_2-2r)}$$

$\therefore$ Mean $=\mu_1'(0)=\dfrac{f\nu_2}{\nu_2-2}>1.$

Put $\qquad r=2$

$$\mu_2'(0)=\left(\frac{\nu_2}{\nu_1}\right)^2 \frac{\nu_1(\nu_1+2)}{(\nu_2-2)(\nu_2-4)}$$

$$=\frac{\nu_2^2(\nu_1+2)}{\nu_1(\nu_2-2)(\nu_2-4)}$$

$\therefore \qquad \mu_2=\mu_2'(0)-\{\mu_1'(0)\}^2$

$$=\frac{\nu_2^2(\nu_1+2)}{\nu_1(\nu_2-2)(\nu_2-4)}-\left(\frac{\nu_2}{\nu_2-2}\right)^2$$

$$=\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$$

Mode. The density function of F variate is

$$f(F)=\frac{1}{\beta\left(\frac{\nu_1}{2},\ \frac{\nu_2}{2}\right)}\ \frac{\nu_1^{\frac{\nu_1}{2}}\ \nu_2^{\frac{\nu_2}{2}}\ F^{\frac{\nu_1}{2}-1}}{(\nu_2+\nu_1 F)^{\frac{\nu_1+\nu_2}{2}}}$$

$$\frac{df}{dF}=\frac{\nu_1^{\frac{\nu_1}{2}}\ \nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\ \frac{\nu_2}{2}\right)}\ \frac{\left\{\left(\frac{\nu_1}{2}-1\right) F^{\frac{\nu_1}{2}-2}\ (\nu_2+\nu_1 F)^{\frac{\nu_1+\nu_2}{2}} - F^{\frac{\nu_1}{2}-1}\left(\frac{\nu_1+\nu_2}{2}\right)(\nu_2+\nu_1F)^{\frac{\nu_1+\nu_2}{2}-1}.\nu_1\right\}}{(\nu_2+\nu_1F)^{\nu_1+\nu_2}}$$

$$\frac{df}{dF}=\frac{\nu_1^{\frac{\nu_1}{2}}\ \nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\quad F^{\frac{\nu_1}{2}-2}\ (\nu_2+\nu_1 F)^{\frac{\nu_1+\nu_2}{2}-1}$$

$$\cdot\left\{\frac{\left(\frac{\nu_1}{2}-1\right)(\nu_2+\nu_1 F)-F\left(\frac{\nu_1+\nu_2}{2}\right)\nu_1}{(\nu_2+\nu_1 F)^{\nu_1+\nu_2}}\right\}$$

$$\frac{df}{dF}=0\Rightarrow$$

$$F=0,\quad \text{and}$$

$$\left(\frac{\nu_1}{2}-1\right)(\nu_2+\nu_1 F)-F\left(\frac{\nu_1+\nu_2}{2}\right)\nu_1=0$$

This distribution is called the distribution of the variance ratio F with ν_1 and ν_2 *d. f.*

Ex. *Let ψ_1^2 and ψ_2^2 be two independent chi-square variates with n_1 and n_2 d. f. Find the distribution of* $F=\dfrac{\psi_1^2/n_1}{\psi_2^2/n_2}$.

16·4·1. Constants of F-Distribution

$\mu_r'(0)=E\{F^r\}$

$$=\frac{\nu_1^{\frac{\nu_1}{2}}\nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\int_0^\infty F^r\frac{F^{\frac{\nu_1}{2}-1}}{(\nu_2+\nu_1F)^{\frac{\nu_1+\nu_2}{2}}}\,dF$$

$$=\frac{\nu_1^{\frac{\nu_1}{2}}\nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\int_0^\infty \frac{F^{\frac{\nu_1}{2}+r-1}}{(\nu_2+\nu_1F)^{\frac{\nu_1+\nu_2}{2}}}\,dF$$

Put $\nu_1F=\nu_2x \Rightarrow dF=\dfrac{\nu_2dx}{\nu_1}$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\int_0^\infty\frac{x^{\frac{\nu_1}{2}+r-1}}{(1+x)^{\left(\frac{\nu_1}{2}+r\right)+\left(\frac{\nu_2}{2}-r\right)}}\,dx$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\,\beta\left(\frac{\nu_1}{2}+r,\frac{\nu_2}{2}-r\right)$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{\Gamma\left(\frac{\nu_1}{2}+r\right)\Gamma\left(\frac{\nu_2}{2}-r\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{\left(\frac{\nu_1}{2}+r-1\right)\ldots\ldots\left(\frac{\nu_1}{2}\right)}{\left(\frac{\nu_2}{2}-1\right)\ldots\ldots\left(\frac{\nu_2}{2}-r\right)}$$

$$=\left(\frac{\nu_2}{\nu_1}\right)^r\frac{\{\nu_1+2(r-1)\}\ldots\ldots(\nu_1)}{(\nu_2-2)\ldots\ldots(\nu_2-2r)}$$

$\therefore\quad \text{Mean}=\mu_1'(0)=\dfrac{\nu_2}{\nu_2-2}>1.$

Put $\qquad r=2$

$$\mu_2'(0)=\left(\frac{\nu_2}{\nu_1}\right)^2\frac{\nu_1(\nu_1+2)}{(\nu_2-2)(\nu_2-4)}$$

$$=\frac{\nu_2^2(\nu_1+2)}{\nu_1(\nu_2-2)(\nu_2-4)}$$

$$\therefore \qquad \mu_2=\mu_2'(0)-\{\mu_1'(0)\}^2$$

$$=\frac{\nu_2^2(\nu_1+2)}{\nu_1(\nu_2-2)(\nu_2-4)}-\left(\frac{\nu_2}{\nu_2-2}\right)^2$$

$$=\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$$

Mode. The density function of F variate is

$$f(F)=\frac{1}{\beta\left(\frac{\nu_1}{2},\ \frac{\nu_2}{2}\right)}\ \frac{\nu_1^{\frac{\nu_1}{2}}\nu_2^{\frac{\nu_2}{2}}F^{\frac{\nu_1}{2}-1}}{(\nu_2+\nu_1 F)^{\frac{\nu_1+\nu_2}{2}}}$$

$$\frac{df}{dF}=\frac{\nu_1^{\frac{\nu_1}{2}}\nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\ \frac{\nu_2}{2}\right)}\ \frac{\left\{\left(\frac{\nu_1}{2}-1\right)F^{\frac{\nu_1}{2}-2}(\nu_2+\nu_1 F)^{\frac{\nu_1+\nu_2}{2}}-F^{\frac{\nu_1}{2}-1}\left(\frac{\nu_1+\nu_2}{2}\right)(\nu_2+\nu_1F)^{\frac{\nu_1+\nu_2}{2}-1}.\nu_1\right\}}{(\nu_2+\nu_1F)^{\nu_1+\nu_2}}$$

$$\frac{df}{dF}=\frac{\nu_1^{\frac{\nu_1}{2}}\nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\ F^{\frac{\nu_1}{2}-2}(\nu_2+\nu_1 F)^{\frac{\nu_1+\nu_2}{2}-1}\cdot\left\{\frac{\left(\frac{\nu_1}{2}-1\right)(\nu_2+\nu_1 F)-F\left(\frac{\nu_1+\nu_2}{2}\right)\nu_1}{(\nu_2+\nu_1 F)^{\nu_1+\nu_2}}\right\}$$

$$\frac{df}{dF}=0\Rightarrow$$

$$F=0,\quad \text{and}$$

$$\left(\frac{\nu_1}{2}-1\right)(\nu_2+\nu_1 F)-F\left(\frac{\nu_1+\nu_2}{2}\right)\nu_1=0$$

$$\Rightarrow \qquad F=\frac{\nu_2(\nu_1-2)}{\nu_1(\nu_2+2)}$$

for $F=0$, $f(F)=0$ which is minimum value of $f(F)$.

$\therefore$ for $F=\dfrac{\nu_2(\nu_1-2)}{\nu_1(\nu_2+2)}$ $f(F)$ is maximum

$$\therefore \quad \text{Mode}=\frac{\nu_2(\nu_1-2)}{\nu_1(\nu_2+2)}$$

15.4.2. Chief features of F-Probability Curve

The equation of the F-probability curve is

$$y=\frac{\nu_1^{\frac{\nu_1}{2}}\nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\ \frac{F^{\frac{\nu_1}{2}-1}}{(\nu_2+\nu_1 F)^{\frac{\nu_1+\nu_2}{2}}}\ .\ 0<F<\infty.$$

(1) At $F=0$, $y=0$

and as $F\to\infty$, $y\to 0$

$\therefore$ F-axis is asymptote to the curve at positive extremity.

(2) Mode is at the point

$$F=\frac{\nu_2(\nu_1-2)}{\nu_1(\nu_2+2)}$$

which exist only when $\nu_1>2$ $(\because \ F\geqslant 0)$

$$\text{Now} \qquad \text{Mode}-1=\frac{\nu_2(\nu_1-2)}{\nu_1(\nu_2+2)}-1$$

$$=\frac{-2(\nu_1+\nu_2)}{\nu_1(\nu_2+\nu_2)}<0$$

$$\Rightarrow \qquad \text{Mode}<1.$$

(3) Karl Pearson's coefficient of skewness is

$$\frac{\text{mean}-\text{mode}}{s.d}$$

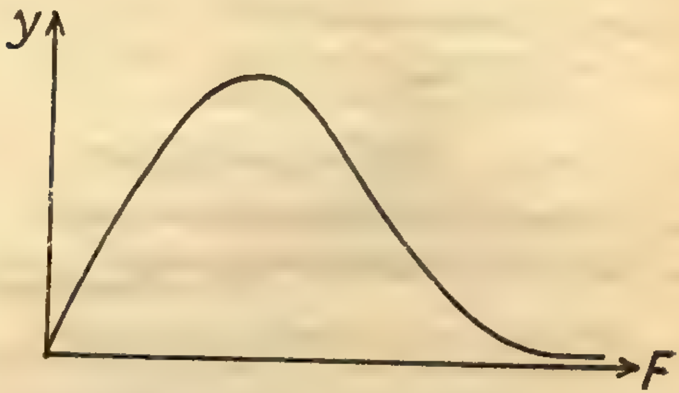
Since mean >1 and mode <1.

F probability curve is highly positively skewed.

(4) The pts of inflextion of F curve exist when $\nu_1>4$ and are equidistant from mode.

(5) y increases steadily at first until it reaches is maximum value and then decreases slowly.

∴ The shape of the probability curve is approximately as shown.



16.4.3. Point of Inflextion

The equation of the probability curve is

$$y=\frac{\nu_1^{\frac{\nu_1}{2}}\ \nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\ \frac{F^{\frac{\nu_1}{2}-1}}{(\nu_2+\nu_1 F)^{\frac{\nu_1+\nu_2}{2}}}$$

Put $\nu_1 F=\nu_2 x.$

Then $y=c\cdot\dfrac{x^{l-1}}{(1+x)^{l+m}}=c\cdot\dfrac{x^{l-1}}{(1+x)^{l+m}}$

where $c=\dfrac{\nu_1}{\nu_2}\ \dfrac{1}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)},\ l=\dfrac{\nu_1}{2},\ m=\dfrac{\nu_2}{2}$

∴ $\log\ y=\log c+(l-1)\log x-(l+m)\log(1+x)$

Differentiating w.r.t. x

$$\frac{1}{y}\ \frac{dy}{dx}=(l-1)\frac{1}{x}-(l+m)\ \frac{1}{1+x}$$

Differentiating again

$$-\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2+\frac{1}{y}\frac{d^2y}{dx^2}=\frac{-(l-1)}{x^2}+\frac{l+m}{(1+x)^2}$$

At points of inflextion

$$\frac{d^2y}{dF^2}=\left(\frac{\nu_1}{\nu_2}\right)^2\ \frac{d^2y}{dx^2}=0 \text{ and } \frac{d^3y}{dF^3}=\left(\frac{\nu_1}{\nu_2}\right)^3\ \frac{d^3y}{dx^3}\neq 0$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2}=0, \quad \frac{d^3y}{dx^3}\neq 0$$

$\therefore$ Points of inflextion are given by

$$\therefore \qquad \frac{-1}{y^2}\left(\frac{dy}{dx}\right)^2=-\frac{(l-1)}{x^2}+\frac{l+m}{(1+x)^2}$$

i.e., $\left\{(l-1)\frac{1}{x}-\frac{l+m}{1+x}\right\}^2-\frac{l-1}{x^2}+\frac{l+m}{(1+x)^2}=0$

i.e., $\{(l-1)(1+x)-(l+m)x\}^2-(l-1)(1+x)^2+(l+m)x^2=0$

i.e., $\{(l-1)^2-(l-1)\}(1+x)^2+\{(l+m)^2+(l+m)\}x^2$
$-2(l-1)(l+m)x(1+x)=0$

i.e., $x^2\{(l-1)^2-(l-1)+(l+m)^2+(l+m)-2(l-1)(l+m)\}$
$+2x\{(l-1)^2-(l-1)-(l-1)(l+m)\}+\{(l-1)^2-(l-1)\}=0$

The roots of this equation give two points of inflextion. Let these be x_1 and x_2

$$\therefore \quad x_1+x_2=\frac{-2\{(l-1)^2-(l-1)-(l-1)(l+m)\}}{(l-1)^2-(l-1)+(l+m)^2+(l+m)-2(l-1)(l+m)}$$

$$=\frac{2(l-1)(m+2)}{-(l-1)(m+2)+(l+m)(m+2)}$$

$$=\frac{2(l-1)}{m+1}$$

$$\therefore \qquad x_1+x_2=\frac{2(\nu_1-2)}{\nu_2+2}$$

Let F_1 and F_2 be the corresponding values of F. Then F_1 and F_2 are pts. of inflextion.

Then $\qquad \nu_1F_1=\nu_2x_1 \quad \Rightarrow \quad F_1=\frac{\nu_2}{\nu_1}\,x_1$

and $\qquad F_2=\frac{\nu_2}{\nu_1}\,x_2$

$$\therefore \qquad F_1+F_2=\frac{\nu_2}{\nu_1}\,(x_1+x_2)$$

$$=\frac{\nu_2}{\nu_1}\cdot\frac{2(\nu_1-2)}{\nu_2+2}$$

$=2$ (mode).

$\therefore \qquad F_1-\text{mode}=\text{mode}-F_2.$

$\therefore$ F_1 and F_2 are equidistant from mode.

The condition $\frac{d^3y}{dx^3} \neq 0 \Rightarrow \nu_1 > 4.$*

$\therefore$ Pts. of inflextion exist if $\nu_1 > 4$.

Ex. 16-19. (*a*) *If x has a F-distribution with (m, n) d.f. show that $\frac{1}{x}$ has a F-distribution with (n, m.) d.f.*

(*b*) *Deduce that, for any $k>0$*

$$P\{x \leqslant k\} + P\left\{y \leqslant \frac{1}{k}\right\} = 1$$

where x and y are F distributed with (m, n) and (n, m) d.f.s. respectively.

Sol. (*a*) Dist. of x is

$$dP = \frac{m^{\frac{m}{2}}\, n^{\frac{n}{2}}}{\beta\left(\frac{m}{2}, \frac{n}{2}\right)} \frac{x^{\frac{m}{2}-1}}{(n+mx)^{\frac{m+n}{2}}}\, dx,\ 0 \leqslant x \leqslant \infty$$

Put $x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2}\, dy$

$\therefore$ Dist. of y is

$$dP = \frac{m^{\frac{m}{2}}\, n^{\frac{n}{2}}}{\beta\left(\frac{m}{2}, \frac{n}{2}\right)} \frac{\left(\frac{1}{y}\right)^{\frac{m}{2}-1}}{\left(n+\frac{m}{y}\right)^{\frac{m+n}{2}}} \frac{1}{y^2}\, dy$$

$$= \frac{m^{\frac{m}{2}}\, n^{\frac{n}{2}}}{\beta\left(\frac{n}{2}, \frac{m}{2}\right)} \frac{y^{\frac{n}{2}-1}}{(m+ny)^{\frac{m+n}{2}}}\, dy$$

$$0 \leqslant y \leqslant \infty$$

$\Rightarrow \quad y = \frac{1}{x}$ is F distributed with (n, m) *d. f.*

(*b*) Since total prob is unity,

$$P\{x \leqslant k\} + P\{x > k\} = 1$$

*The proof of this is left as an exercise for the reader.

Now $$P\{x>k\}=P\left\{\frac{1}{x}\leqslant\frac{1}{k}\right\}$$
$$=P\left\{\ y<\frac{1}{k}\right\}$$

where $y=\dfrac{1}{x}$ is $F-$distributed with (n, m) *d.f.*

$$\therefore \qquad P\{x\leqslant k\}+P\left\{\ y<\frac{1}{k}\right\}=1.$$

Ex. 16-20. *If $\nu_1=\nu_2$, the median of F distribution is at $F=1$, Show also that*

$$Q_1Q_3=1$$

where Q_1, Q_3 are quartiles.

Sol. Let $\nu_1=\nu_2=\nu$.

Let x be F distributed with (ν, ν) *d.f.*

Then $y=\dfrac{1}{x}$ is also F distributed with (ν, ν) *d. f.*

$$\therefore \qquad P\{x<a\}=P\{y<a\}, \text{ for any } a.$$

(*i*) Let k be the median.

Then $$P\{x<k\}=P\{x>k\}$$
$$=P\left\{\ y<\frac{1}{k}\right\}$$
$$=P\left\{\ x<\frac{1}{k}\right\}$$

which is possible only when $k=1$.

$\therefore$ Median$=1$.

(*ii*) Now $P(x<Q_1)=P(x>Q_3)$
$$=P\left(\ y<\frac{1}{Q_3}\right)$$
$$=P\left(\ x<\frac{1}{Q_3}\right)$$

which is possible only when

$$Q_1=\frac{1}{Q_3}$$

i.e., $$Q_1\,Q_3=1$$

EXERCISE

1. If $\nu_1=2$, show that

$$P(F>F_0)=\left(1+\frac{2F_0}{\nu_2}\right)^{-\frac{\nu_2}{2}}.$$

2. If x has a F distribution with (n_1, n_2) $d.f.$, show that

$$\left(1+n_1\frac{x}{n_2}\right)^{-1}$$

has a Beta distribution.

3. If $\nu_1=\nu_2=n-1$, show that

$$\text{H.M.}=\frac{n-1}{n-3}.$$

16.4.4. Relation between t and F distributions

Student's t-distribution is

$$dP=\frac{1}{\sqrt{\nu}}\,\frac{1}{\beta\left(\frac{\nu}{2},\frac{1}{2}\right)}\,\frac{dt}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}},\ -\infty<t<\infty$$

where ν=No. of $d.f.$

Put $t^2=x$ i.e., $t=\sqrt{x}$

$$\therefore\quad dt=\frac{1}{2\sqrt{x}}\,dx$$

$$\therefore\quad dP=\frac{1}{2\sqrt{\nu}}\,\frac{1}{\beta\left(\frac{\nu}{2},\frac{1}{2}\right)}\,\frac{x^{-\frac{1}{2}}\,dx}{\left(1+\frac{x}{\nu}\right)^{\frac{\nu+1}{2}}}$$

$$=\frac{1}{\sqrt{\nu}}\cdot\frac{1}{\beta\left(\frac{\nu}{2},\frac{1}{2}\right)}\,\frac{x^{\frac{1}{2}-1}}{\left(1+\frac{x}{\nu}\right)^{\frac{\nu+1}{2}}}\,dx,\ 0<x<\infty$$

$$=\frac{\nu^{\frac{\nu}{2}}}{\beta\left(\frac{\nu}{2},\frac{1}{2}\right)}\,\frac{x^{\frac{1}{2}-1}}{(\nu+x)^{\frac{\nu+1}{2}}}\,dx$$

which $\Rightarrow$ x is a F variate with $d.f.$ 1 and ν.

Remark. All tests of significance based on t-distribution can be done by using F-distribution.

16.4.5. Relation between F and ψ^2.

F-distribution with ν_1, ν_2 $d.f.$ is

$$dP=\frac{1}{\beta\left(\frac{\nu_1}{2},\ \frac{\nu_2}{2}\right)}\,\nu_1^{\frac{\nu_1}{2}}\,\nu_2^{\frac{\nu_2}{2}}\,\frac{F^{\frac{\nu_1}{2}-1}}{(\nu_2+\nu_1F)^{\frac{\nu_1+\nu_2}{2}}}\,dF$$

$$=\left\{\frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\quad\frac{1}{\nu_2^{\frac{\nu_1}{2}}}\right\}\frac{1}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\nu_1^{\frac{\nu_1}{2}}\,F^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1F}{\nu_2}\right)^{\frac{\nu_1+\nu_2}{2}}}\,dF$$

Let $\quad\lambda=\dfrac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\cdot\dfrac{1}{\nu_2^{\frac{\nu_1}{2}}}$

$$\approx\frac{\sqrt{2\pi}\ e^{-\left\{\frac{\nu_1+\nu_2}{2}-1\right\}}\cdot\left\{\frac{\nu_1+\nu_2}{2}-1\right\}^{\frac{\nu_1+\nu_2}{2}-\frac{1}{2}}}{\sqrt{2\pi}\ e^{-\left\{\frac{\nu_2}{2}-1\right\}}\left\{\frac{\nu_2}{2}-1\right\}^{\frac{\nu_2}{2}-\frac{1}{2}}}\cdot\frac{1}{\nu_2^{\frac{\nu_1}{2}}}$$

$$=\frac{e^{-\frac{\nu_1}{2}}}{2^{\frac{\nu_1}{2}}}\quad\frac{(\nu_1+\nu_2-2)^{\frac{\nu_1+\nu_2-1}{2}}}{(\nu_2-2)^{\frac{\nu_2}{2}-\frac{1}{2}}}\cdot\frac{1}{\nu_2^{\frac{\nu_1}{2}}}$$

$$=\frac{e^{-\frac{\nu_1}{2}}\cdot 2^{-\frac{\nu_1}{2}}\left\{1+\frac{\nu_1-2}{\nu_2}\right\}^{\frac{\nu_1+\nu_2-1}{2}}}{\left(1-\frac{2}{\nu_2}\right)^{\frac{\nu_2}{2}-\frac{1}{2}}}$$

$$=e^{-\frac{\nu_1}{2}}\cdot 2^{-\frac{\nu_1}{2}}\frac{\left(1+\frac{\nu_1-2}{\nu_2}\right)^{\frac{\nu_1-1}{2}}\left[\left\{\left(1+\frac{\nu_1-2}{\nu_2}\right)\right\}^{\frac{\nu_2}{\nu_1-2}}\right]^{\frac{\nu_1-2}{2}}}{\left\{\left(1-\frac{2}{\nu_2}\right)^{-\frac{\nu_2}{2}}\right\}^{-1}\cdot\left\{1-\frac{2}{\nu_2}\right\}^{-\frac{1}{2}}}.$$

$$\rightarrow e^{-\frac{\nu_1}{2}} \cdot 2^{-\frac{\nu_1}{2}} \frac{e^{\frac{\nu_1}{2}-1}}{e} = 2^{-\frac{\nu_1}{2}}$$

as $\nu_2 \rightarrow \infty$

$$\text{Also} \quad \left(1+\frac{\nu_1 F}{\nu_2}\right)^{\frac{\nu_1+\nu_2}{2}} = \left[\left\{1+\frac{\nu_1 F}{\nu_2}\right\}^{\frac{\nu_2}{\nu_1 F}}\right]^{\frac{\nu_1 F}{2}} \left(1+\frac{\nu_1 F}{\nu_2}\right)^{\frac{\nu_1}{2}}$$

$$\rightarrow e^{\frac{\nu_1 F}{2}} \quad \text{as } \nu_2 \rightarrow \infty$$

$\therefore$ As $\nu_2 \rightarrow \infty$, probability differential is of the form

$$dP = \frac{1}{\Gamma\left(\frac{\nu_1}{2}\right)} \, 2^{-\frac{\nu_1}{2}} \, \nu_1^{\frac{\nu_1}{2}} \, F^{\frac{\nu_1}{2}-1} \, e^{-\frac{\nu_1 F}{2}} \, dF$$

$$= \frac{1}{2^{\frac{\nu_1}{2}} \Gamma\left(\frac{\nu_1}{2}\right)} \, e^{-\frac{\nu_1 F}{2}} \, (\nu_1 F)^{\frac{\nu_1}{2}-1} \, d(\nu_1 F)$$

$\Rightarrow$ $\nu_1 F$ is a ψ^2-variate with ν_1 *d f.*

16-5. F-tests

Tests of significance based on *F*-distribution are called *F*-tests. Various *F*-tests are :

(*i*) For equality of population variance.

(*ii*) For the significance of an observed multiple correlation coefficient.

(*iii*) For the significance of an observed sample correlation ratio.

(*iv*) For testing the linearity of regression.

All these tests are for small samples.

Rules of Decision

Let $P = P\{F > F_0\ (\nu_1, \nu_2)\}$

For a given value of P and for ν_1, ν_2 *d fs*, values of F_0 have been tabulated in the form of *F*-tables.

The value F_0 is called the critical value of F for ν_1, ν_2 *d. fs.* at level of significance P.

To test the significance the calculated value of F is compared with tabulated value at certain specified level of significance. Generally 5% or 1% levels are taken.

If $F_{cal} > F_{tab}$, the null hypothesis is rejected and the difference is said to be significant and if $F_{cal} < F_{tab}$, the hypothesis is accepted at the level of significance adopted.

Ex. 16.21. *When $\nu_1=2$, show that the significance level of F corresponding to a significant probability p is*

$$F=\frac{\nu_2}{2}\left(p^{\frac{-2}{\nu_2}}-1\right)$$

where ν_1 and ν_2 have their usual meanings.

Sol. The dist. of F is

$$dP=\frac{2.\nu_2^{\frac{\nu_2}{2}}}{\beta\left(1,\frac{\nu_2}{2}\right)}\cdot\frac{dF}{(\nu_2+2F)^{\frac{\nu_2}{2}+1}}$$

Let F_0 be the significance level of F corresponding to the probability p.

$$\text{Then}\quad p=P\{F>F_0\}=\int_{F_0}^{\infty}dP$$

$$=\frac{2\nu_2^{\frac{\nu_2}{2}}}{\beta\left(1,\frac{\nu_2}{2}\right)}\int_{F_0}^{\infty}\frac{dF}{(\nu_2+2F)^{\frac{\nu_2}{2}+1}}$$

$$=\frac{2\nu_2^{\frac{\nu_2}{2}}}{\beta\left(1,\frac{\nu_2}{2}\right)}\left\{-\frac{1}{\nu_2}\cdot\frac{1}{(\nu_2+2F)^{\frac{\nu_2}{2}}}\right\}_{F_0}^{\infty}$$

$$=\frac{2\nu_2^{\frac{\nu_2}{2}-1}}{\beta\left(1,\frac{\nu_2}{2}\right)}\frac{1}{(\nu_2+2F_0)^{\frac{\nu_2}{2}}}$$

$$=2\nu_2^{\frac{\nu_2}{2}-1}\frac{\Gamma\left(\frac{\nu_2}{2}+1\right)}{\Gamma(1)\,\Gamma\left(\frac{\nu_2}{2}\right)}\cdot\frac{1}{(\nu_2+2F_0)^{\frac{\nu_2}{2}}}$$

$$=2\nu_2^{\frac{\nu_2}{2}-1}\frac{\frac{\nu_2}{2}\,\Gamma\left(\frac{\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)}\frac{1}{(\nu_2+2F_0)^{\frac{\nu_2}{2}}}\quad(\because\ \Gamma(1)=1)$$

$$=\nu_2^{\frac{\nu_2}{2}} \Big/ (\nu_2+2F_0)^{\frac{\nu_2}{2}}$$

16.5.1. Test of significance for equality of population variance

Consider two independent random samples $x_1, x_2, \ldots\ldots x_{n_1}$ and $y_1, y_2, \ldots\ldots y_{n_2}$ from normal populations. The hypothesis to be tested is : 'The population variances are same'.

Assuming this hypothesis, the statistics

$$F=\frac{S_1^2}{S_2^2} \; (S_1^2 > S_2^2)$$

where $S_1^2=\dfrac{1}{n_1-1}\displaystyle\sum_{i=1}^{n_1}(x_i-\bar{x})^2$ and $S_2^2=\dfrac{1}{n_2-1}\displaystyle\sum_{j=1}^{n_2}(y_j-\bar{y})^2$, follows F-distribution with $\nu_1=n_1-1$ and $\nu_2=n_2-1$ degrees of freedom.

By comparing the calculated value of F with the tabulated value for ν_1 and ν_2 *d.f.* at certain level of significance (5% or 1%) the significance is tested.

Ex. 16.22. *It is known that the mean diameters of rivets produced by two firms A and B are practically the same but the standard deviations may differ. For 22 rivets produced by A the s.d. is 2·9 mm, while for 16 rivets manufactured by B the s.d. is 3·8. Test whether the products of A have the same variability as those of B.*

Sol. $n_1=22$, $n_2=16$, $s_1=2{\cdot}9$ and $s_2=3{\cdot}8$

$$\therefore \qquad S_1^2=\frac{n_1 s_1^2}{n_1-1}=\frac{(22)(2{\cdot}9)^2}{21}\simeq 8{\cdot}81$$

and
$$S_2^2=\frac{(16)(3{\cdot}8)^2}{15}\simeq 15{\cdot}40$$

$$\therefore \qquad F=\frac{S_2^2}{S_1^2}=\frac{15{\cdot}4}{8{\cdot}81}\simeq 1{\cdot}748$$

Nos. of degrees of freedom are $16-1=15$ and $22-1=21$

$$\therefore \qquad F_{0{\cdot}05}=2{\cdot}18 > F_{cal}$$

$\therefore$ Variability for two types of products may be same.

Ex 16.23. *Given below are the qualities of ten items (in proper units) produced by two processes A and B. Test whether the variability of quality may be taken to be the same for the two processes.*

Process A :	3	7	5	6	5	4	4	5	3	3
Process B :	8	5	7	8	3	2	7	6	5	7

(*F-value for $n_1=9$ and $n_2=9$ degrees of freedoms is 3·18 at 5% level of significance and 5·35 at 1% level of significance*).

Sol.

Process A : $x \to$	3	7	5	6	5	4	4	5	3	3	Total	$=45$
$X=(x-\bar{x}) \to$	$-1·5$	2·5	0·5	1·5	0·5	$-0·5$	$-0·5$	0·5	$-1·5$	$-1·5$		
$X^2 \to$	2·25	6·25	0·25	2·25	0·25	0·25	0·25	0·25	2·25	2·25		$=16·5$
Process B : $y \to$	8	5	7	8	3	2	7	6	5	7		$=58$
$Y=(y-\bar{y}) \to$	2·2	$-0·8$	1·2	2·2	$-2·8$	$-3·8$	1·2	0·2	$-0·8$	1·2		
$Y^2 \to$	4·84	0·64	1·44	4·84	7·84	14·44	1·44	0·04	0·64	1·44		$=37·6$

$$\bar{x}=4·5,\ \bar{y}=5·8,\ S_1^2=\frac{16·5}{9} \text{ and } S_2^2=\frac{37·6}{9}$$

$\therefore$ $$F=\frac{37·6}{16·5}=2·28$$

Degrees of freedom are $10-1=9$ and $10-1=9$.

$\therefore$ $F_{0·05}=3·18 > F_{cal}$

$\therefore$ Variability of quality may be taken to be the same for two processes.

Ex. 16.24. *Two random samples of sizes 3 und 11, drawn from two normal populations, are characterized as follows :*

Population from which the sample is drawn	*Size of the sample*	*Sum of observations*	*Sum of squares of observations*
I	8	9·6	61·52
II	11	16·5	73·26

You are to decide if the two populations can be taken to have the same variance.

Sol. Let x and y be the observations for two samples.

Now $$\Sigma(x-\bar{x})^2=\Sigma x^2-N_1\bar{x}^2=\Sigma(x)^2-\frac{1}{N_1}(\Sigma x)^2$$

$$=61·52-\frac{1}{8}(9·6)^2=50$$

and $$\Sigma(y-\bar{y})^2=73·26-\frac{1}{11}(16·5)^2=48·51$$

$$\therefore \qquad S_1^2 = \frac{50}{7} \text{ and } S_2^2 = \frac{48 \cdot 51}{10}$$

$$\therefore \qquad F = \frac{S_1^2}{S_2^2} = \frac{500}{339 \cdot 57} \simeq 1 \cdot 47$$

Nos. of *d. f* are $8-1=7$ and $11-1=10$

$\therefore \qquad F_{0 \cdot 05} = 3 \cdot 14 > F_{cal}$

$\therefore$ Variances of two populations may be same.

EXERCISES

1. Two independent samples of 8 and 7 items respectively had the following values of the variable (weight in ounces)

Sample I :	9	11	13	11	15	9	12	14
Sample II :	10	12	10	14	9	8	10	

Test whether the estimates of the population variances differ significantly. (Given that $F_{0 \cdot 05}$ for 7 and 6 *d. f.* is 4·21).

[**Ans.** Not significant]

2. In two groups of ten children each, increases in weight due to two different diets in the same period were in pounds,

8, 5, 7, 8, 3, 2, 7, 6, 5, 7,
3, 7, 5, 6, 5, 4, 4, 5, 3, 6

Test whether the variances differ significantly. (Given that $F_{0 \cdot 05}$ for 9 and 9 *d. f.* is 3 18). [**Ans.** Not significant]

3. Two random samples drawn from two normal populations are :

Sample I : 20, 16, 26, 27, 23, 22, 18, 24, 25 and 19

Sample II : 27, 33, 42, 35, 32, 34, 38, 28, 41, 43, 30 and 37

Test whether the two populations have the same variance.

[**Ans.** 2·14, Not significant]

4. The students of the same age of two different colleges were tested for variability of intelligence. The *I.Q*'s of 10 students from one college showed a variance of 20 and those of an equal number from the other college had a variance of 15. Discuss whether there is any significant difference in variability.

[**Ans.** Not significant]

5. In one sample of 8 observations the sum of the squares of the deviations of the sample values from the sample mean was 84·4 and in the other sample of 10 observations it was 1026. Test whether this difference is significant at 5% level, given that $F_{0 \cdot 05}$ for 7 and 9 degrees of freedom is 3·29.

[**Ans.** Not significant]

6. Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means equal to 160 inches squares

and 91 inches squares respectively. Can they be regarded as drawn from the same normal population ?

[**Ans.** 1·54, Not significant]

7. Two random samples gave the following results :

Size	Mean	S.D.
10	3·0	2·9
12	4·0	3 2

Test whether the samples come from the same normal population.

8. Two chemists A and B repeat a protein analysis 20 times. If X_i and Y_i are the values obtained by A and B respectively and if

$$\Sigma X_i=196,\ \Sigma X_i^2=1928,\ \Sigma Y_i=205 \text{ and } \Sigma Y_i^2=2105$$

Determine whether there is a significant difference in precision between the two sets of results, the precision being measured by the inverse of the variance. [**Ans.** Not significant]

16.5.2. Test for the Significance of an Observed Multiple Correlation Coefficient

Consider a random sample of size n from a $(k+1)$ variate normal population. Let R be the multiple correlation coefficient of a variate with k other variates. Hypothesis to be tested is

"The multiple correlation coefficient in the population is zero".

Assuming this hypothesis, the statistic

$$F=\frac{R^2}{1-R^2}\cdot\frac{n-k-1}{k}$$

follows F-distribution with k, $n-k-1$ $d.f.$

16.5.3. Test for the Significance of an Observed Samples Correlation Ratio

Let (x_i, y_{ij}), $(i=1, 2 \ldots\ldots h,\ j=1, \ldots\ldots n_i)$ be a random sample from a bivariate normal population and let

$$N=\sum_{i=1}^{h} n_i$$

Let η be the correlation ratio of y on x.

The hypothesis to be tested is :

"Population correlation ratio is zero" Assuming this hypothesis, the statistic

$$F=\frac{\eta^2}{1-\eta^2}\cdot\frac{N-h}{h-1}$$

follows F distribution with $h-1$ and $N-h$ $d.f.$

16.5.4. Testing the Linearity of Regression

Let η be the correlation ratio and r the correlation coefficient for a sample of size N arranged in h arrays, from a bivariate normal population.

The test statistic for testing the hypothesis of linearity of regression is

$$F=\frac{\eta^2-r^2}{1-\eta^2}\cdot\frac{N-h}{h-2}$$

which follows F-distribution with $h-2$ and $N-h$ $d.f.$s.

16.6. Fisher's z-Distribution

Fisher's z-variate is defined by

$$z=\frac{1}{2}\log_e F$$

where F follows F distribution with ν_1, ν_2 $d.f.$s.

$\therefore$ $F=e^{2z}$

$\therefore$ Dist. of z is

$$dP=\frac{2\nu_1^{\frac{\nu_1}{2}}\nu_2^{\frac{\nu_2}{2}}}{\beta\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\cdot\frac{e^{\nu_1 z}}{(\nu_2+\nu_1 e^{2z})^{\frac{\nu_1+\nu_2}{2}}}\,dz$$

$$-\infty<z<\infty$$

This distribution is called Fisher's z-distribution.

16.7. z-tests

Tests of significance based on z-distribution are called z-tests.

z-tables provide critical values of z for various values of ν_1, ν_2 at 5% or 1% levels.

Significance is tested by comparing the calculated value of z with tabulated value at certain level of significance (5% or 1%). Rules of decision are :

If $z_{0.05} > z_{cal}$, the population variances may be same and if $z_{0.05} < z_{cal}$, the ratio is significant at 5% level. Similarly for 1% level.

Some z-tests are as below :

16.7.1. Test for Equality of Variance

Consider two independent random samples $x_1, x_2 \ldots x_{n_1}$ and $y_1, y_2, \ldots\ldots y_{n_2}$ drawn from normal populations. The hypothesis to be tested is :

'The population variances are same'

Assuming this hypothesis, the statistic

$$z=\frac{1}{2}\log_e\frac{S_1^2}{S_2^2}=\frac{1}{2}\left\{(\log_e 10)\log_{10}\left(\frac{S_1^2}{S_2^2}\right)\right.$$

$$=\frac{(2\cdot3026)}{2}\log_{10}\frac{S_1^2}{S_2^2}=1\cdot1513\log_{10}\frac{S_1^2}{S_2^2}$$

where $S_1^2=\frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_i-\bar{x})^2$ and $S_2^2=\frac{1}{n_2-1}\sum_{j=1}^{n_2}(y_j-\bar{y})^2$

follows z-distribution with $\nu_1=n_1-1$ and $\nu_2=n_2-1$ *d f.*

Ex. 16·25. *Show how you would use student's t-test and Fisher's z-test to decide whether the two sets of observations*

17, 27, 18, 25, 27, 29, 27, 23, 17,
16, 16, 20, 16, 20 17, 15, 21

indicate samples drawn from the same universe.

Sol. Let x and y be the variables for two samples respectively.

x :	17	27	18	25	27	29	27	23	17	Total
$X=x-23$:	-6	4	-5	2	4	6	4	0	-6	$=3$
X^2 :	36	16	25	4	16	36	16	0	36	$=185$
y :	16	16	20	16	20	17	15	21		
$Y=y-16$:	0	0	4	0	4	1	-1	5		$=13$
Y^2 :	0	0	16	0	16	1	1	25		$=59$

$$\bar{x}=23+\frac{3}{9}=\frac{70}{3},\ \bar{y}=16+\frac{13}{8}=\frac{141}{8}$$

$$\Sigma(x-\bar{x})^2=\Sigma(X-\bar{X})^2=\Sigma X^2-\frac{1}{n_1}(\Sigma X)^2$$

$$=185-\frac{1}{9}(9)=184$$

and $$\Sigma(y-\bar{y})^2=\Sigma(Y-\bar{Y})^2$$

$$=(\Sigma Y^2)-\frac{1}{n_2}(\Sigma Y)^2=59-\frac{1}{8}(169)=\frac{303}{8}$$

∴ $$S_1^2=\frac{184}{8},\ S_2^2=\frac{303}{56}$$

Firstly the equality of population variances will be tested by applying z-test.

Now $$z=1\cdot1513\log_{10}\frac{S_1^2}{S_2^2}=(1\cdot1513)\log_{10}\frac{1288}{303}$$

$$=(1{\cdot}1513)\{\log_{10} 1288-\log_{10} 303\}$$
$$=(1{\cdot}1513)\{3{\cdot}1099-2{\cdot}4814\}\simeq 0{\cdot}724$$

Now $\quad z_{0\cdot05}$ for 8 and 7 $d.f.=0{\cdot}6576 < z_{cal.}$

and $\quad z_{0\cdot01}$ for 8 and 7 $d.f.=0{\cdot}9614 > z_{cal.}$

$\therefore$ At 5% level the variance ratio is significant and at 1% level not significant.

$\therefore$ At 1% level, the two population variances may be same.

Now t-test will be applied to test the significance of the difference between the means.

$$S^2=\frac{\sum_{i=1}^{n_1}(x_i-\bar{x})^2+\sum_{j=1}^{n_2}(y_j-\bar{y})^2}{n_1+n_2-2}=\frac{184+\frac{303}{8}}{15}$$
$$=\frac{1775}{120}$$

$$\therefore \quad |t|=\frac{\bar{x}\sim\bar{y}}{S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}=\frac{\frac{70}{3}-\frac{141}{8}}{\sqrt{\frac{1775}{120}}\sqrt{\frac{1}{8}+\frac{1}{9}}}\simeq 3{\cdot}05$$

No. of $d.f.=9+8-2=15$

$\therefore \quad t_{0\cdot05}=2{\cdot}13$ and $t_{0\cdot05}=2{\cdot}95$

$\therefore \quad t_{cal} > t_{0\cdot01}$ and $t_{0\cdot01}$

$\therefore$ The difference between the means is significant.

$\therefore$ The two samples do not belong to the same universe.

Ex. 16·26. (*i*) *Give the test of significance of correlation co-efficient based on Fishers z-transformation.*

(*ii*) *A correlation co-efficients of 0·7 is discovered in a sample of 28 pairs. Apply z-transformation to find out if this differs significantly (a) from '0' (b) from 0·5.*

Sol. (*i*) Let 'r' and 'ρ' be the correlation coefficient for sample and the population respectively and n the sample size

Fisher z-transformation is

$$z=\frac{1}{2}\log_e\frac{1+r}{1-r}=1{\cdot}1513\log_{10}\frac{1+r}{1-r}$$

For large values of 'n', 'z' is distributed asymptotically normally about the mean 'ζ' where

$$\zeta=\frac{1}{2}\log_e\frac{1+\rho}{1-\rho}=1{\cdot}1513\log_{10}\frac{1+\rho}{1-\rho}$$

and variance $\dfrac{1}{n-3}$

$\therefore$ For large values of n, the variate

$$u=\frac{z-\zeta}{\sqrt{\frac{1}{n-3}}}$$

is asymptotically standard normal variate. Thus the significance between 'r' and 'ρ' is tested by calculating 'u' and using normal tables.

Thus if $|u|>1{\cdot}96$, the difference is significant at 5% level and if $|u|>2{\cdot}58$, the difference is significant at 1% level and if $|u|>3$, the difference is highly significant.

Note. The symbol 'z' used here is different from Fisher's z-distribution.

(*ii*) (*a*) $n=28$, $\rho=0$, ; $r=0{\cdot}7$

$$\therefore \quad \zeta=0, \quad z=(1{\cdot}1513)\log_{10}\frac{1{\cdot}7}{0{\cdot}3}$$
$$=(1{\cdot}1513)\{\log_{10}17-\log_{10}3\}$$
$$=(1{\cdot}1513)\{1{\cdot}2304-0{\cdot}4771\}=0{\cdot}87$$

$$\therefore \quad u=(0{\cdot}87)\sqrt{25}=4{\cdot}35>3.$$

$\therefore$ The hypothesis of zero correlation is refuted and hence the population is correlated.

(*b*) $n=28$. $\rho=0{\cdot}5$, $r=0{\cdot}7$

$$\therefore \quad \zeta=(1{\cdot}1513)\log_{10}\frac{1{\cdot}5}{0{\cdot}5}=(1{\cdot}1513)\log_{10}3$$
$$=(1{\cdot}1513)(0{\cdot}4771)\simeq 0{\cdot}55$$

Also $\quad z=0{\cdot}87$

$$\therefore \quad u=(0{\cdot}32)\sqrt{25}=1{\cdot}60<1{\cdot}96.$$

$\therefore$ The difference between r and ρ is not significant and hence the hypothesis that $\rho=0{\cdot}5$ is acceptable.

Ex. 16-27. *What is the probability that a correlation coefficient of 0·75 or less can arise in a sample of 30 from a normal population in which the true correlation is 0·9 ?*

Sol. $r=0{\cdot}75$, $\rho=0{\cdot}9$, $n=30$

$$\therefore \quad \zeta=(1{\cdot}1513)\log_{10}\frac{1{\cdot}9}{0{\cdot}1}=1{\cdot}1513\log_{10}19$$
$$=(1{\cdot}1513)(1{\cdot}2788)\simeq 1{\cdot}472$$
$$z=(1{\cdot}1513)\log_{10}\left(\frac{1{\cdot}75}{0{\cdot}25}\right)=(1{\cdot}1513)\log_{10}7$$
$$=(1{\cdot}1513)(0{\cdot}8451)\simeq 0{\cdot}973$$
$$\therefore \quad u=(0{\cdot}973-1{\cdot}472)\sqrt{27}\simeq -2{\cdot}59$$

Now $\quad P\{r \leqslant 0{\cdot}75\}=P\{1+r \leqslant 1{\cdot}75\}=P\left\{\frac{1+r}{1-r} \leqslant \frac{1{\cdot}75}{0{\cdot}25}=7\right\}$

$=P\{z \leqslant 0{\cdot}973\}=P\{u < -2{\cdot}59\}$

$=0{\cdot}5-P\{-2{\cdot}59 < u < 0\}=0{\cdot}5-P\{0 < u < 2{\cdot}59\}$

$=0{\cdot}5-0{\cdot}4952=0{\cdot}0048.$

Ex. 16·28. *In a random sample of 19 pairs of values from a bivariate normal population, the correlation was found to be 0·7. Is this value consistent with the assumption that the correlation in the population is 0·5 ?*

[**Ans** : $u=1{\cdot}28$]

Ex. 16·29. (*a*) *Give the procedure of testing the significance of the difference between two independent correlation co-efficients.*

(*b*) *The correlation coefficient between temperature of rice and broackage percentage calculated from two samples of 12 and 16 are 0·8912 and 0·8482 respectively. Do the two estimates differ significantly ?*

Sol. (*a*) Let there be two independent random samples of sizes n_1, n_2 and correlation coefficients r_1, r_2. The hypothesis to be tested is :

'Can the samples be regarded as drawn from the same population or from two populations with the same correlation co-efficients'.

Assuming this hypothesis, the statistic

$$u=\frac{z_1-z_2}{\sqrt{\dfrac{1}{n_1-3}+\dfrac{1}{n_2-3}}}$$

where $z_i=(1{\cdot}1513) \log_{10} \dfrac{1+r_i}{1-r_i} \qquad (i=1, 2)$

is asymptotically standard normal variate.

Thus the significance of the difference between r_1 aud r_2 is tested by calculating u and using normal tables.

(*c*) $\quad n_1=12,\ n_2=16,\ r_1=0{\cdot}8912,\ r_2=0{\cdot}8482$

$$z_1=(1{\cdot}1513)\left\{\log_{10} \frac{1{\cdot}8912}{0{\cdot}1088}\right\}=(1{\cdot}1513)\{\log_{10} 18912 - \log_{10} 1088\}$$

$=(1{\cdot}1513)\{4{\cdot}2767-3{\cdot}0366\}=(1{\cdot}1513)(1{\cdot}2401)$

$\simeq 1{\cdot}428$

$$z_2=(1{\cdot}1513)\left\{\log_{10} \frac{1{\cdot}8482}{0{\cdot}1518}\right\}$$

$=(1{\cdot}1513)\{\log_{10} 18482-\log_{10} 1518\}$

$=(1{\cdot}1513)\{4{\cdot}2667-3{\cdot}1813\}=(1{\cdot}1513)(1{\cdot}0854)$

$\simeq 1{\cdot}250.$

$$u=\frac{1{\cdot}428-1{\cdot}250}{\sqrt{\frac{1}{12-3}+\frac{1}{16-3}}}\simeq 0{\cdot}41 < 1{\cdot}96$$

$\therefore$ Difference between r_1 and r_2 is not significant.

Ex. 16·30. *The first of two samples consists of 23 pairs and gives a correlation of 0·5 while the second of 28 pairs has a correlation of 0·8. Are these values significantly different?*

Sol. $n_1=23$, $n_2=28$, $r_1=0{\cdot}5$ and $r_2=0{\cdot}8$

$$\therefore \quad z_1=(1{\cdot}1513)\left\{\log_{10}\frac{1{\cdot}5}{0{\cdot}5}\right\}=(1{\cdot}1513)\log_{10}3$$

$$=(1{\cdot}1513)(0{\cdot}4771)\simeq 0{\cdot}5493$$

$$z_2=(1{\cdot}1513)\log_{10}\frac{1{\cdot}8}{0{\cdot}2}=1{\cdot}1513\log_{10}9$$

$$=(1{\cdot}1513)(0{\cdot}9542)\simeq 1{\cdot}0986$$

$$|u|=\frac{1{\cdot}0986-0{\cdot}5493}{\sqrt{\frac{1}{20}+\frac{1}{25}}}\simeq 1{\cdot}83 < 1{\cdot}96$$

$\therefore$ Difference is not significant.

Ex. 16·31. *The correlation coefficient between Mathematics aptitude and Physics aptitude for a group of 20 girls is 0·42 and for a group of 25 boys is 0·75. Is the difference significant?*

[**Ans.** 1·6, Not significant].

17

Multiple and Partial Correlations

17.1. Introduction

Here the theory of correlation involving more than two variables will be discussed. The aim of the theory of **multiple correlation** is to study the joint effect of a group of variables upon a variable not included in that group.

In case, the study of relationship between only two variables is to be made, the effect of remaining variables on these two variables should be eliminated. As it is not possible to eliminate the entire influence only the linear effect is eliminated. Then the correlation between the two variables is called **partial correlation**.

Here only three variables will be taken.

17.2. Notations

Let x_1, x_2, x_3 be the variables. It is assumed that these denote the deviations from their respective means, so that

$$\bar{x}_1=\bar{x}_2=\bar{x}_3=0$$

The multiple correlation coefficient between x_1 (dependent variable) and x_2, x_3 (independent variables) is denoted by $R_{1\cdot 23}$. Where figures after dot (.) correspond to independent variables and figure before dot refer to dependent variable. Thus $R_{2\cdot 13}$ denote the multiple correlation between x_2 and x_1, x_3 and so on.

The partial correlation coefficient between x_1 and x_2 is denoted by $r_{12\cdot 3}$ Where figure after dot refer to variable whose effect has been eliminated or is kept constant. Thus $r_{23\cdot 1}$ denote the partial correlation between x_2 and x_3.

17.3 Plane of Regression

The equation of plane of regression of x_1 on x_2, x_3 is of the form

$$x_1=a+b_{12\cdot 3}\,x_2+b_{13\cdot 2}\,x_3 \qquad \ldots(1)$$

where a, $b_{12\cdot 3}$ and $b_{13\cdot 2}$ are constants. The quantities $b_{12\cdot 3}$ and $b_{13\cdot 2}$ are called partial regression coefficients of x_1 and x_2 for fixed x_3 and of x_1 on x_3 for fixed x_2 respectively. The first subscript attached to the b's is the subscript of the letter on the left (the dependent variable) and the second subscript is that of x to which it is attached. These subscripts are called **primary subscripts**. The subscript separated from the primary subscript by a dot (.) is that of x which has been left. These are called **secondary subscripts**.

Now (1) $\Rightarrow$

$$\bar{x}_1 = a + b_{12\cdot 3}\,\bar{x}_2 + b_{13\cdot 2}\,\bar{x}_3 \quad \Rightarrow \quad a = 0$$

$\therefore$ (1) takes the form $x_1 = b_{12\cdot 3}\,x_2 + b_{13\cdot 2}\,x_3$ ...(2)

Here the coefficients b's are to be obtained so as to minimize

$$S = \Sigma\,(x_1 - b_{12\cdot 3}\,x_2 - b_{13\cdot 2}\,x_3)^2$$

which is the sum of the squares of the residuals, the summation is over the given values of the variables.

The normal equations are

$$0 = \frac{\partial S}{\partial b_{12\cdot 3}} = -2\Sigma x_2\,(x_1 - b_{12\cdot 3}\,x_2 - b_{13\cdot 2}\,x_3)$$

$$\Rightarrow \quad \Sigma x_1 x_2 = b_{12\cdot 3}\,\Sigma x_2^2 + b_{13\cdot 2}\,\Sigma x_2\,x_3$$

$$\Rightarrow \quad n\,r_{12}\,\sigma_1\sigma_2 = b_{12\cdot 3}\,n\,\sigma_2^2 + b_{13\cdot 2}\,n\,r_{23}\,\sigma_2\sigma_3$$

$$\Rightarrow \quad r_{12}\,\sigma_1 = b_{12\cdot 3}\,\sigma_2 + b_{13\cdot 2}\,r_{23}\,\sigma_3 \qquad \text{.. (3)}$$

and

$$0 = \frac{\partial S}{\partial b_{13\cdot 2}} = -2\Sigma x_3\,(x_1 - b_{12\cdot 3}\,x_2 - b_{13\cdot 2}\,x_3)$$

$$\Rightarrow \quad \Sigma x_1 x_3 = b_{12\cdot 3}\,\Sigma x_2 x_3 + b_{13\cdot 2}\,\Sigma x_3^2$$

$$\Rightarrow \quad r_{13}\,\sigma_1 = b_{12\cdot 3}\,\sigma_2\,r_{23} + b_{13\cdot 2}\,\sigma_3 \qquad \text{.. (4)}$$

Solving (3) and (4)

$$b_{12\cdot 3} = \frac{\begin{vmatrix} r_{12}\,\sigma_1 & r_{23}\,\sigma_3 \\ r_{13}\,\sigma_1 & \sigma_3 \end{vmatrix}}{\begin{vmatrix} \sigma_2 & r_{23}\,\sigma_3 \\ r_{23}\sigma_2 & \sigma_3 \end{vmatrix}} = \frac{\sigma_1}{\sigma_2}\,\frac{\begin{vmatrix} r_{12} & r_{23} \\ r_{13} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & r_{23} \\ r_{23} & 1 \end{vmatrix}}$$

and

$$b_{13\cdot 2} = \frac{\sigma_1}{\sigma_3}\,\frac{\begin{vmatrix} 1 & r_{12} \\ r_{23} & r_{13} \end{vmatrix}}{\begin{vmatrix} 1 & r_{23} \\ r_{23} & 1 \end{vmatrix}}$$

For convenience and simplicity, let

$$\omega = \begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix}$$

and observe $r_{11}=r_{22}=r_{33}=1$,

$r_{12}=r_{21},\ r_{13}=r_{31},\ r_{23}=r_{32}$.

Let ω_{ij}=cofactor of (i, j)th place.

Then

$$\omega_{11}=\begin{vmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{vmatrix}=\begin{vmatrix} 1 & r_{23} \\ r_{23} & 1 \end{vmatrix}$$

$$\omega_{12}=-\begin{vmatrix} r_{21} & r_{23} \\ r_{31} & r_{33} \end{vmatrix}=-\begin{vmatrix} r_{12} & r_{23} \\ r_{13} & 1 \end{vmatrix}$$

$$\omega_{13}=\begin{vmatrix} r_{21} & r_{22} \\ r_{31} & r_{32} \end{vmatrix}=-\begin{vmatrix} 1 & r_{12} \\ r_{23} & r_{13} \end{vmatrix}$$

$$\therefore \quad b_{12\cdot 3}=-\frac{\sigma_1}{\sigma_2}\cdot\frac{\omega_{12}}{\omega_{11}},\quad b_{13\cdot 2}=-\frac{\sigma_1}{\sigma_3}\,\frac{\omega_{13}}{\omega_{11}} \qquad \text{...(5)}$$

Substituting these values in (2), Eq. of plane of regression of x_1 on x_2, x_3 is

$$x_1=-\frac{\sigma_1}{\sigma_2}\,\frac{\omega_{12}}{\omega_{11}}\,x_2-\frac{\sigma_1}{\sigma_3}\,\frac{\omega_{13}}{\omega_{11}}\,x_3$$

i.e.,

$$\frac{x_1}{\sigma_1}\omega_{11}+\frac{x_2}{\sigma_2}\omega_{12}+\frac{x_3}{\sigma_3}\,\omega_{13}=0. \qquad \text{...(6)}$$

Similarly eqs of planes of regression of x_2 on x_1, x_3 and x_3 on x_1, x_{12} respectively are

$$\frac{x_1}{\sigma_1}\omega_{21}+\frac{x_2}{\sigma_2}\omega_{22}+\frac{x_3}{\sigma_3}\omega_{23}=0 \qquad \text{...(7)}$$

and

$$\frac{x_1}{\sigma_1}\omega_{31}+\frac{x_2}{\sigma_2}\omega_{32}+\frac{x_3}{\sigma_3}\omega_{33}=0 \qquad \text{...(8)}$$

Remark. Eliminating $b_{12\cdot 3}$ and $b_{13\cdot 2}$ between (2), (3) and (4).

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ r_{12}\sigma_1 & \sigma_2 & r_{23}\sigma_3 \\ r_{13}\sigma_1 & \sigma_2 r_{23} & \sigma_3 \end{vmatrix}=0.$$

i.e.

$$\begin{vmatrix} \dfrac{x_1}{\sigma_1} & \dfrac{x_2}{\sigma_2} & \dfrac{x_3}{\sigma_3} \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{vmatrix}=0.$$

which is the eq. of plane of regression of x_1 on x_2, x_3 in determinent form.

Remark. (1) x_1, x_2, x_3 are also considered as the observed values of variates respectively. Then value of x_1 as estimated by plane of regression is $b_{12\cdot 3}\,x_2+b_{13\cdot 2}\,x_2$. Let this value be denoted by $e_{1\cdot 23}$. Thus

$$e_{1\cdot 23}=b_{12\cdot 3}\,x_2+b_{13\cdot 2}\,x_3$$

Similarly

$$\epsilon_{2\cdot13}=b_{21\cdot3}\,x_1+b_{23\cdot1}\,x_3$$
$$\epsilon_{3\cdot12}=b_{31\cdot2}\,x_1+b_{32\cdot1}\,x_2$$

The difference $x_1-\epsilon_{1\cdot23}$ is the residual of x_1. It is denoted by $x_{1\cdot23}$.

Thus $\quad x_{1\cdot23}=x_1-\epsilon_{1\cdot23}$
$$=x_1-b_{12\cdot3}\,x_2-b_{13\cdot2}\,x_3$$

Similarly $\quad x_{2\cdot13}=x_2-b_{21\cdot3}\,x_1-b_{23\cdot1}\,x_3$
$$x_{3\cdot12}=x_3-b_{31\cdot2}\,x_1-b_{32\cdot1}\,x_2.$$

(2) In a quantity the subscripts before dot (.) are known as **primary subscripts** and those after dot are called **secondary subscripts**. The secondary subscripts can be written in any order but the order of primary subscripts is important. First primary subscript from the left refers to the dependent variable and other to independent variable.

The order of the quantity is determined by the number of secondary subscripts in it. Thus $x_{1\cdot23}$ is of order two, $b_{12\cdot3}$ is order one and so on.

Ex. 17.1. Using the following data :

$$\bar{x}_1=40 \qquad \bar{x}_2=70 \qquad \bar{x}_3=90$$
$$\sigma_1=3 \qquad \sigma_2=6 \qquad \sigma_3=7$$
$$r_{12}=0{\cdot}4 \qquad r_{23}=0{\cdot}5 \qquad r_{13}=0{\cdot}6$$

find the equation of plane of regression of x_2 on x_1 and x_2. Also find the value of x_2 for $x_1=30$, $x_3=40$.

Solution. We have

$$\omega=\begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{vmatrix}=\begin{vmatrix} 1 & 0{\cdot}4 & 0{\cdot}6 \\ 0{\cdot}4 & 1 & 0{\cdot}5 \\ 0{\cdot}6 & 0{\cdot}5 & 1 \end{vmatrix}$$

Eq. of plane of regression of x_2 on x_1 and x_3 is

$$\frac{x_1}{\sigma_1}\,\omega_{21}+\frac{x_2}{\sigma_2}\,\omega_{22}+\frac{x_3}{\sigma_3}\,\omega_{23}=0$$

Here x_1, x_2, x_3 are with zero means. So these are to be replaced by

$$x_1-\bar{x}_1,\; x_2-\bar{x}_2,\; x_3-\bar{x}_3$$

respectively.

$\therefore$ Eq. of plane of regression of x_2 on x_1 and x_3 is

$$\frac{(x_1-\bar{x}_1)}{\sigma_1}\,\omega_{21}+\frac{(x_2-\bar{x}_2)}{\sigma_2}\,\omega_{22}+\frac{(x_3-\bar{x}_3)}{\sigma_3}\,\omega_{23}=0 \quad \text{...(1)}$$

Now $\omega_{21}=-\begin{vmatrix} 0.4 & 0.6 \\ 0.5 & 1 \end{vmatrix}=0.3-0.4=-0.1$

$\omega_{22}=\begin{vmatrix} 1 & 0.6 \\ 0.6 & 1 \end{vmatrix}=1-.36=0.64$

$\omega_{23}=-\begin{vmatrix} 1 & 0.4 \\ 0.6 & 0.5 \end{vmatrix}=0.24-0.5$
$=-0.26$

$\therefore$ Substituting values in (1), eq. of plane of regression of x_2 on x_1 and x_3 is

$$\frac{(x_1-40)}{3}(-0.1)+\left(\frac{x_2-70}{6}\right)(0.64)+\left(\frac{x_3-90}{7}\right)(-0.26)=0$$

$$-0.03(x-40)+0.11(x_2-70)-0.04(x_3-90)=0$$

$$-0.03x_1+0.11x_2-0.04x_3-2.9=0$$

$\therefore$ $0.03x_1-0.11x_2+0.04x_3+2.9=0.$

Put $x_1=30,\ x_3=40$

$\therefore$ $0.9-0.11x_2+1.6+2.9=0$

$0.11x_2=5.4 \quad \therefore \quad x_2=49.09.$

17.4. Properties of Residuals

(1) In the derivation of plane of regression of x_1 on x_2, x_3, normal equations are

$$\Sigma x_2(x_1-b_{12.3}x_2-b_{13.2}x_3)=0$$

and $$\Sigma x_3(x_1-b_{12.3}x_2-b_{13.2}x_3)=0$$

These equations $\Rightarrow$ $\Sigma x_2 x_{1.23}=0$ and $\Sigma x_3 x_{1.23}=0$

Similarly, $\Sigma x_1 x_{2.13}=0=\Sigma x_3 x_{2.13}$

$\Sigma x_1 x_{3.12}=0=\Sigma x_2 x_{3.12}$

Thus "the sum of the product of any residual of zero order with any other higher order residual (having the subscripts of the former as one of its secondary subscripts) is zero."

(2) $\Sigma x_{1.2}x_{1.23}=\Sigma(x_1-b_{12}x_2)x_{1.23}$

$=\Sigma x_1 x_{1.23}-b_{12}\,\Sigma x_2 x_{1.23}$

$=\Sigma x_1 x_{1.23}$ {using property (1)}

Similarly $\Sigma x_{1.3}x_{1.23}=\Sigma x_1 x_{1.23}.$

Also $\Sigma x_{1.23}^2=\Sigma x_{1.23}x_{1.23}$

$=\Sigma(x_1-b_{12.3}x_2-b_{13.2}x_3)x_{1.23}$

$=\Sigma x_1 x_{1.23}-b_{12.3}\Sigma x_2 x_{1.23}-b_{13.2}\Sigma x_3 x_{1.23}$

$=\Sigma x_1 x_{1.23}$

Thus $\Sigma x_{1.2}x_{1.23}=\Sigma x_{1.3}x_{1.23}=\Sigma x_{1.23}^2=\Sigma x_1 x_{1.23}$

Similarly $\Sigma x_{2.1}x_{2.13}=\Sigma x_{2.3}x_{2.13}=\Sigma x_{2.13}^2=\Sigma x_2 x_{2.13}$

$\Sigma x_{3.1}x_{3.12}=\Sigma x_{3.2}x_{3.12}=\Sigma x_{3.12}^2=\Sigma x_3 x_{3.12}.$

Thus, "in the sum of product of any two residuals in which all the secondary subscripts of first occur among the secondary subscripts of the second, all the secondary subscripts of the first can be omitted."

$$\text{(3)}\quad \Sigma x_{1\cdot2}x_{3\cdot12}=\Sigma(x_1-b_{12}x_2)x_{3\cdot12}$$
$$=\Sigma x_1x_{3\cdot12}-b_{12}\Sigma x_2x_{3\cdot12}=0$$

Similarly

$$\Sigma x_{1\cdot3}x_{2\cdot13}=0,\ \Sigma x_{2\cdot1}x_{3\cdot12}=0,\ \Sigma x_{2\cdot3}x_{1\cdot23}=0 \text{ etc.}$$

Thus "the sum of the product of two residuals is zero provided all the subscripts (primary as well as secondary) of one occur among the secondary subscripts of the other."

17.5. Multiple Correlation Coefficient

Multiple correlation coefficient of x_1 on x_2, x_3 is the simple correlation coefficient between x_1 and its value as given by the plane of regression of x_1 on x_2, x_3 *viz*,

$$e_{1\cdot23}=b_{12\cdot3}x_2+b_{13\cdot2}x_3$$

Let N be the total number of observations for each variate.

Now $\quad \bar{e}_{1\cdot23}=b_{12\cdot3}\bar{x}_2+b_{13\cdot2}\bar{x}_3$
$$=0$$

$$\therefore\ \text{Var}(e_{1\cdot23})=\frac{1}{N}\Sigma\{e_{1\cdot23}\}^2$$
$$=\frac{1}{N}\Sigma\{b_{12\cdot3}x_2+b_{13\cdot2}x_3\}^2$$
$$=b_{12\cdot3}^2\frac{\Sigma(x_2^2)}{N}+b_{13\cdot2}^2\frac{\Sigma(x_3)^2}{N}+2b_{12\cdot3}b_{13\cdot2}\frac{\Sigma(x_2x_3)}{N}$$
$$=b_{12\cdot3}^2\sigma_2^2+b_{13\cdot2}^2\sigma_3^2+2b_{12\cdot3}b_{13\cdot2}\,\text{cov}(x_2,x_3)$$
$$=\left(-\frac{\sigma_1}{\sigma_2}\frac{\omega_{12}}{\omega_{11}}\right)^2\sigma_2^2+\left(-\frac{\sigma_1}{\sigma_3}\frac{\omega_{13}}{\omega_{11}}\right)^2\sigma_3^2+2\left(-\frac{\sigma_1}{\sigma_2}\frac{\omega_{12}}{\omega_{11}}\right)\left(-\frac{\sigma_1}{\sigma_3}\frac{\omega_{13}}{\omega_{11}}\right)r_{23}\sigma_2\sigma_3$$
$$=\frac{\sigma_1^2}{\omega_{11}^2}\{\omega_{12}^2+\omega_{13}^2+2r_{23}\omega_{12}\omega_{13}\}$$
$$=\frac{\sigma_1^2}{\omega_{11}^2}\{(r_{13}r_{23}-r_{12})^2+(r_{12}r_{23}-r_{13})^2+2r_{23}(r_{13}r_{23}-r_{12})(r_{12}r_{23}-r_{13})\}$$
$$=\frac{\sigma_1^2}{\omega_{11}^2}\{(r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23})(1-r_{23}^2)\}$$

$$=\frac{\sigma_1^2}{\omega_{11}}\{r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23}\}$$

$$=\frac{\sigma_1^2}{\omega_{11}}\{\omega_{11}-\omega\}=\sigma_1^2\left\{1-\frac{\omega}{\omega_{11}}\right\} \qquad \ldots(1)$$

where
$$\omega=\begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix}$$

$$=1-r_{12}^2-r_{13}^2-r_{23}^2+2r_{12}r_{13}r_{23}.$$

$$\text{cov}(x_1, \epsilon_{1\cdot23})=\frac{1}{N}\,\Sigma x_1\epsilon_{1\cdot23}$$

$$=\frac{1}{N}\,\Sigma x_1(b_{12\cdot3}x_2+b_{13\cdot2}x_3)$$

$$=b_{12\cdot3}\,\text{cov}(x_1, x_2)+b_{13\cdot2}\,\text{cov}(x_1, x_3)$$

$$=\left(-\frac{\sigma_1}{\sigma_2}\,\frac{\omega_{12}}{\omega_{11}}\right)\sigma_1\sigma_2r_{12}+\left(-\frac{\sigma_1}{\sigma_3}\,\frac{\omega_{13}}{\omega_{11}}\right)\sigma_1\sigma_3r_{13}$$

$$=-\frac{\sigma_1^2}{\omega_{11}}\{\omega_{12}r_{12}+\omega_{13}r_{13}\}$$

$$=-\frac{\sigma_1^2}{\omega_{11}}\{(r_{13}r_{23}-r_{12})r_{12}+(r_{12}r_{23}-r_{13})r_{13}\}$$

$$=\frac{\sigma_1^2}{\omega_{11}}\{r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23}\}$$

$$=\sigma_1^2\left\{1-\frac{\omega}{\omega_{11}}\right\} \qquad \ldots(2)$$

$$\therefore \qquad R_{1\cdot23}^2=1-\frac{\omega}{\omega_{11}}$$

$$=\frac{r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23}}{1-r_{23}^2}$$

Remark (I). (1) and (2) $\Rightarrow$

$\text{Cov}(x_1, \epsilon_{1\cdot23})=\text{Var}(\epsilon_{1\cdot23})\geqslant 0$

$\therefore \qquad R_{1\cdot23}\geqslant 0$

Also since $R_{1\cdot23}$ is simple correlation coefficient, $R_{1\cdot23}\leqslant 1$

$\therefore \qquad 0\leqslant R_{1\cdot23}\leqslant 1.$

Ex. 17.2. Three variables have in pairs simple correlation coefficients given by $r_{12}=0{\cdot}8 \qquad r_{13}=-0{\cdot}7 \qquad r_{23}=-0{\cdot}9.$

Find the multiple correlation coefficient $R_{1\cdot23}$ of x_1 on x_2 and x_3.

Solution. We have

$$\omega=\begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{12} & 1 & r_{23} \\ r_{13} & r_{23} & 1 \end{vmatrix}=\begin{vmatrix} 1 & 0\cdot8 & -0\cdot7 \\ 0\cdot8 & 1 & -0\cdot9 \\ -0\cdot7 & -0\cdot9 & 1 \end{vmatrix}=\cdot07$$

$$\omega_{11}=\begin{vmatrix} 1 & -0\cdot9 \\ -0\cdot9 & 1 \end{vmatrix}=1-0\cdot81=0\cdot19$$

$$\therefore\quad R_{1\cdot23}^2=1-\frac{\omega}{\omega_{11}}=1-\frac{\cdot07}{\cdot19}=\frac{12}{19}=0\cdot63$$

$$\therefore\quad R_{1\cdot23}=0\cdot8$$

Ex. 17.3. Show that $R_{1\cdot23}^2=1-\dfrac{\sigma_{1\cdot23}^2}{\sigma_1^2}$

where $\sigma_{1\cdot23}$ denotes the *s.d.* of $x_{1\cdot23}$.

Solution. We have $\epsilon_{1\cdot23}=b_{12\cdot3}\,x_2+b_{13\cdot2}\,x_3=x_1-x_{1\cdot23}$.

$$\therefore\quad \text{Var}\,(\in_{1\cdot23})=\frac{1}{N}\Sigma\,(x_1-x_{1\cdot23})^2$$

$$=\frac{1}{N}\Sigma\,\{x_1^2+x_{1\cdot23}^2-2x_1\,x_{1\cdot23}\}$$

$$=\frac{1}{N}\,\Sigma x_1^2+\frac{1}{N}\Sigma x_{1\cdot23}^2-2.\frac{1}{N}\,\Sigma x_1\,x_{1\cdot23}$$

$$=\sigma_1^2+\sigma_{1\cdot23}^2-2\,\frac{1}{N}\,\Sigma x_{1\cdot23}^2$$

$$=\sigma_1^2-\sigma_{1\cdot23}^2$$

$$\text{Cov}\,(x_1,\in_{1\cdot23})=\frac{1}{N}\,\Sigma x_1\,(x_1-x_{1\cdot23})$$

$$=\frac{1}{N}\,\Sigma x_1^2-\frac{1}{N}\Sigma x_1\,x_{1\cdot23}$$

$$=\sigma_1^2-\frac{1}{N}\Sigma x_{1\cdot23}^2=\sigma_1^2-\sigma_{1\cdot23}^2$$

$$\therefore\quad R_{1\cdot23}^2=\left\{\frac{\text{Cov}\,(x_1,\epsilon_{1\cdot23})}{(\text{s.d. of }x_1)(\text{s.d. of }\epsilon_{1\cdot23})}\right\}^2$$

$$=1-\frac{\sigma_{1\cdot23}^2}{\sigma_1^2}=1-\frac{\omega}{\omega_{11}}$$ (See Ex. 17.4)

Ex. 17.4. Show that $\sigma_{1\cdot32}^2=\sigma_1^2\,\dfrac{\omega}{\omega_{11}}$

Solution. We have $\sigma_{1\cdot32}^2=\dfrac{1}{N}\,\Sigma x_{1\cdot32}^2$

$$=\frac{1}{N}\Sigma x_{1\cdot32}\,x_{1\cdot32}$$

$$=\frac{1}{N}\Sigma x_1\, x_{1.23}$$

$$=\frac{1}{N}\Sigma x_1\,(x_1-b_{12.3}\,x_2-b_{13.2}\,x_3)$$

$$=\sigma_1^2-b_{12.3}\,\text{Cov}\,(x_1, x_2)-b_{13.2}\,\text{Cov}\,(x_1, x_3)$$

$$=\sigma_1^2-r_{12}\sigma_1\sigma_2\left(-\frac{\sigma_1}{\sigma_2}\,\frac{\omega_{12}}{\omega_{11}}\right)-r_{13}\sigma_1\sigma_3\left(-\frac{\sigma_1}{\sigma_3}\,\frac{\omega_{13}}{\omega_{11}}\right)$$

$$=\sigma_1^2\left(1+r_{12}\,\frac{\omega_{12}}{\omega_{11}}+r_{13}\,\frac{\omega_{13}}{\omega_{11}}\right)$$

$$=\frac{\sigma_1^2}{\omega_{11}}(r_{11}\,\omega_{11}+r_{12}\,\omega_{12}+r_{13}\,\omega_{13})$$

Since w_{ij}'s are cofactor in ω,

$$r_{11}\,\omega_{11}+r_{12}\,\omega_{12}+r_{13}\,\omega_{13}=\omega$$

$$\therefore\quad \sigma_{1.23}^2=\sigma_1^2\,\frac{\omega}{\omega_{11}}$$

17.6 Partial Correlation Coefficient

As already defined, the partial correlation between x_1 and x_2 is the simple correlation between x_1 and x_2 after the linear effect of x_3 on them has been eliminated.

Now linear effect of x_3 on x_1 as indicated by regression of x_1 on x_3 is $b_{13}=r_{13}\,\frac{\sigma_1}{\sigma_3}$ and linear effect of x_3 on x_2 is $b_{23}=r_{23}\,\frac{\sigma_2}{\sigma_3}$.

$$\therefore\quad x_{1.3}=x_1-r_{13}\,\frac{\sigma_1}{\sigma_3}\,x_3$$

and $\quad x_{2.3}=x_2-r_{23}\,\frac{\sigma_2}{\sigma_3}\,x_3$

are parts of x_1, x_2 respectively, which remain after the elimination of linear effect of x_3.

Thus partial correlation coefficient between x_1, x_2 is the simple correlation coefficient between $x_{1.3}$, $x_{2.3}$.

Now $\text{Cov}\,(x_{1.3}, x_{2.3})=\frac{1}{N}\Sigma x_{1.3}\,x_{2.3}$ $\qquad\{\because\ \bar{x}_{1.3}=\bar{x}_{2.3}=0\}$

$$=\frac{1}{N}\sum\left(x_1-r_{13}\frac{\sigma_1}{\sigma_3}\,x_3\right)\left(x_2-r_{23}\,\frac{\sigma_2}{\sigma_3}\,x_3\right)$$

$$=\frac{1}{N}\sum\left\{x_1x_2-r_{13}\,\frac{\sigma_1}{\sigma_3}\,x_3x_2-r_{23}\frac{\sigma_2}{\sigma_3}\,x_1x_3 +r_{13}\,r_{23}\,\frac{\sigma_1}{\sigma_3}\cdot\frac{\sigma_2}{\sigma_3}\,x_3^2\right\}$$

$$=\text{Cov}(x_1, x_2)-r_{13}\ \frac{\sigma_1}{\sigma_3}\ \text{Cov}(x_2, x_3)-r_{23}\ \frac{\sigma_2}{\sigma_3}\text{Cov}(x_1, x_3)$$

$$+r_{13}\ r_{23}\ \frac{\sigma_1\sigma_2}{\sigma_3^2}\cdot\sigma_3^2$$

$$=\sigma_1\sigma_2\,(r_{12}-r_{13}\ r_{23})$$

$$\text{Var}(x_{1\cdot3})=\frac{1}{N}\Sigma x_{1\cdot3}^2$$

$$=\frac{1}{N}\Sigma x_{1\cdot3}\ x_{1\cdot3}$$

$$=\frac{1}{N}\Sigma x_1\ x_{1\cdot3}$$

$$=\frac{1}{N}\ \Sigma x_1\left[\ x_1-r_{13}\frac{\sigma_1}{\sigma_3}x_3\ \right]=\sigma_1^2\,(1-r_{13}^2)$$

Similarly

$$\text{Var}(x_{2\cdot3})=\sigma_2^2\,(1-r_{23}^2)$$

$$\therefore\qquad r_{12\cdot3}=\frac{\text{Cov}(x_{1\cdot3}, x_{2\cdot3})}{(\text{s. d. of } x_{1\cdot3})(\text{s. d. of } x_{2\cdot3})}$$

$$=\frac{r_{12}-r_{13}\ r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}=\frac{-\omega_{12}}{\sqrt{\omega_{11}\ \omega_{22}}}$$

Remark. If $r_{12\cdot3}=0$, then $r_{12}=r_{13}\ r_{23}$.

$\therefore$ If x_3 is correlated with x_1, x_2 both *i.e.*, $r_{23}\neq0$, $r_{13}\neq0$, then

$$r_{12}\neq0$$

$\Rightarrow$ x_1, x are not uncorrelated.

$\therefore$ x_1, x_2 are correlated even though they are uncorrelated after the effect of x_3 is eliminated.

This is because x_1, x_2 carry the effect of x_3 on them.

Ex. 17 5. Show that

(*i*) $r_{23\cdot1}=\dfrac{r_{23}-r_{21}\ r_{31}}{\sqrt{(1-r_{21}^2)\,(1-r_{31}^2)}}$

(*ii*) $r_{13\cdot2}=\dfrac{r_{13}-r_{12}\ r_{32}}{\sqrt{(1-r_{12}^2)\,(1-r_{32}^2)}}$.

Solution. (*i*) We have

$$0=\Sigma x_{2\cdot13}\ x_{3\cdot1}$$
$$=\Sigma\,(x_2-b_{21\cdot3}\ x_1-b_{23\cdot1}\ x_3)\ x_{3\cdot1}$$
$$=\Sigma x_2\ x_{3\cdot1}-b_{21\cdot3}\ \Sigma x_1\ x_{3\cdot1}-b_{23\cdot1}\ \Sigma x_3\ x_{3\cdot1}$$
$$=\Sigma x_{2\cdot1}\ x_{3\cdot1}-b_{23\cdot1}\Sigma x_{3\cdot1}^2$$

$$\Rightarrow \qquad b_{23\cdot1}=\frac{\Sigma x_{2\cdot1}\, x_{3\cdot1}}{\Sigma x_{3\cdot1}^2}=\frac{\text{Cov}\,(x_{2\cdot1},\, x_{3\cdot1})}{\sigma_{3\cdot1}^2}$$

$$=r_{23\cdot1}\,\frac{\sigma_{2\cdot1}}{\sigma_{3\cdot1}}$$

Similarly

$$b_{32\cdot1}=r_{23\cdot1}\,\frac{\sigma_{3\cdot1}}{\sigma_{2\cdot1}}$$

$$\therefore \quad (b_{23\cdot1})\,(b_{32\cdot1})=r_{23\cdot1}^2$$

$$\therefore \qquad r_{23\cdot1}^2=\left[-\frac{\sigma_2}{\sigma_3}\,\frac{\omega_{23}}{\omega_{22}}\right]\left[-\frac{\sigma_3}{\sigma_2}\,\frac{\omega_{32}}{\omega_{33}}\right]$$

$$=\frac{\omega_{23}^2}{\omega_{22}\,\omega_{33}}$$

$$\therefore \qquad r_{23\cdot1}=\frac{-\omega_{23}}{\sqrt{\omega_{22}\,\omega_{33}}}\,(\because\ b_{23\cdot1},\, b_{32\cdot1},\ \text{both are with } -\text{ve sign})$$

Now $\qquad \omega_{23}=-\begin{vmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{vmatrix}$

$$=(r_{31}\, r_{12}-r_{32}) \qquad\qquad (\because\ r_{11}=1)$$

$$\omega_{22}=1-r_{13}^2,\quad \omega_{33}=1-r_{12}^2$$

$$\therefore \qquad r_{23\cdot1}=\frac{r_{32}-r_{31}\, r_{12}}{\sqrt{(1-r_{13}^2)\,(1-r_{12}^2)}}$$

Similarly prove (*ii*)

Ex. 17.6. In a trivariate distribution

$$\sigma_1=3 \qquad \sigma_2=4, \qquad \sigma_3=5$$

$$r_{12}=0{\cdot}7 \qquad r_{13}=0{\cdot}61, \qquad r_{23}=0{\cdot}4$$

find $\qquad r_{23\cdot1}, \qquad b_{12\cdot3}$ and $\sigma_{1\cdot23}$.

Solution. We have $\omega=\begin{vmatrix} 1 & 0{\cdot}7 & 0{\cdot}61 \\ 0{\cdot}7 & 1 & 0{\cdot}4 \\ 0{\cdot}61 & 0{\cdot}4 & 1 \end{vmatrix}=0{\cdot}32$

$$\omega_{11}=\begin{vmatrix} 1 & 0{\cdot}4 \\ 0{\cdot}4 & 1 \end{vmatrix}=0{\cdot}84$$

$$\omega_{22}=\begin{vmatrix} 1 & 0{\cdot}61 \\ 0{\cdot}61 & 1 \end{vmatrix}=0{\cdot}63$$

$$\omega_{33}=\begin{vmatrix} 1 & 0{\cdot}7 \\ 0{\cdot}7 & 1 \end{vmatrix}=0{\cdot}51$$

$$\omega_{23} = -\begin{vmatrix} 1 & 0{\cdot}7 \\ 0{\cdot}61 & 0{\cdot}4 \end{vmatrix} = 0{\cdot}027$$

$$\omega_{12} = -\begin{vmatrix} 0{\cdot}7 & 0{\cdot}4 \\ 0{\cdot}61 & 1 \end{vmatrix} = -0{\cdot}46$$

$$\therefore \qquad r_{23\cdot1} = \frac{-\omega_{23}}{\sqrt{\omega_{22}\,\omega_{33}}} = \frac{-{\cdot}027}{\sqrt{(0{\cdot}63)(0{\cdot}51)}}$$

$$= -0{\cdot}05$$

$$b_{12\cdot3} = -\frac{\sigma_1}{\sigma_2} \cdot \frac{\omega_{12}}{\omega_{11}} = \frac{3}{4} \cdot \frac{0{\cdot}46}{0{\cdot}84}$$

$$= 0{\cdot}54$$

$$\sigma_{1\cdot23} = \sigma_1 \sqrt{\frac{\omega}{\omega_{11}}} = 3\sqrt{\frac{0{\cdot}32}{0{\cdot}84}} = 1{\cdot}85.$$

Ex. 17.7. Show that $1-R_{1\cdot23}^2 = (1-r_{12}^2)(1-r_{13\cdot2}^2)$

Deduce that $R_{1\cdot23} \geq r_{12}$

and $\quad 1+2r_{12}\,r_{13}\,r_{23} \geq r_{12}^2 + r_{13}^2 + r_{23}^2$

Solution. We have

$$1-R_{1\cdot23}^2 = \frac{\omega}{\omega_{11}}$$

$$1-r_{13\cdot2}^2 = 1-\frac{\omega_{13}^2}{\omega_{11}\,\omega_{33}}$$

$$= \frac{\omega_{11}\,\omega_{33}-\omega_{13}^2}{\omega_{11}\,\omega_{33}}$$

$$= \frac{1}{\omega_{11}\,\omega_{33}}\{(1-r_{23}^2)(1-r_{12}^2)-(r_{21}\,r_{32}-r_{31})^2\}$$

$$= \frac{1}{\omega_{11}\,\omega_{33}}\{1-r_{23}^2-r_{12}^2-r_{31}+2r_{31}\,r_{21}\,r_{32}\} \qquad \ldots(1)$$

$$= \frac{\omega}{\omega_{11}\,\omega_{33}}$$

$$\therefore \qquad \frac{1-R_{1\cdot23}^2}{1-r_{13\cdot2}^2} = \omega_{33} = 1-r_{12}^2$$

$$\Rightarrow \qquad (1-R_{1\cdot23}^2) = (1-r_{12}^2)(1-r_{13\cdot2}^2)$$

Now $\qquad 0 \leq r_{13\cdot2}^2 \leq 1$

$$\Rightarrow \qquad 0 \leq 1-r_{13\cdot2}^2 \leq 1 \qquad \ldots(2)$$

$\therefore \quad 1-R_{1\cdot 23}^2 < 1-r_{12}^2$

$\Rightarrow \quad R_{1\cdot 23}^2 > r_{12}^2$

$\Rightarrow \quad R_{1\cdot 23} > r_{12}.$

Also (1) and (2) $\Rightarrow$

$$1-r_{12}^2-r_{23}^2-r_{31}^2+2r_{31}\, r_{21}\, r_{32}>0$$

$$\Rightarrow \quad 1+2r_{12}\, r_{13}\, r_{23}>r_{12}^2+r_{13}^2+r_{23}^2$$

Ex. 17.8. Show that three regression planes coincide iff

$$\begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix}=0.$$

Solution. The equation of three planes of regression are

$$\frac{x_1}{\sigma_1}\omega_{11}+\frac{x_2}{\sigma_2}\omega_{12}+\frac{x_3}{\sigma_3}\omega_{13}=0 \qquad \text{...(1)}$$

$$\frac{x_1}{\sigma_1}\omega_{21}+\frac{x_2}{\sigma_2}\omega_{22}+\frac{x_3}{\sigma_3}\omega_{23}=0 \qquad \text{...(2)}$$

$$\frac{x_1}{\sigma_1}\omega_{31}+\frac{x_2}{\sigma_2}\omega_{32}+\frac{x_3}{\sigma_3}\omega_{33}=0 \qquad \text{...(3)}$$

Planes (1) and (2) will coincide iff

$$\frac{\omega_{11}}{\omega_{21}}=\frac{\omega_{12}}{\omega_{22}}=\frac{\omega_{13}}{\omega_{23}} \qquad \text{...(4)}$$

and planes (2) and (3) coincide iff

$$\frac{\omega_{21}}{\omega_{31}}=\frac{\omega_{22}}{\omega_{32}}=\frac{\omega_{23}}{\omega_{33}} \qquad \text{...(5)}$$

First two ratios in (4) $\Rightarrow$

$$\omega_{11}\, \omega_{22}=\omega_{21}\, \omega_{12}$$

i.e., $\quad (1-r_{23}^2)\,(1-r_{13}^2)=(r_{32}\, r_{13}-r_{12})^2$

i.e., $\quad 1-r_{23}^2-r_{13}^2+r_{23}^2\, r_{13}^2=r_{23}^2\, r_{13}^2+r_{12}^2-2r_{12}\, r_{32}\, r_{13}$

i.e., $\quad 1-r_{23}^2-r_{13}^2-r_{12}^2+2r_{12}\, r_{32}\, r_{13}=0$

$$\Rightarrow \quad \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix}=0.$$

Similarly other ratios in (4) and (5) also imply this condition.

Ex. 17.9. Show that

$$b_{12\cdot 3}\, b_{23\cdot 1}\, b_{31\cdot 2}=r_{12\cdot 3}\, r_{23\cdot 1}\, r_{31\cdot 2}.$$

Solution. R.H S.$=r_{12\cdot 3}\, r_{23\cdot 1}\, r_{31\cdot 2}$

$$=\left[\frac{-\omega_{12}}{\sqrt{\omega_{11}\, \omega_{22}}}\right]\left[\frac{-\omega_{23}}{\sqrt{\omega_{22}\, \omega_{33}}}\right]\left[\frac{-\omega_{31}}{\sqrt{\omega_{33}\, \omega_{11}}}\right]$$

$$=\left[-\frac{\sigma_1}{\sigma_2}\ \frac{\omega_{12}}{\omega_{11}}\right]\left[-\frac{\sigma_2}{\sigma_3}\ \frac{\omega_{23}}{\omega_{22}}\right]\left[-\frac{\sigma_3}{\sigma_1}\ \frac{\omega_{31}}{\omega_{33}}\right]$$

$$=b_{12\cdot3}\ b_{23\cdot1}\ b_{31\cdot2}.$$

Ex. 17.10. Show that the correlation coefficient between $x_{1\cdot23}$ and $x_{2\cdot13}$ is equal and opposite to that between $x_{1\cdot3}$ and $x_{2\cdot3}$.

Solution. We have

$$\text{Cov}\,(x_{1\cdot23},\ x_{2\cdot13})=\frac{1}{N}\,\Sigma x_{1\cdot23}\,x_{2\cdot13}$$

$$=\frac{1}{N}\,\Sigma x_{1\cdot23}\,(x_2-b_{21\cdot3}\,x_1-b_{23\cdot1}\,x_3)$$

$$=-b_{21\cdot3}\ \frac{1}{N}\ \Sigma x_{1\cdot23}\,x_1$$

$$=-b_{21\cdot3}\,\frac{1}{N}\ \Sigma x_{1\cdot23}^2$$

$$=-b_{21\cdot3}\,\sigma_{1\cdot23}^2$$

$$\therefore\quad r\,(x_{1\cdot23},\ x_{2\cdot13})=\frac{\text{Cov}\,(x_{1\cdot23},\ x_{2\cdot13})}{\sigma_{1\cdot23}\ \sigma_{2\cdot13}}$$

$$=-b_{21\cdot3}\ \frac{\sigma_{1\cdot23}}{\sigma_{2\cdot13}}.$$

$$=-\left[-\frac{\sigma_2}{\sigma_1}\ \frac{\omega_{21}}{\omega_{22}}\right]\left[\frac{\sigma_1^2\ \omega/\omega_{11}}{\sigma_2^2\ \omega/\omega_{22}}\right]^{1/2}$$

$$=-\left\{-\frac{\omega_{21}}{(\omega_{11}\ \omega_{22})^{1/2}}\right\}$$

$$=-r_{21\cdot3}$$

$$=-r\,(x_{1\cdot3},\ x_{2\cdot3}).$$

Ex. 17.11. Show that if $x_3=ax_1+bx_2$, the three partial correlations are numerically equal to unity, $r_{13\cdot2}$ having the sign of a, $r_{23\cdot1}$, the sign of b and $r_{12\cdot3}$ the opposite sign of $\frac{a}{b}$.

Solution. Here x_1, x_2 can be regarded as independent and x_3 is dependent on both of them

$$\therefore\qquad r_{12}=0\quad\Rightarrow\quad \text{Cov}\,(x_1,\ x_2)=0$$

$$\Rightarrow\quad \Sigma x_1x_2=0$$

Now $\text{Var}\,(x_3)=\text{Var}\,(ax_1+bx_2)$

$$\therefore\qquad \sigma_3^2=a^2\ \text{Var}\,(x_1)+b^2\ \text{Var}\,(x_2)$$

$$=a^2\sigma_1^2+b^2\sigma_2^2$$

$$\text{Cov}(x_1, x_3) = \frac{1}{N} \Sigma x_1 x_3$$

$$= \frac{1}{N} \Sigma x_1 (ax_1 + bx_2) = a\,\sigma_1^2$$

$$\therefore \qquad r_{13} = \frac{a\,\sigma_1^2}{\sigma_1\,\sigma_3} = a\,\frac{\sigma_1}{\sigma_3}$$

Similarly

$$r_{23} = b\,\frac{\sigma_2}{\sigma_3}$$

$$\therefore \qquad r_{13\cdot2} = \frac{r_{13} - r_{12}\,r_{32}}{\sqrt{(1-r_{12}^2)(1-r_{32}^2)}}$$

$$= \frac{a\,\sigma_1/\sigma_3}{\sqrt{1 - \dfrac{b^2\sigma_2^2}{\sigma_3^2}}} = \frac{a\,\sigma_1}{\sqrt{\sigma_3^2 - b^2\sigma_2^2}}$$

$$= \frac{a\sigma_1}{\sqrt{a^2\sigma_1^2 + b^2\sigma_2^2 - b^2\sigma_2^2}} = \frac{a\sigma_1}{|\,a\,|\,\sigma_1}$$

$= \pm 1$ according as $a >$ or < 0.

Similarly $r_{23\cdot1} = \pm 1$ according as $b >$ or < 0.

Now

$$r_{12\cdot3} = \frac{r_{12} - r_{13}\,r_{23}}{\sqrt{(1-r_{13}^2)(1-r_{23}^2)}}$$

$$= \frac{-\dfrac{a\sigma_1}{\sigma_3} \cdot \dfrac{b\sigma_2}{\sigma_3}}{\sqrt{\left[1 - \dfrac{a^2\sigma_1^2}{\sigma_3^2}\right]\left[1 - \dfrac{b^2\sigma_2^2}{\sigma_3^2}\right]}}$$

$$= \frac{-ab\,\sigma_1\sigma_2}{\sqrt{(\sigma_3^2 - a^2\sigma_1^2)(\sigma_3^2 - b^2\sigma_2^2)}}$$

$$= \frac{-ab}{|\,ab\,|} = \mp 1$$

according as $ab >$ or < 0 *i.e.*, a and b are of same or opposite sign

i.e., $\qquad \dfrac{a}{b} >$ or < 0.

$\therefore$ $r_{12\cdot3}$ has sign opposite of $\dfrac{a}{b}$.

Ex. 17.12. If $r_{23} = 1$, show that

$$R_{1\cdot23}^2 = r_{12}^2 = r_{13}^2$$

and

$$\sigma_{1\cdot23}^2 = \sigma_1^2\,(1 - r_{12}^2)$$

Solution. We have $R_{1\cdot 23}^2(1-r_{23}^2)=r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23}$

Put $\quad r_{23}=1$

$$(r_{12}-r_{13})^2=0 \quad \Rightarrow \quad r_{12}=r_{13}.$$

$\therefore \quad R_{1\cdot 23}^2(1-r_{23}^2)=2r_{12}^2(1-r_{23})$

$\Rightarrow \quad R_{1\cdot 23}^2 \to \left[\dfrac{2r_{12}^2}{1+r_{23}}\right]_{r_{23}=1}=r_{12}^2=r_{13}^2$

$\therefore$ By Ex. 17.3,

$$\sigma_{1\cdot 23}^2=\sigma_1^2(1-R_{1\cdot 23}^2)=\sigma_1^2(1-r_{12}^2)$$

Ex. 17.13. Show that $R_{1\cdot 23}^2=b_{12\cdot 3}\,r_{12}\dfrac{\sigma_2}{\sigma_1}+b_{13\cdot 2}\,r_{13}\dfrac{\sigma_3}{\sigma_1}$

Solution. R.H.S.$=b_{12\cdot 3}\,r_{12}\dfrac{\sigma_2}{\sigma_1}+b_{13\cdot 2}\,r_{13}\dfrac{\sigma_3}{\sigma_1}$

$$=\left[-\frac{\sigma_1}{\sigma_2}\frac{\omega_{12}}{\omega_{11}}\right]r_{12}\frac{\sigma_2}{\sigma_1}+\left[-\frac{\sigma_1}{\sigma_3}\frac{\omega_{13}}{\omega_{11}}\right]r_{13}\frac{\sigma_3}{\sigma_1}$$

$$=-\frac{1}{\omega_{11}}\{\omega_{12}r_{12}+\omega_{13}r_{13}\}$$

$$=-\frac{1}{1-r_{23}^2}\{r_{12}(r_{31}r_{23}-r_{21})+r_{13}(r_{21}r_{32}-r_{31})\}$$

$$=\frac{1}{1-r_{23}^2}\{r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{32}\}$$

$$=R_{1\cdot 23}^2.$$

Ex. 17.14. Show that $R_{1\cdot 23}=0 \Rightarrow r_{12}=r_{13}=0.$

Solution. We have $1-R_{1\cdot 23}^2=(1-r_{12}^2)(1-r_{13\cdot 2}^2)$

(See Ex. 17.7)

Put $\quad R_{1\cdot 23}^2=0$

$\therefore \quad (1-r_{12}^2)(1-r_{13\cdot 2}^2)=1 \quad$ ...(1)

Since $0\leqslant r_{12}^2\leqslant 1$ and $0\leqslant r_{13\cdot 2}^2\leqslant 1$,

(1) is possible only when

$1-r_{12}^2=1 \Rightarrow r_{12}=0 \quad$ ...(2)

and $\quad 1-r_{13\cdot 2}^2=1 \Rightarrow r_{13\cdot 2}=0 \quad$ ...(3)

(3) $\Rightarrow \quad r_{13}-r_{12}r_{32}=0$

$\Rightarrow \quad r_{13}=0$ (using 2)

EXERCISES

Ex. 1. Find the regression equation of x_3 on x_1 and x_2 given that

$r_{12}=0\cdot 28 \qquad r_{23}=0\cdot 49 \qquad r_{31}=0\cdot 51$

$\sigma_1=2\cdot 7 \qquad \sigma_2=2\cdot 4 \qquad \sigma_3=2\cdot 7$

Also find $R_{1\cdot 23}$, $r_{23\cdot 1}$.

Ex. 2. Calculate the multiple correlation coefficient of x_1 on x_2 and x_3 from the following data :

x_1	1	2	3	4	5	8
x_2	2	2	4	2	2	4
x_3	13	15	21	17	21	32

Also find the regression equation of x_1 on x_2, x_3.

Ex. 3. Let x_1=seed-hay crop, x_2=rainfall and x_3=accumulated temperature. The following means, s.ds and correlations are found

$$\bar{x}_1=28{\cdot}02, \qquad \bar{x}_2=4{\cdot}9, \qquad \bar{x}_3=594$$
$$\sigma_1=4{\cdot}4, \qquad \sigma_2=1{\cdot}1, \qquad \sigma_3=85$$
$$x_{12}=0{\cdot}8, \qquad r_{13}=-0{\cdot}4, \qquad r_{23}=-0{\cdot}56.$$

Find all partial correlations and the regression equations for hay-crop on rainfall and accumulated temperature.

Ex. 4. Let x_1, x_2, x_3 are variates with zero means and

$$\sigma_1=1, \qquad \sigma_2=1{\cdot}3, \qquad \sigma_3=1{\cdot}9$$
$$r_{12}={\cdot}37, \qquad r_{13}=-0{\cdot}641, \qquad r_{23}=-0{\cdot}736.$$

Verify that $\quad r_{13\cdot2}=r_{43\cdot2}$ where $x_4=x_1+x_2$.

Ex. 5. If $x_1=y_1+y_2$, $x_2=y_2+y_3$, $x_3=y_3+y_1$ where y_1, y_2 y_3 are uncorrelated variables each of which has zero mean and unit standard deviation, find $R_{1\cdot23}$.

$$\left[\textbf{Ans.}\ \frac{1}{\sqrt{3}}\right]$$

Ex. 6. If $r_{12}=r_{13}=r_{23}=\rho(\neq1)$, show that each partial correlation coefficient is $\dfrac{\rho}{1+\rho}$ and each multiple correlation coefficient is

$$\frac{\rho\sqrt{2}}{\sqrt{1+\rho}}.$$

Also show that

$$1-R_{1\cdot23}^2=\frac{(1-\rho)\,(1+2\rho)}{1+\rho}$$

Ex. 7. x_1, x_2, x_3 are uncorrelated variates with same variance. Let

$$y_1=\frac{x_1-x_3}{\sqrt{2}}, \quad y_2=\frac{x_1+x_2+x_3}{\sqrt{3}}, \quad y_3=\frac{x_1+2x_2+x_3}{\sqrt{6}}$$

show that y_1, y_2, y_3 are standard variates. Also find $r_{12\cdot3}$ and $R_{1\cdot23}$ for y's.

Ex. 8. If $a_1x_1+a_2x_2+a_3x_3=k$, prove that

$$r_{12}=(a_3^2\sigma_3^2-a_1^2\sigma_1^2-a_2^2\sigma_2^2)/2a_1\,a_2\sigma_1\,\sigma_2$$

with two similar expressions for r_{13} and r_{23}.

Also show that all the partial quotients are equal to -1 provided that a's are all positive.

Ex. 9. If x_1, x_2, x_3 are three variates measured from their respective means and if e_1 is the expected value of x_1 for given values of x_2 and x_3 from the linear regression of x_1 on x_2 and x_3, prove that

$$\text{Cov}\,(x_1, e_1)=\text{Var}\,(e_1)=\text{Var}\,(x_1)-\text{Var}\,(x_1-e_1)$$

Ex. 10. If x_1, x_2, x_3 are standard variates and

$E(x_2x_3)=E(x_1x_3)=\dfrac{1}{2}$. Show that

$$E(x_1x_2)\geq -5/2.$$

Ex. 11. If $r_{23}=0$, show that

$$R_{1\cdot23}^2=r_{12}^2+r_{13}^2$$

and

$$\sigma_{1\cdot23}^2=1-r_{12}^2-r_{13}^2$$

Ex. 12. Suppose a computer has found, for a given set of values of x_1, x_2 and x_3

$$r_{12}=0\cdot6, \qquad r_{23}=0\cdot7, \qquad r_{31}=-0\cdot4.$$

Examine whether his computations may be said to be free from error. (**Ans.** No)

(**Hint.** Find $r_{12\cdot3}$)

Ex. 13. Comment on the consistancy of

$$r_{12}=0\cdot6, \qquad r_{23}=0\cdot8, \qquad r_{31}=-0\cdot5.$$

Ex. 14. For what value of $R_{1\cdot23}$, will x_2 and x_3 be uncorrelated !

Ex. 15. If r_{12} and r_{13} are given, show that r_{23} must lie in the range

$$r_{12}\,r_{13}\pm(1-r_{12}^2-r_{13}^2+r_{12}^2\,r_{13}^2)^{1/2}$$

(**Hint.** Use $r_{12\cdot3}^2\leq 1$)

Ex. 16. If $r_{12}=k$; $r_{23}=-k$, show that r_{13} will lie between -1 and $1-2k^2$.

Ex. 17. A number of persons are measured for heights x_1, weights x_2 and chest expansions x_3 and product moment correlation coefficients are calculated. Show that

$$r_{12}+r_{23}+r_{31}\geq -3/2.$$

$$\left[\textbf{Hint.}\quad \text{Use } E\left[\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2}+\frac{x_3}{\sigma_3}\right]^2\geq 0\right]$$

Table I

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170						5 9 13	17 21 26	30 34 38
						0212	0253	0294	0334	0374	4 8 12	16 20 24	28 32 36
11	0414	0453	0492	0531	0569						4 8 12	16 20 23	27 31 35
						0607	0645	0682	0719	0755	4 7 11	15 18 22	26 29 33
12	0792	0828	0864	0899	0934						3 7 11	14 18 21	25 28 32
						0969	1004	1038	1072	1106	3 7 10	14 17 20	24 27 31
13	1139	1173	1206	1239	1271						3 6 10	13 16 19	23 26 29
						1303	1335	1367	1399	1430	3 7 10	13 16 19	22 25 29
14	1461	1492	1523	1553	1584						3 6 9	12 15 19	22 25 28
						1614	1644	1673	1703	1732	3 6 9	12 14 17	20 23 26
15	1761	1790	1818	1847	1875						3 6 9	11 14 17	20 23 26
						1903	1931	1959	1987	2014	3 6 8	11 14 17	19 22 25
16	2041	2068	2095	2122	2148						3 6 8	11 14 16	19 22 24
						2175	2201	2227	2253	2279	3 5 8	10 13 16	18 21 23
17	2304	2330	2355	2380	2405						3 5 8	10 13 15	18 20 23
						2430	2455	2480	2504	2529	3 5 8	10 12 15	17 20 22
18	2553	2577	2601	2625	2648						2 5 7	9 12 14	17 19 21
						2672	2695	2718	2742	2765	2 4 7	9 11 14	16 18 21
19	2788	2810	2833	2856	2878						2 4 7	9 11 13	16 18 20
						2900	2923	2945	2967	2989	2 4 6	8 11 13	15 17 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 3 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 13 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4	6 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 6 8	9 10 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 3 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 3	5 6 7	8 9 10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3	4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 5	6 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 4 5	6 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8

Table II

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

Table III

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	2	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	1	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	2	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	5	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Table IV

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3231	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	[illegible]	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	[illegible]	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Table V

HYPERBOLIC OR NAPERIAN LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
1·0	0·0000	0099	0198	0296	0392	0488	0583	0677	0770	0862	10	19	29	38	48	57	67	76	86
1·1	·0953	1044	1133	1222	1310	1398	1484	1570	1655	1740	9	17	26	35	44	52	61	70	78
1·2	·1823	1906	1989	2070	2151	2231	2311	2390	2469	2546	8	16	24	32	40	48	56	64	72
1·3	·2624	2700	2776	2852	2927	3001	3075	3148	3221	3293	7	15	22	30	37	44	52	59	67
1·4	·3365	3436	3507	3577	3646	3716	3784	3853	3920	3988	7	14	21	28	35	41	48	55	62
1·5	·4055	4121	4187	4253	4318	4383	4447	4511	4574	4637	6	13	19	26	32	39	45	52	58
1·6	·4700	4762	4824	4886	4947	5008	5068	5128	5188	5247	6	12	18	24	30	36	42	48	55
1·7	·5306	5365	5423	5481	5539	5596	5653	5710	5766	5822	6	11	17	24	29	34	40	46	51
1·8	·5878	5933	5988	6043	6098	6152	6206	6259	6313	6366	5	11	16	22	27	32	38	43	49
1·9	·6419	6471	6523	6575	6627	6678	6729	6780	6831	6881	5	10	15	20	26	31	36	41	46
2·0	·6931	6981	7031	7080	7129	7178	7227	7275	7324	7372	5	10	15	20	24	29	34	39	44
2·1	·7419	7467	7514	7561	7608	7655	7701	7747	7793	7839	5	9	14	19	23	28	33	37	42
2·2	·7885	7930	7975	8020	8065	8109	8154	8198	8242	8286	4	9	13	18	22	27	31	36	40
2·3	·8329	8372	8416	8459	8502	8544	8587	8629	8671	8713	4	9	13	17	21	26	30	34	38
2·4	·8755	8796	8838	8879	8920	8961	9002	9042	9083	9123	4	8	12	16	20	24	29	33	37
2·5	·9163	9203	9243	9282	9322	9361	9400	9439	9478	9517	4	8	12	16	20	24	27	31	35
2·6	·9555	9594	9632	9670	9708	9746	9783	9821	9858	9895	4	8	11	15	19	23	26	30	34
2·7	·9933	9969	1·0006	0043	0080	0116	0152	0188	0225	0260	4	7	11	15	18	22	25	29	33
2·8	1·0296	0332	0367	0403	0438	0473	0508	0543	0578	0613	4	7	11	14	18	21	25	28	32
2·9	1·0647	0682	0716	0750	0784	0818	0852	0886	0919	0953	3	7	10	14	17	20	24	27	31
3·0	1·0986	1019	1053	1086	1119	1151	1184	1217	1249	1282	3	7	10	13	16	20	23	26	30
3·1	1·1314	1346	1378	1410	1442	1474	1506	1537	1569	1600	3	6	10	13	16	19	22	25	29
3·2	1·1632	1663	1694	1725	1756	1787	1817	1848	1878	1909	3	6	9	12	15	18	22	25	28
3·3	1·1939	1969	1·2000	2030	2060	2090	2119	2149	2179	2208	3	6	9	12	15	18	21	24	27
3·4	1·2238	2267	2296	2326	2355	2384	2413	2442	2470	2499	3	6	9	12	15	17	20	23	26
3·5	1·2528	2556	2585	2613	2641	2669	2698	2726	2754	2782	3	6	8	11	14	17	20	23	25
3·6	1·2809	2837	2865	2892	2920	2947	2975	3002	3029	3056	3	5	8	11	14	16	19	22	25
3·7	1·3083	3110	3137	3164	3191	3218	3244	3271	3297	3324	3	5	8	11	13	16	19	21	24
3·8	1·3350	3376	3403	3429	3455	3481	3507	3533	3558	3584	3	5	8	10	13	16	18	21	23
3·9	1·3610	3635	3661	3686	3712	3737	3762	3788	3813	3838	3	5	8	10	13	15	18	20	23
4·0	1·3863	3888	3913	3938	3962	3987	4012	4036	4061	4085	2	5	7	10	12	15	17	20	22
4·1	1·4110	4134	4159	4183	4207	4231	4255	4279	4303	4327	2	5	7	10	12	14	17	19	22
4·2	1·4351	4375	4398	4422	4446	4469	4493	4516	4540	4563	2	5	7	9	12	14	16	19	21
4·3	1·4586	4609	4633	4656	4679	4702	4725	4748	4770	4793	2	5	7	9	12	14	16	18	21
4·4	1·4816	4839	4861	4884	4907	4929	4951	4974	4996	5019	2	5	7	9	11	14	16	18	20
4·5	1·5041	5063	5085	5107	5129	5151	5173	5195	5217	5239	2	4	7	9	11	13	15	18	20
4·6	1·5261	5282	5304	5326	5347	5369	5390	5412	5433	5454	2	4	6	9	11	13	15	17	19
4·7	1·5476	5497	5518	5539	5560	5581	5602	5623	5644	5665	2	4	6	8	11	13	15	17	19
4·8	1·5686	5707	5728	5748	5769	5790	5810	5831	5851	5872	2	4	6	8	10	12	14	16	19
4·9	1·5892	5913	5933	5953	5974	5994	6014	6034	6054	6074	2	4	6	8	10	12	14	16	18
5·0	1·6094	6114	6134	6154	6174	6194	6214	6233	6253	6273	2	4	6	8	10	12	14	16	18
5·1	1·6292	6312	6332	6351	6371	6390	6409	6429	6448	6467	2	4	6	8	10	12	14	16	18
5·2	1·6487	6506	6525	6544	6563	6582	6601	6620	6639	6658	2	4	6	8	10	11	13	15	17
5·3	1·6677	6696	6715	6734	6752	6771	6790	6808	6827	6845	2	4	6	7	9	11	13	15	17
5·4	1·6864	6882	6901	6919	6938	6956	6974	6993	7011	7029	2	4	5	7	9	11	13	15	17

Hyperbolic or Naperian Logarithms of 10^{+n}.

n	1	2	3	4	5	6	7	8	9
$\log_e 10^n$	2·3026	4·6052	6·9078	9·2103	11·5129	13·8155	16·1181	18·4207	20·7233

Table VI

HYPERBOLIC OR NAPERIAN LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1 2 3	4 5 6	7 8 9
5·5	1·7047	7066	7084	7102	7120	7138	7156	7174	7192	7210	2 4 5	7 9 11	13 14 16
5·6	1·7228	7246	7263	7281	7299	7317	7334	7352	7370	7387	2 4 5	7 9 11	12 14 16
5·7	1·7405	7422	7440	7457	7475	7492	7509	7527	7544	7561	2 3 5	7 9 10	12 14 16
5·8	1·7579	7596	7613	7630	7647	7664	7681	7699	7716	7733	2 3 5	7 9 10	12 14 15
5·9	1·7750	7766	7783	7800	7817	7834	7851	7867	7884	7901	2 3 5	7 8 10	12 13 15
6·0	1·7918	7934	7951	7967	7984	8001	8017	8034	8050	8066	2 3 5	7 8 10	12 13 15
6·1	1·8083	8099	8116	8132	8148	8165	8181	8197	8213	8229	2 3 5	6 8 10	11 13 15
6·2	1·8245	8262	8278	8294	8310	8326	8342	8358	8374	8390	2 3 5	6 8 10	11 13 14
6·3	1·8405	8421	8437	8453	8469	8485	8500	8516	8532	8547	2 3 5	6 8 9	11 13 14
6·4	1·8563	8579	8594	8610	8625	8641	8656	8672	8687	8703	2 3 5	6 8 9	11 12 14
6·5	1·8718	8733	8749	8764	8779	8795	8810	8825	8840	8856	2 3 5	6 8 9	11 12 14
6·6	1·8871	1·8886	1·8901	8916	8931	8946	8961	8976	8991	9006	2 3 5	6 8 9	11 12 14
6·7	1·9021	9036	9051	9066	9081	9095	9110	9125	9140	9155	1 3 4	6 7 9	10 12 13
6·8	1·9169	9184	9199	9213	9228	9242	9257	9272	9286	9301	1 3 4	6 7 9	10 12 13
6·9	1·9315	9330	9344	9359	9373	9387	9402	9416	9430	9445	1 3 4	6 7 9	10 12 13
7·0	1·9459	9473	9488	9502	9516	9530	9544	9559	1·9573	9587	1 3 4	6 7 9	10 11 13
7·1	1·9601	9615	9629	9643	9657	9671	9685	9699	9713	9727	1 3 4	6 7 8	10 11 13
7·2	1·9741	9755	9769	9782	9796	9810	9824	9838	9851	9865	1 3 4	6 7 8	10 11 12
7·3	1·9879	9892	9906	9920	9933	9947	9961	9974	9988	2·0001	1 3 4	5 7 8	10 11 12
7·4	2·0015	0028	0042	0055	0069	0082	0096	0109	0122	0136	1 3 4	5 7 8	9 11 12
7·5	2·0149	0162	0176	0189	0202	0215	0229	0242	0255	0268	1 3 4	5 7 8	9 11 12
7·6	2·0281	0295	0308	0321	0334	0347	0360	0373	0386	0399	1 3 4	5 7 8	9 10 12
7·7	2·0412	0425	0438	0451	0464	0477	0490	0503	0516	0528	1 3 4	5 6 8	9 10 12
7·8	2·0541	0554	0567	0580	0592	0605	0618	0631	0643	0656	1 3 4	5 6 8	9 10 11
7·9	2·0669	0681	0694	0707	0719	0732	0744	0757	0769	0782	1 3 4	5 6 8	9 10 11
8·0	2·0794	0807	0819	0832	0844	0857	0869	0882	0894	0906	1 3 4	5 6 7	9 10 11
8·1	2·0919	0931	0943	0956	0968	0980	0992	1005	1017	1029	1 2 4	5 6 7	9 10 11
8·2	2·1041	1054	1066	1078	1090	1102	1114	1126	1138	1150	1 2 4	5 6 7	9 10 11
8·3	2·1163	1175	1187	1199	1211	1223	1235	1247	1258	1270	1 2 4	5 6 7	8 10 11
8·4	2·1282	1294	1306	1318	1330	1342	1353	1365	1377	1389	1 2 4	5 6 7	8 9 11
8·5	2·1401	1412	1424	1436	1448	1459	1471	1483	1494	1506	1 2 4	5 6 7	8 9 11
8·6	2·1518	1529	1541	1552	1564	1576	1587	1599	1610	1622	1 2 3	5 6 7	8 9 10
8·7	2·1633	1645	1656	1668	1679	1691	1702	1713	1725	1736	1 2 3	5 6 7	8 9 10
8·8	2·1748	1759	1770	1782	1793	1804	1815	1827	1838	1849	1 2 3	5 6 7	8 9 10
8·9	2·1861	1872	1883	1894	1905	1917	1928	1939	1950	1961	1 2 3	4 6 7	8 9 10
9·0	2·1972	1983	1994	2006	2017	2028	2039	2050	2061	2072	1 2 3	4 6 7	8 9 10
9·1	2·2083	2094	2105	2116	2127	2138	2148	2159	2170	2181	1 2 3	4 5 7	8 9 10
9·2	2·2192	2203	2214	2225	2235	2246	2257	2268	2279	2289	1 2 3	4 5 6	8 9 10
9·3	2·2300	2311	2322	2332	2343	2354	2364	2375	2386	2396	1 2 3	4 5 6	7 9 10
9·4	2·2407	2418	2428	2439	2450	2460	2471	2481	2492	2502	1 2 3	4 5 6	7 8 10
9·5	2·2513	2523	2534	2544	2555	2565	2576	2586	2597	2607	1 2 3	4 5 6	7 8 9
9·6	2·2618	2628	2638	2649	2659	2670	2680	2690	2701	2711	1 2 3	4 5 6	7 8 9
9·7	2·2721	2732	2742	2752	2762	2773	2783	2793	2803	2814	1 2 3	4 5 6	7 8 9
9·8	2·2824	2834	2844	2854	2865	2875	2885	2895	2905	2915	1 2 3	4 5 6	7 8 9
9·9	2·2925	2935	2946	2956	2966	2976	2986	2996	3006	3016	1 2 3	4 5 6	7 8 9
10·0	2·3026												

Hyperbolic or Naperian Logarithms of 10^{-n}.

n	1	2	3	4	5	6	7	8	9
$\log_e 10^{-n}$	$\bar{3}$·6974	$\bar{5}$·3948	$\bar{7}$·0922	$\overline{10}$·7897	$\overline{12}$·4871	$\overline{14}$·1845	$\overline{17}$·8819	$\overline{19}$·5793	$\overline{21}$·2767

Table VII

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

x	e^x	e^{-x}	sinh x	cosh x
·02	1·0202	·9802	·0200	1·0002
·04	1·0408	·9608	·0400	1·0008
·06	1·0618	·9418	·0600	1·0018
·08	1·0833	·9231	·0801	1·0032
·10	1·1052	·9048	·1002	1·0050
·11	1·1163	·8958	·1102	1·0061
·12	1·1275	·8869	·1203	1·0072
·13	1·1388	·8781	·1304	1·0085
·14	1·1503	·8694	·1405	1·0098
·15	1·1618	·8607	·1506	1·0113
·16	1·1735	·8521	·1607	1·0128
·17	1·1853	·8437	·1708	1·0145
·18	1·1972	·8353	·1810	1·0162
·19	1·2092	·8270	·1911	1·0181
·20	1·2214	·8187	·2013	1·0201
·21	1·2337	·8106	·2115	1·0221
·22	1·2461	·8025	·2218	1·0243
·23	1·2586	·7945	·2320	1·0266
·24	1·2712	·7866	·2423	1·0289
·25	1·2840	·7788	·2526	1·0314
·26	1·2969	·7711	·2629	1·0340
·27	1·3100	·7634	·2733	1·0367
·28	1·3231	·7558	·2837	1·0395
·29	1·3364	·7483	·2941	1·0423
·30	1·3499	·7408	·3045	1·0453
31	1·3634	·7335	·3150	1·0484
·32	1·3771	·7261	·3255	1·0516
·33	1·3910	·7189	·3360	1·0550
·34	1·4050	·7118	·3466	1·0584
·35	1·4191	·7047	·3572	1·0619
36	1·4333	·6977	·3678	1·0655
37	1·4477	·6907	·3785	1·0692
·38	1·4623	·6839	·3892	1·0731
39	1·4770	·6771	·4000	1·0770
·40	1·4918	·6703	·4107	1·0811
41	1·5068	·6636	·4216	1·0852
·42	1·5220	·6570	·4325	1·0895
43	1·5373	·6505	·4434	1·0939
·44	1·5527	·6440	·4543	1·0984
45	1·5683	·6376	·4653	1·1030
46	1·5841	·6313	·4764	1·1077
47	1·6000	·6250	·4875	1·1125
48	1·6161	·6188	·4986	1·1174
49	1·6323	·6126	·5098	1·1225
·50	1·6487	·6065	·5211	1·1276
·6	1·8221	·5488	·6367	1·1855
·7	2·0138	·4966	·7586	1·2552
·8	2·2255	·4493	·8881	1·3374
·9	2·4596	·4066	1·0265	1·4331

x	e^x	e^{-x}	sinh x	cosh x
1·0	2·7183	·3679	1·1752	1·5431
1·1	3·0042	·3329	1·3356	1·6685
1·2	3·3201	·3012	1·5095	1·8107
1·3	3·6693	·2725	1·6984	1·9709
1·4	4·0552	·2466	1·9043	2·1509
1·5	4·4817	·2231	2·1293	2·3524
1·6	4·9530	·2019	2·3756	2·5775
1·7	5·4739	·1827	2·6456	2·8283
1·8	6·0497	·1653	2·9422	3·1075
1·9	6·6859	·1496	3·2682	3·4177
2·0	7·3891	·1353	3·6269	3·7622
2·1	8·1662	·1225	4·0219	4·1443
2·2	9·0250	·1108	4·4571	4·5679
2·3	9·9742	·1003	4·9370	5·0372
2·4	11·023	·0907	5·4662	5·5569
2·5	12·182	·0821	6·0502	6·1323
2·6	13·464	·0743	6·6947	6·7690
2·7	14·880	·0672	7·4063	7·4735
2·8	16·445	·0608	8·1919	8·2527
2·9	18·174	·0550	9·0596	9·1146
3·0	20·085	·0498	10·018	10·068
3·1	22·198	·0450	11·076	11·121
3·2	24·532	·0408	12·246	12·287
3·3	27·113	·0369	13·538	13·575
3·4	29·964	·0334	14·965	14·999
3·5	33·115	·0302	16·543	16·573
3·6	36·598	·0273	18·285	18·313
3·7	40·447	·0247	20·211	20·236
3·8	44·701	·0224	22·339	22·362
3·9	49·402	·0202	24·691	24·711
4·0	54·598	·0183	27·290	27·308
4·1	60·340	·0166	30·162	30·178
4·2	66·686	·0150	33·336	33·351
4·3	73·700	·0136	36·843	36·857
4·4	81·451	·0123	40·719	40·732
4·5	90·017	·0111	45·003	45·014
4·6	99·484	·0100	49·737	49·747
4·7	109·95	·00910	54·969	54·978
4·8	121·51	·00823	60·751	60·759
4·9	134·29	·00745	67·141	67·149
5·0	148·41	·00674	74·203	74·210
5·1	164·02	·00610	82·008	82·014
5·2	181·27	·00552	90·633	90·639
5·3	200·34	·00499	100·17	100·17
5·4	221·41	·00452	110·70	110·71
5·5	244·69	·00409	122·34	122·35
5·6	270·43	·00370	135·21	135·21
5·7	298·87	·00335	149·43	149·43
5·8	330·30	·00303	165·15	165·15
5·9	365·04	·00274	182·52	182·52
6·0	403·43	·00248	201·71	201·72

$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),\quad \sinh x = \tfrac{1}{2}(e^x - e^{-x}).$

Table VIII

POWERS, ROOTS AND RECIPROCALS

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$\sqrt{10n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$
1	1	1	1	1	3·162	2·154	4·642	1
2	4	8	1·414	1·260	4·472	2·714	5·848	·5000
3	9	27	1·732	1·442	5·477	3·107	6·694	·3333
4	16	64	2	1·587	6·325	3·420	7·368	·2500
5	25	125	2·236	1·710	7·071	3·684	7·937	·2000
6	36	216	2·449	1·817	7·746	3·915	8·434	·1667
7	49	343	2·646	1·913	8·367	4·121	8·879	·1429
8	64	512	2·828	2·000	8·944	4·309	9·283	·1250
9	81	729	3·000	2·080	9·487	4·481	9·655	·1111
10	100	1000	3·162	2·154	10·0	4·642	10·000	·1000
11	121	1331	3·317	2·224	10·488	4·791	10·323	·09091
12	144	1728	3·464	2·289	10·954	4·932	10·627	·08333
13	169	2197	3·606	2·351	11·402	5·066	10·914	·07692
14	196	2744	3·742	2·410	11·832	5·192	11·187	·07143
15	225	3375	3·873	2·466	12·247	5·313	11·447	·06667
16	256	4096	4·000	2·520	12·649	5·429	11·696	·06250
17	289	4913	4·123	2·571	13·038	5·540	11·935	·05882
18	324	5832	4·243	2·621	13·416	5·646	12·164	·05556
19	361	6859	4·359	2·668	13·784	5·749	12·386	·05263
20	400	8000	4·472	2·714	14·142	5·848	12·599	·0500
21	441	9261	4·583	2·759	14·491	5·944	12·806	·04762
22	484	10648	4·690	2·802	14·832	6·037	13·006	·04545
23	529	12167	4·796	2·844	15·166	6·127	13·200	·04348
24	576	13824	4·899	2·884	15·492	6·214	13·389	·04167
25	625	15625	5·000	2·924	15·811	6·300	13·572	·0400
26	676	17576	5·099	2·962	16·125	6·383	13·751	·03846
27	729	19683	5·196	3·000	16·432	6·463	13·925	·03704
28	784	21952	5·292	3·037	16·733	6·542	14·095	·03571
29	841	24389	5·385	3·072	17·029	6·619	14·260	·03448
30	900	27000	5·477	3·107	17·321	6·694	14·422	·03333
31	961	29791	5·568	3·141	17·607	6·768	14·581	·03226
32	1024	32768	5·657	3·175	17·889	6·840	14·736	·03125
33	1089	35937	5·745	3·208	18·166	6·910	14·888	·03030
34	1156	39304	5·831	3·240	18·439	6·980	15·037	·02941
35	1225	42875	5·916	3·271	18·708	7·047	15·183	·02857
36	1296	46656	6·000	3·302	18·974	7·114	15·326	·02778
37	1369	50653	6·083	3·332	19·235	7·179	15·467	·02703
38	1444	54872	6·164	3·362	19·494	7·243	15·605	·02632
39	1521	59319	6·245	3·391	19·748	7·306	15·741	·02564
40	1600	64000	6·325	3·420	20·00	7·368	15·874	·0250
41	1681	68921	6·403	3·448	20·248	7·429	16·005	·02439
42	1764	74088	6·481	3·476	20·494	7·489	16·134	·02381
43	1849	79507	6·557	3·503	20·736	7·548	16·261	·02326
44	1936	85184	6·633	3·530	20·976	7·606	16·386	·02273
45	2025	91125	6·708	3·557	21·213	7·663	16·510	·02222
46	2116	97336	6·782	3·583	21·448	7·719	16·631	·02174
47	2209	103823	6·856	3·609	21·679	7·775	16·751	·02128
48	2304	110592	6·928	3·634	21·909	7·830	16·869	·02083
49	2401	117649	7·000	3·659	22·136	7·884	16·985	·02041
50	2500	125000	7·071	3·684	22·361	7·937	17·100	·020

Table IX

POWERS ROOTS AND RECIPROCALS

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$\sqrt{10n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$\frac{1}{n}$
51	2601	132651	7·141	3·708	22·583	7·990	17·213	·01961
52	2704	140608	7·211	3·733	22·804	8·041	17·325	·01923
53	2809	148877	7·280	3·756	23·022	8·093	17·435	·01887
54	2916	157464	7·348	3·780	23·238	8·143	17·544	·01852
55	3025	166375	7·416	3·803	23·452	8·193	17·052	·01818
56	3136	175616	7·483	3·826	23·664	8·243	17·758	·01786
57	3249	185193	7·550	3·849	23·875	8·291	17·863	·01754
58	3364	195112	7·616	3·871	24·083	8·340	17·967	·01724
59	3481	205379	7·681	3·893	24·290	8·387	18·070	·01695
60	3000	216000	7·746	3·915	24·495	8·434	18·171	·01667
61	3721	226981	7·810	3·936	24·698	8·481	18·272	·01639
62	3844	238328	7·874	3·958	24·900	8·527	18·371	·01613
63	3969	250047	7·937	3·979	25·100	8·573	18·469	·01587
64	4096	262144	8·000	4·000	25·298	8·618	18·566	·01562
65	4225	274625	8·062	4·021	25·495	8·662	18·663	·01538
66	4356	287496	8·124	4·041	25·690	8·707	18·758	·01515
67	4489	300763	8·185	4·062	25·884	8·750	18·852	·01493
68	4624	314432	8·246	4·082	26·077	8·794	18·945	·01471
69	4761	328509	8·307	4·102	26·268	8·837	19·038	·01449
70	4900	343000	8·367	4·121	26·458	8·879	19·129	·01429
71	5041	357911	8·426	4·141	26·646	8·921	19·220	·01408
72	5184	373248	8·485	4·160	26·833	8·963	19·310	·01389
73	5329	389017	8·544	4·179	27·019	9·004	19·399	·01370
74	5476	405224	8·602	4·198	27·203	9·045	19·487	·01351
75	5625	421875	8·660	4·217	27·386	9·086	19·574	·01333
76	5776	438976	8·718	4·236	27·568	9·126	19·661	·01316
77	5929	456533	8·775	4·254	27·749	9·166	19·747	·01299
78	6084	474552	8·832	4·273	27·928	9·205	19·832	·01282
79	6241	493039	8·888	4·291	28·107	9·244	19·916	·01266
80	6400	512000	8·944	4·309	28·284	9·283	20·000	·01250
81	6561	531441	9·000	4·327	28·460	9·322	20·083	·01235
82	6724	551368	9·055	4·344	28·636	9·360	20·165	·01220
83	6889	571787	9·110	4·362	28·810	9·398	20·247	·01205
84	7056	592704	9·165	4·380	28·983	9·435	20·328	·01190
85	7225	614125	9·220	4·397	29·155	9·473	20·408	·01176
86	7396	636056	9·274	4·414	29·326	9·510	20·488	·01163
87	7569	658503	9·327	4·431	29·496	9·546	20·567	·01149
88	7747	681472	9·381	4·448	29·665	9·583	20·646	·01136
89	7921	704969	9·434	4·465	29·833	9·619	20·724	·01124
90	8100	729000	9·487	4·481	30·000	9·655	20·801	·01111
91	8281	753571	9·539	4·498	30·166	9·691	20·878	·01099
92	8464	778688	9·592	4·514	30·332	9·726	20·954	·01087
93	8649	804357	9·644	4·531	30·496	9·761	21·029	·01075
94	8836	830584	9·695	4·547	30·659	9·796	21·105	·01064
95	9025	857375	9·747	4·563	30·822	9·830	21·179	·01053
96	9216	884736	9·798	4·579	30·984	9·865	21·253	·01042
97	9409	912673	9·849	4·595	31·145	9·899	21·327	·01031
98	9604	941192	9·899	4·610	31·305	9·933	21·400	·01020
99	9801	970299	9·950	4·626	31·464	9·967	21·472	·01010
100	10000	1000000	10·000	4·642	31·623	10·000	21·544	·0100

Table—X

u	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.01595	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.07535
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	[illegible]
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.22575	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.26115	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.29955	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.44845	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.48645	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.49795	.4980	.4981
2.9	.4981	.4982	.49825	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.49865	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.49975	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.49995	.49995	.49995
3.9	.49995	.49995	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

Table – XI

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	3989	.3989	3989	.3988	.3986	.3984	.3962	.3980	.3977	.3973
0.1	.3970	.3965	.3961	.3956	.3951	.3945	.3939	.3932	.3925	.3918
0.2	.3910	.3902	.3894	.3885	.3876	.3867	.3857	.3847	.3836	.3825
0.3	.3814	.3802	.3790	.3778	.3765	.3752	.3739	.3725	3712	.3697
0.4	.3683	.3668	.3653	.3637	.3621	.3605	.3589	.3572	.3555	.3538
0.5	.3521	.3503	3485	.3467	.3448	.3429	.3410	.3391	.3372	.3352
0.6	.3332	.3312	.3292	.3271	.3251	.3230	.3209	.3187	.3166	.3144
0.7	.3123	.3101	.3079	.3056	.3034	.3011	.2989	2966	.2943	.2920
0.8	.2897	.2874	.2850	.2827	.2803	.2780	.2756	.2732	.2709	2685
0.9	.2661	.2637	.2613	.2589	.2565	.2541	.2516	.2492	.2468	.2444
1.0	.2420	.2396	.2371	.2347	.2323	.2299	.2275	.2251	.2227	.2203
1.1	.2179	.2155	.2131	.2107	.2083	.2059	.2036	.2012	.1989	.1965
1.2	.1942	.1919	.1895	.1872	.1849	.1826	.1804	.1781	1758	.1736
1.3	.1714	.1691	.1669	.1647	.1626	.1604	.1582	.1561	.1539	.1518
1.4	.1497	.1476	.1456	.1435	.1415	.1394	.1374	.1354	.1334	.1315
1.5	.1295	.1276	.1257	.1238	.1219	.1200	.1182	.1163	.1145	.1127
1.6	.1109	.1092	.1074	.1057	.1040	.1023	.1006	.0989	.0973	.0957
1.7	.0940	.0925	.0909	.0893	.0878	.0863	.0848	.0833	.0818	.0804
1.8	.0790	.0775	.0761	.0748	.0734	.0721	.0707	.0694	.0681	.0669
1.9	.0656	.0644	.0632	.0620	.0608	.0596	.0584	.0573	.0562	.0551
2.0	0540	.0529	.0519	.0508	.0498	.0488	.0478	.0468	.0459	.0449
2.1	.0440	.0431	.0422	.0413	.0404	0396	.0387	.0379	.0371	.0363
2.2	.0355	.0347	.0339	.0332	.0325	.0317	.0310	.0303	.0297	.0290
2.3	.0283	.0277	.0270	.0264	.0258	.0252	.0246	.0241	.0235	.0229
2.4	.0224	.0219	.0213	.0208	.0203	.0198	.0194	.0189	.0184	.0180
2.5	.0175	.0171	.0167	.0163	.0158	.0154	.0151	.0147	.0143	.0139
2.6	.0136	.0132	.0129	.0126	.0122	.0119	.0116	0113	.0110	.0107
2.7	.0104	.0101	.0099	.0096	.0093	.0091	.0088	.0086	.0084	.0081
2.8	.0079	.0077	.0075	.0073	.0071	.0069	.0067	.0065	.0063	.0061
2.9	0060	.0058	.0056	.0055	.0053	.0051	.0050	.0048	.0047	.0046
3.0	.0044	.0043	.0042	.0040	.0039	.0038	.0037	.0036	.0035	.0034
3.1	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026	0025	.0025
3.2	.0024	.0023	.0022	.0022	.0021	.0020	.0020	.0019	.0018	.0018
3.3	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014	.0013	.0013
3.4	.0012	.0012	.0012	.0011	.0011	.0010	0010	.0010	.0009	.0009
3.5	.0009	.0008	.0008	.0008	.0008	0007	.0007	.0007	.0007	0006
3.6	.0006	.0006	.0006	.0005	.0005	.0005	0005	.0005	.0005	0004
3.7	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003	.0003	0003
3.8	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0002	.0002	.0002
3.9	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001

Table—XII

Pobability

n	·99	·95	·90	·70	·50	·30	·10	·05	·01
1	$\cdot 0^{3}157$	·00393	·0158	·148	·455	1·074	2·706	3·841	6·635
2	·0201	·103	·211	·713	1·386	2·408	4·605	5·991	9·210
3	·115	·352	·584	1·424	2·366	3·665	6·251	7·815	11·345
4	·297	·711	1·064	2·195	3·357	4·878	7·779	9·438	13·277
5	·554	1·145	1·610	3·000	4·351	6·064	9·236	11·070	15·086
6	·872	1·635	2·204	3·828	5·348	7·231	10·645	12·592	16·812
7	1·239	2·167	2·833	4·671	6·346	8·383	12·017	14·067	18·475
8	1·646	2·733	3·490	5·527	7·344	9·524	13·362	15·507	20·090
9	2·088	3·325	4·168	6·393	8·343	10·656	14·684	16·919	21·666
10	2·558	3·940	4·865	7·267	9·342	11·781	15·987	18·307	23·209
11	3·053	4·575	5·578	8·148	10·341	12·899	17·275	19·675	24·725
12	3·571	5·226	6·304	9·034	11·340	14·011	18·549	21·026	26·217
13	4·107	5·892	7·042	9·926	12·340	15·119	19·812	22·362	27·688
14	4·660	6·571	7·790	10·821	13·339	16·222	21·064	23·685	29·141
15	5·229	7·261	8·547	11·721	14·339	17·322	22·307	24·996	30·578
16	5·812	7·962	9·312	12·624	15·338	18·418	23·542	26·296	32·000
17	6·408	8·672	10·085	13·531	16·338	19·511	24·769	27·587	33·409
18	7·015	9·390	10·865	14·440	17·338	20·601	25·989	28·869	34·805
19	7·633	10·117	11·651	15·352	18·338	21·689	27·204	30·144	36·191
20	8·260	10·851	12·443	16·266	19·337	22·775	28·412	31·410	37·566
21	8·897	11·591	13·240	17·182	20·337	23·858	29·615	32·671	38·932
22	9·542	12·338	14·041	18·101	21·337	24·939	30·813	33·924	40·289
23	10·196	13·091	14·848	19·021	22·337	26·018	32·007	35·172	41·638
24	10·856	13·848	15·659	19·943	23·337	27·096	33·196	36·415	42·980
25	11·524	14·611	16·473	20·867	24·337	28·172	34·382	37·652	44·314
26	12·198	15·379	17·292	21·792	25.336	29·246	35·563	38·885	45·642
27	12·879	16·151	18·114	22·719	26.336	30·319	36·741	40·113	46·963
28	13·565	16·928	18·939	23·647	27.336	31·391	37·916	41·337	48·278
29	14·256	17·708	19·768	24·577	28.336	32·461	39·087	42·557	49·588
30	14·953	18·493	20·599	25·508	29·336	33·530	40·256	43·773	50·892

Table—XIII

Probability

n	·9	·5	·4	·1	·05	·01
1	·158	1·000	1·376	6·314	12·706	63·657
[illegible]	·142	·816	1·061	2·920	4·303	9·925
[illegible]	·137	·765	·978	2·353	3·182	5·841
4	·134	·741	·941	2·132	2·776	4·604
5	·132	·727	·920	2·015	2·571	4·032
6	·131	·718	·906	1·943	2·447	3·707
7	·130	·711	·896	1·895	2·365	3·499
[illegible]	·130	·706	·889	1·860	2·306	3·355
9	·129	·703	·883	1·833	2·262	3·250
10	·129	·700	·879	1·812	2·228	3·169
11	·129	·697	·876	1·796	2·201	3·106
12	·128	·695	·873	1·782	2·179	3·055
13	·128	·694	·870	1·771	2·160	3·012
14	·128	·692	·868	1·761	2·145	2·977
15	·128	·691	·866	1·753	2·131	2·947
16	·128	·690	·865	1·746	2·120	2·921
17	·128	·689	·863	1·740	2·110	2·898
18	·127	·688	·862	1·734	2·101	2·878
19	·127	·688	·861	1·729	2·093	2·861
20	·127	·687	·860	1·725	2·086	2·845
21	·127	·686	·859	1·721	2·080	2·831
22	·127	·686	·858	1·717	2·074	2·819
23	·127	·685	·858	1·714	2·069	2·807
24	·127	·685	·857	1·711	2·064	2·797
25	·127	·684	·856	1·708	2·060	2·787
26	·127	·684	·856	1·706	2·056	2·779
27	·127	·684	·855	1·703	2·052	2·771
28	·127	·683	·855	1·701	2·048	2·763
29	·127	·683	·854	1·699	2·045	2·756
30	·127	·683	·854	1·697	2·042	2·750
40	·126	·681	·851	1·684	2·021	2·704
60	·126	·679	·848	1·671	2·000	2·660
120	·126	·677	·845	1·658	1·980	2·617
∞	·126	·674	·842	1.645	1·960	2·576

Table—XIV (5% points)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	∞
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.95	248.01	249.05	254.32
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.84	2.77	2.74	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.00

Table XV (1% points of F)

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	∞
1	4,052.4	4,999.5	5,403.3	5,624.6	5,763.7	5,859.0	5,928.3	5,981.6	6,022.5	6,055.8	6,106.3	6,157.3	6,2 8.7	6,234.6	6,366.0
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.34	27.23	27.05	26.87	26.69	26.60	26.12
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.02
6	13.74	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.31	4.19	4.10	3.96	3.82	3.66	3.59	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.42
21	8.0	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.52	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.00

Table XVI

5% points of z

ν_2 \ ν_1	1	2	3	4	5	6	8	12	24	∞
1	2·5421	2·6479	2·6870	2·7071	2·7194	2·7276	2·7380	2·7484	2·7588	2·7693
2	1·4592	1·4722	1·4765	1·4787	1·4800	1·4808	1·4819	1·4830	1·4840	1·4851
3	1·1577	1·1284	1·1137	1·1051	1·0994	1·0953	1·0899	1·0842	1·0781	1·0716
4	1·0212	·9690	·9429	·9272	·9168	·9093	·8993	·8885	·8767	·8639
5	·9441	·8777	·8441	·8236	·8097	·7997	·7862	·7714	·7550	·7368
6	·8948	·8188	·7798	·7558	·7394	·7274	·7112	·6931	·6729	·6409
7	·8606	·7777	·7347	·7080	·6896	·6761	·6576	·6369	·6134	·5862
8	·8355	·7475	·7014	·6725	·6525	·6378	·6175	·5945	·5682	·5371
9	·8163	·7242	·6757	·6450	·6238	·6080	·5862	·5613	·5324	·4979
10	·8012	·7058	·6553	·6232	·6009	·5843	·5611	·5346	·5035	·4657
11	·7889	·6909	·6387	·6055	·5822	·5648	·5406	·5126	·4795	·4387
12	·7788	·6786	·6250	·5907	·5666	·5487	·5234	·4941	·4592	·4156
13	·7703	·6682	·6134	·5783	·5535	·5350	·5089	·4785	·4419	·3957
14	·7630	·6594	·6036	·5677	·5423	·5233	·4964	·4649	·4269	·3782
15	·7568	·6518	·5950	·5585	·5326	·5131	·4855	·4532	·4138	·3628
16	·7514	·6451	·5876	·5505	·5241	·5042	·4763	·4428	·4022	·3490
17	·7466	·6393	·5811	·5434	·5166	·4964	·4676	·4337	·3919	·3366
18	·7424	·6341	·5753	·5371	·5099	·4894	·4602	·4255	·3827	·3253
19	·7386	·6295	·5701	·5315	·5040	·4832	·4535	·4182	·3743	·3151
20	·7352	·6254	·5654	·5265	·4986	·4776	·4474	·4116	·3668	·3057
21	·7322	·6216	·5612	·5219	·4938	·4725	·4420	·4055	·3599	·2971
22	·7294	·6182	·5574	·5178	·4894	·4679	·4370	·4001	·3536	·2892
23	·7269	·6151	·5540	·5140	·4854	·4636	·4325	·3950	·3478	·2818
24	·7246	·6123	·5508	·5106	·4817	·4598	·4283	·3904	·3425	·2749
25	·7225	·6097	·5478	·5074	·4783	·4562	·4244	·3862	·3376	·2685
26	·7205	·6073	·5451	·5045	·4752	·4529	·4209	·3823	·3330	·2625
27	·7187	·6051	·5427	·5017	·4723	·4499	·4176	·3786	·3287	·2569
28	·7171	·6030	·5403	·4992	·4696	·4471	·4146	·3752	·3248	·2516
29	·7155	·6011	·5382	·4969	·4671	·4444	·4117	·3720	·3211	·2466
30	·7141	·5994	·5362	·4947	·4648	·4420	·4090	·3691	·3176	·2419
40	·7037	·5866	·5217	·4789	·4479	·4242	·3897	·3475	·2920	·2057
60	·6933	·5738	·5073	·4632	·4311	·4064	·3702	·3255	·2654	·1644
120	·6830	·5611	·4930	·4475	·4143	·3885	·3506	·3032	·2376	·1131
∞	·6729	·5486	·4787	·4319	·3974	·3706	·3309	·2804	·2085	0

Table XVII

1% points of z

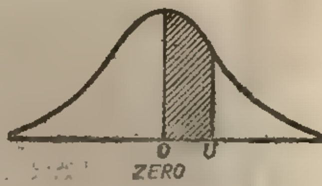
ν_2 \ ν_1	1	2	3	4	5	6	8	12	24	∞
1	4·1535	4·2585	4·2974	4·3175	4·3297	4·3379	4·3482	4·3585	4·3689	4·3794
2	2·2950	2·2976	2·2984	2·2988	2·2991	2·2992	2·2994	2·2997	2·2999	2·3001
3	1·7649	1·7140	1·6915	1·6786	1·6703	1·6645	1·6569	1·6489	1·6404	1·6314
4	1·5270	1·4452	1·4075	1·3856	1·3711	1·3609	1·3473	1·3327	1·3170	1·3000
5	1·3943	1·2929	1·2449	1·2164	1·1974	1·1838	1·1656	1·1457	1·1239	1·0997
6	1·3103	1·1955	1·1401	1·1068	1·0843	1·0680	1·0460	1·0218	·9948	·9643
7	1·2526	1·1281	1·0672	1·0300	1·0048	·9864	·9614	·9335	·9020	·8658
8	1·2106	1·0787	1·0135	·9734	·9459	·9259	·8983	·8673	·8319	·7904
9	1·1786	1·0411	·9724	·9299	·9006	·8791	·8494	·8157	·7769	·7305
10	1·1535	1·0114	·9399	·8954	·8646	·8419	·8104	·7744	·7324	·6816
11	1·1333	·9874	·9136	·8674	·8354	·8116	·7785	·7405	·6958	·6408
12	1·1166	·9677	·8919	·8443	·8111	·7864	·7520	·7122	·6649	·6061
13	1·1027	·9511	·8737	·8248	·7907	·7652	·7295	·6882	·6386	·5761
14	1·0909	·9370	·8581	·8082	·7732	·7471	·7103	·6675	·6159	·5500
15	1·0807	·9249	·8448	·7939	·7582	7314	·6937	·6496	·5961	·5269
16	1·0719	·9144	·8331	·7814	·7450	·7177	·6791	·6339	·5786	·5064
17	1·0641	·9051	·8229	·7705	·7335	·7057	·6663	·6199	·5630	·4879
18	1·0572	·8970	·8138	·7607	·7232	·6950	·6549	·6075	·5491	·4712
19	1·0511	·8897	·8057	·7521	·7140	·6854	·6447	·5964	·5366	·4560
20	1·0457	·8831	·7985	·7443	·7058	·6768	·6355	·5864	·5253	·4421
21	1·0408	·8772	·7920	·7372	·6984	·6690	·6272	·5773	·5150	·4294
22	1·0363	·8719	·7860	·7309	·6916	·6620	·6196	·5691	·5056	·4176
23	1·0322	·8670	·7806	·7251	·6855	·6555	·6127	·5615	·4969	·4068
24	1·0285	·8626	·7757	·7197	·6799	·6496	·6064	·5545	·4890	·3967
25	1·0251	·8585	·7712	·7148	·6747	·6442	·6006	·5481	·4816	·3872
26	1·0220	·8548	·7670	·7103	·6699	·6392	·5952	·5422	·4748	·3784
27	1·0191	·8513	·7631	·7062	·6655	·6346	·5902	·5367	·4685	·3701
28	1·0164	·8481	·7595	·7023	·6614	·6303	·5856	·5316	·4626	·3624
29	1·0139	·8451	·7562	·6987	·6576	·6263	·5813	·5269	·4570	·3550
30	1·0116	·8423	·7531	·6954	·6540	·6226	·5773	·5224	·4519	·3481
40	·9949	·8223	·7307	·6712	·6283	·5956	·5481	·4901	·4138	·2952
60	·9784	·8025	·7086	·6472	·6028	·5687	·5189	·4574	·3746	·2352
120	·9622	·7829	·6867	·6234	·5774	5419	·4897	·4243	·3339	·1612
∞	·9462	·7636	·6651	·5959	·5522	·5152	·4604	·3908	·2913	0

TABLE X

Areas under the Standard Normal Curve

The area is measured from the mean '0' to any ordinate 'U'.

The table gives the shaded area.



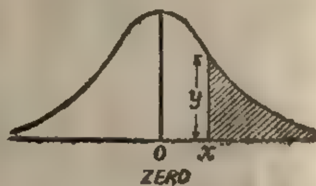
The results are given for values of 'U' at intervals 0·01.

TABLE XI

Ordinates of the Standard Normal Curve

The table gives ordinates (y) erected at a distance 'x' from the mean *i.e.*,

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

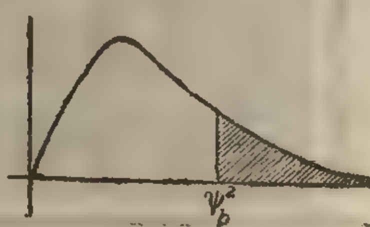


The results are given for values of 'x' at intervals 0·01.

TABLE XII

Significance Points of ψ^2

The table gives the values of ψ_p^2 for different p's and degrees of freedom 'n'.



Shaded area=p

For large values of n, the expression $\sqrt{2\psi^2} - \sqrt{2n-1}$ may be used as a normal variate with unit variance, remembering that the probability for ψ^2 corresponds with that of a single tail of the normal curve.

TABLE XIII

Values of 'mod. t'

Significant values t_0 of t for given probabilities P_F and $d.f.v.$ where

$$P_F = P(\,|\,t\,|\,) > t_0)$$

P_F is the shaded area.

TABLE XIV

Variance Ratio

Upper 5% points of F

v_1 is the number of degrees of freedom for the greater estimate of variance and v_2 for the smaller.

Lower 5% poiats are found by interchanging v_1 and v_2.

TABLE XV

Variance Ratio

Upper 1% points of F

v_1 is the number of degrees of freedom for the greater estimate of variance and v_2 for the smaller.

Lower 1% points are found by interchanging v_1 and v_2.